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Monotonicity properties of interpolation spaces

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MONOTONICITY PROPERTIES OF INTERPOLATION SPACES

by Michael Cwikel

ABSTRACT.- For any interpolation pair (A_o, A_1), Peetre's K-functional is defined by :

$$K(t,a\ ;\ A_o,A_1) = \inf_{a=a_o+a_1} \|a_o\|_{A_o} + t\|a_1\|_{A_1}.$$

We show that all interpolation spaces A for the pair (L^p, L^q) are characterised by the property of K-monotonicity, that is, if $a\in A$ and $K(t,b\ ;\ L^p,L^q) \leq K(t,a\ ;\ L^p,L^q)$ for all positive t then $b\in A$ also. This extends results of Calderon and of Lorentz and Shimogaki. Sedaev and Semenov also showed that all the interpolation spaces for a pair of weighted L^p spaces and for a pair of Hilbert spaces have analogous characterisations. We give a necessary (but not sufficient) condition for an interpolation pair to have its interpolation spaces characterised by K-monotonicity. We describe a weaker form of K-monotonicity which holds for all the interpolation spaces of any interpolation pair and show that it is in a sense the strongest form of monotonicity which holds in such generality.

O. INTRODUCTION.

In the study of interpolation spaces the point of departure is usually a pair of Banach spaces A_o and A_1 which are both continuously embedded in some Hausdorff topological vector space $\mathcal{A}$. We refer to the couple (A_o, A_1) as an interpolation pair.

For such a pair the vector spaces $A_o \cap A_1$ and $A_o + A_1$ are well defined and, when normed by $\|a\|_{A_o\cap A_1} = \max(\|a\|_{A_o}, \|a\|_{A_1})$ and

$\|a\|_{A_o+A_1} = \inf_{a=a_o+a_1} (\|a_o\|_{A_o} + \|a_1\|_{A_1})$, become Banach spaces continuously embedded in $\mathcal{A}$.

$A_o + A_1$ can be equivalently renormed by Peetre's "K-functional"

$$K(t,a\ ;\ A_o,A_1) = \inf_{a=a_o+a_1} (\|a_o\|_{A_o} + t\|a_1\|_{A_1})$$

for any positive number t. The abbreviated notation $K(t,a)$ is also used where there is no risk of ambiguity. For each fixed $a \in A_o+A_1$, $K(t,a)$ is a continuous non decreasing concave function of t. (see [1] p. 167).

A vector space A is called <u>intermediate</u> if $A_o \cap A_1 \subset A \subset A_o+A_1$, the inclusions being continuous embeddings if A is topologised. An intermediate space A is an <u>interpolation space</u> if all linear operators on $A_o + A_1$ which map A_o continuously into itself and A_1 continuously into itself also map A into itself (continuously if A is topologised).

We shall be concerned here with the characterisation of all interpolation spaces for a given couple (A_o,A_1). The first result of this type was obtained by Calderón [4] for the pair (L^1,L^∞). Subsequently Lorentz and Shimogaki [9] treated the pair (L^p,L^∞) with $1 < p < \infty$, and Sedaev and Semenov [13], [14], dealt with a pair of L^p spaces with different weights, and also with a pair of Hilbert spaces. For each of these interpolation pairs it was found that the corresponding interpolation spaces could be characterised as those spaces possessing a property which we shall call <u>K-monotonicity.</u>

DEFINITION 1. The space A is <u>K-monotone</u> with respect to the pair (A_o,A_1) if whenever $a \in A$, $b \in A_o+A_1$ and $K(t,b\ ;\ A_o,A_1) \leq K(t,a\ ;\ A_o,A_1)$ for all positive t it follows that $b \in A$.

In view of the above series of results we also introduce the following terminology.

DEFINITION 2. The interpolation pair (A_0,A_1) will be called a <u>Calderón pair</u> if every intermediate space is an interpolation space if and only if it is K-monotone.

In section II of this paper we show that (L^p,L^q) is a Calderon pair for any choice of p and q in $[1,\infty]$. The proof is given for an arbitrary measure space, thus dispensing with some restrictions imposed in the above-mentioned studies of (L^1,L^∞) and (L^p,L^∞). We remark that this result enables a reformulation of a theorem about norm convergence of Fourier series in rearrangement invariant Banach spaces. (See [5]. (We refer to [9] for an alternative characterisation of the interpolation spaces of $(L^1,L^p$ obtained by dualising the results for $(L^{p'},L^\infty)$.)

In sectionsIII and IV we study the interplay of K-monotonicity and interpolation in the general setting. A necessary condition for an interpolation pair to be Calderón is describec in section III. This condition is not sufficient. In section IV we show that for an arbitrary interpolation pair (A_0,A_1), every interpolation space A satisfies a weak form of K-monotonicity : if $a\in A$ and $b\in A_0+A_1$, then b is also in A if the inequality $K(t,b)\le w(t)K(t,a)$ holds for all positive t, where w(t) is a positive measurable function satisfying $\int_0^\infty \min(\varepsilon,w(t))dt/t<\infty$ for some positive constant ε. This result seems very close to the best possible. It will be seen that the hypothesis on w(t) cannot be weakened to $\int_0^\infty \min(\varepsilon,w(t)^p)\,dt/t<\infty$ for some $p>1$.

I. PRELIMINARIES.

For any pair of Banach spaces A and B, $\mathcal{L}(A,B)$ will denote the class of all bounded linear operators mapping A into B, and $\mathscr{L}_\lambda(A,B)$ will denote the subclass of $\mathscr{L}(A,B)$ of operators with norm not exceeding λ. Let $\mathscr{L}(A) = \mathscr{L}(A,A)$ and $\mathscr{L}_\lambda(A) = \mathscr{L}_\lambda(A,A)$.

Let R_+ denote the positive real line equipped with Lebesgue measure. Where it is necessary to indicate the underlying measure space of the space L^p we shall write $L^p(R_+)$, or $L^p(X)$ or $L^p(\mu)$ in the case of a measure space (X,Σ,μ).

Given an interpolation pair (A_o,A_1). there are two important special methods of constructing interpolation spaces.

(i) The real method (see for example [1] Chapter 3) : For $0 < \theta < 1$ and $1 \le q < \infty$, the space $(A_o,A_1)_{\theta,q}$ is defined to consist of all elements $a \in A_o + A_1$ such that

$$\|a\|_{(A_o,A_1)_{\theta,q}} = \left(\int_0^\infty \left[t^{-\theta} K(t,a\ ;\ A_o,A_1)\right]^q dt/t\right)^{1/q} < \infty.$$

$(A_o,A_1)_{\theta,\infty}$ is defined similarly by the norm $\sup_{t>0} t^{-\theta} K(t,a)$.

(ii) The complex method (see for example [3]) : Let $\mathscr{F}(A_o,A_1)$ be the space of A_o+A_1 – valued functions $f(z)$ continuous in the strip $0 \le \operatorname{Re} z \le 1$ and analytic in its interior such that

$$\|f\|_{\mathscr{F}} = \max\left\{\sup_{-\infty<y<\infty} \|f(iy)\|_{A_o},\ \sup_{-\infty<y<\infty} \|f(1+iy)\|_{A_1}\right\} < \infty.$$

Then the complex interpolation space $[A_o,A_1]_\theta$ is defined by $[A_o,A_1]_\theta = \{f(\theta)\,|\,f \in \mathscr{F}\}$, and as norm we usually take $\|a\|_\theta = \inf\{\|f(z)\|_{\mathscr{F}}\,|\,f(\theta) = a\}$.

The notation $\Phi(t,f) \sim \Psi(t,f)$ shall mean that there exists a positive constant C independent of t and f such that $C^{-1}\,\Phi(t,f) \le \Psi(t,f) \le C\,\Phi(t,f)$.

II. (L^p, L^q) IS A CALDERÓN PAIR.

It is a simple matter to show that if $T \in \mathcal{L}_\alpha(A_0) \cap \mathcal{L}_\beta(A_1)$ and $a \in A_0 + A_1$, then $K(t, Ta ; A_0, A_1) \leq \max(\alpha, \beta) K(t, a ; A_0, A_1)$. Thus any K-monotone space is necessarilv an interpolation space with respect to (A_0, A_1). The non trivial part of the proof that a given pair (A_0, A_1) is Calderón is to show that if f, g are in $A_0 + A_1$ with $K(t,g) \leq K(t,f)$ for all positive t, then there exists an operator $T \in \mathcal{L}(A_0) \cap \mathcal{L}(A_1)$ with $Tf = g$ and so every interpolation space is K-monotone. Theorem 4 will give such an operator for the pair $(L^p(\mu), L^q(\mu))$ where $1 \leq p, q \leq \infty$ and (X, Σ, μ) is an arbitrary measure space.

For any measurable function f on (X, Σ, μ) we let $f^*(t)$ denote the non-increasing rearrangement of $|f|$ on R_+. Then

$$K(t, f ; L^1, L^\infty) = \int_0^t f^*(s)\, ds \quad \text{(Peetre [10])}$$

and

$$K(t, f ; L^p, L^\infty) \sim \left(\int_0^{t^p} f^*(s)^p\, ds\right)^{1/p} \quad \text{(Krée [8])}.$$

For $0 < p < q < \infty$, Holmstedt [6] has shown that :

$$K(t, f ; L^p, L^q) \sim \left(\int_0^{t^\alpha} f^*(s)^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^*(s)^q\, ds\right)^{1/q}$$

where $1/\alpha = 1/p - 1/q$.

THEOREM 1. <u>Let</u> $p \in [1, \infty)$ <u>and let</u> f, g <u>be non-negative non-increasing simple functions on</u> R_+ <u>such that :</u>

$$\int_0^t g(s)^p\, ds \leq \int_0^t f(s)^p\, ds \quad \underline{\text{for all positive}} \quad t.$$

<u>Then there exists an operator</u> $T \in \mathcal{L}_1(L^p(R_+)) \cap \mathcal{L}_1(L^\infty(R_+))$ <u>such that</u> $Tf = g$.

Proof: This is exactly Lemma 4 of [9]. (The case $p = 1$ was treated in [4]).

THEOREM 2. *Let* $q \in (1,\infty)$ *and let* f, g *be non-negative non-increasing simple functions on* R_+ *such that* :

(1) $$\int_t^\infty g(s)^q \, ds \leq \int_t^\infty f(s)^q \, ds \quad \text{for all positive } t.$$

Then there exists an operator $T \in \mathcal{L}_1(L^1(R_+)) \cap \mathcal{L}_1(L^q(R_+))$ such that $Tf = g$.

Proof: We proceed via two lemmas.

LEMMA 2A. Let φ, ψ be two measurable functions on a finite measure space such that φ is a constant and let $q > 1$. Then $\|\psi\|_{L^q} \le \|\varphi\|_{L^q}$ implies $\|\psi\|_{L^1} \le \|\varphi\|_{L^1}$.

Proof: Simple application of Hölder's inequality.

LEMMA 2B. Let f be a non-negative non-increasing simple function on R_+ taking a constant value α on an interval $[a,b)$. Then for any a', $0 < a' \le a$, there exists an operator $S \in \mathcal{L}_1(L^1(R_+)) \cap \mathcal{L}_1(L^q(R_+))$ such that :

(i) f is non-negative and non-increasing

(ii) $Sf = \alpha$ on $[a', b)$

(iii) $\int_t^\infty (Sf)^q \, ds = \int_t^\infty f^q \, ds$ for all $0 \le t \le a''$

where $[a'', a')$ is the interval of constancy of Sf preceding $[a', b)$

(iv) The number of different values taken by Sf on $[0,a')$ does not exceed the number of different values taken by f on $[0,a)$.

Proof: Let $f = \sum_{j=1}^{N} \alpha_j \chi_{[a_{j-1}, a_j)} + \alpha \chi_{[a,b)} + f\chi_{[b,\infty)}$ where $0 = a_0 < a_1 < \ldots$ $\ldots < a_N = a$, and $\alpha_1 > \alpha_2 \ldots > \alpha_N > \alpha$. For each $u \in [a_{N-1}, a_N)$ define the function f_u to equal α on $[u, a_N)$ and to equal $\lambda(u)\alpha_N$ on $[a_N, u)$, where $\lambda(u) > 1$ is chosen to give

$$\int_{a_{N-1}}^{a_N} f_u^q \, ds = \int_{a_{N-1}}^{a_N} f^q \, ds.$$

By Lemma 2A, $\int_{a_{N-1}}^{a_N} f_u \, ds \le \int_{a_{N-1}}^{a_N} f \, ds.$

Clearly $\lambda(u)$ is a continuous decreasing function of u. Let u_N be the smallest value of u in $[a_{N-1}, a_N)$ for which $\lambda(u)\alpha_N \le a_{N-1}$, and for all $u \in [u_N, a_N)$ define the operator S_u by :

$$S_u h = \frac{f_u}{(a_N - a_{N-1})\,\alpha_N} \int_{a_{N-1}}^{a_N} h\, ds \quad \text{on} \quad [a_{N-1}, a_N)$$

$$= h \quad \text{elsewhere,}$$

for all $h \in L^1 + L^q$.

It is easy to see that $S_u \in \mathcal{L}_1(L^1) \cap \mathcal{L}_1(L^q)$, and that $S_u f = f_u$ on $[a_{N-1}, a_N)$ and equals f elsewhere. Thus $S_u f$ satisfies (i), (ii), (iii) and (iv) with $a' = u$ and $[a'', a') = [a_{N-1}, u)$. If the given number a' satisfies $a' \ge u_N$ this completes the proof of the lemma. If instead $a' < u_N$ the process must be reapplied as follows. Let us redefine a_{N-1} to be u_N. Then

$$S_{u_N} f = \sum_{j=1}^{N-1} \alpha_j \chi[a_{j-1}, a_j) + \alpha \chi[a_{N-1}, b) + f \chi[b, \infty).$$

We may apply the preceding argument to the function $S_{u_N} f$ and construct a new function $S_u(S_{u_N} f)$ which equals α on the interval $[u, b)$. This construction will be valid for all $u \in [u_{N-1}, u_N)$ where u_{N-1} is determined by conditions analogous to those above which fix u_N. Again S_u will be an operator in the class $\mathcal{L}_1(L^1) \cap \mathcal{L}_1(L^q)$ and consequently the composed operator $S_u S_{u_N}$ will also be in this class. Reiterating this argument as many times as necessary we can, so to speak, move the point u back to any point $a' > 0$ by an operator $S = S_{a'} S_{u_M} S_{u_{M+1}} \ldots S_{u_N}$, such that $S \in \mathcal{L}_1(L^1) \cap \mathcal{L}_1(L^q)$ and Sf satisfies (i), (ii), (iii) and (iv).

Proof of Theorem 2: Let f and g be functions satisfying the hypotheses of the theorem. Let $f = \sum_{j=1}^{N} \alpha_j \chi[c_{j-1}, c_j)$, with $0 = c_0 < c_1 < c_2 \ldots < c_N$ and $\alpha_1 > \alpha_2 > \ldots > \alpha_N$

We shall perform induction on N. If $N = 1$, $f = \alpha_1 \chi_{[0,c_1)}$, g must vanish outside $[0,c_1)$ and so $\int_0^{c_1} g^q\, ds \leq \int_0^{c_1} f^q\, ds$. By Lemma 2A we then have $\int_0^{c_1} g\, ds \leq \int_0^{c_1} f\, ds$ and the desired operator T is given by $Th = \left(\frac{1}{\alpha_1 c_1}\int_0^{c_1} h\, ds\right) g$ for all $h \in L^1 + L^q$.

Now suppose the theorem is proven in the case where f has $N-1$ different positive values and consider $f = \sum_{j=1}^{N} \alpha_j \chi_{[c_{j-1},c_j)}$ and g as above such that (1) holds for all $t > 0$. It follows that $g(s)$ must vanish for $s > c_N$ and so :

(2) $$\int_{c_{N-1}}^{c_N} g^q\, ds \leq \int_{c_{N-1}}^{c_N} f^q\, ds = \alpha_N^q(c_N - c_{N-1}).$$

At this point we must consider two possible cases.

CASE 1. Suppose that $\int_0^{c_N} g^q\, ds \leq \alpha_N^q\, c_N$. Then, by Lemma 2A, $\int_0^{c_N} g\, ds \leq \alpha_n\, c_N$ and the operator T can be obtained in the form $Th = \left(\frac{1}{c_N}\int_0^{c_N} h/f\ ds\right)g$.

CASE 2. Alternatively we have :

(3) $$\int_0^{c_N} g^q\, ds > \alpha_N^q\, c_N.$$

From (2) and (3) and the fact that g is non-increasing we deduce that there exists a number $a' \in (0, c_{N-1}]$ for which

(4) $$\int_{a'}^{c_N} g^q\, ds = \alpha_N^q(c_N - a') = \int_{a'}^{c_N} (Sf)^q\, ds\ ,$$

where $S \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ is an operator of the type constructed in Lemma 2B, chosen to give $Sf = \alpha_N$ on $[a', c_N)$. Furthermore Sf is a non-negative non-increasing simple function vanishing on (c_N, ∞) and

(5) $$\int_t^{\infty} g^q\, ds \leq \int_t^{\infty} (Sf)^q\, ds \qquad \text{for all } t \leq a''$$

where $[a'', a')$ is the interval of constancy of Sf preceding a'. In fact (5) will be shown to hold for all positive t. If $t \geq c_N$ $\int_t^{\infty} g^q\, ds = \int_t^{\infty} (Sf)^q\, ds = 0$. If $t \in [a', c_N)$

$$\int_t^\infty (Sf)^q\, ds = \alpha_N^q (c_N - t) \geq \int_t^\infty g^q\, ds$$

from (4) and the fact that g is non-increasing. It remains to consider $t \in [a'',a')$. On this interval $\int_t^\infty (Sf)^q\, ds$ is a linear function and $\int_t^\infty g^q\, ds$ is a convex function since its gradient is increasing (becoming less negative). The inequality (5) holds for $t = a''$, $t = a'$, and so holds for all $t \in [a'',a']$.

Using (4) and the constancy of Sf on $[a',c_N)$ we see that the operator U, defined by

$$Uh = \chi_{[0,a')} h + \left(\frac{1}{\alpha_N(c_N-a')}\int_{a'}^{c_N} h\, ds\right) \chi_{[a',c_N)}\, g$$

is in $\mathcal{L}_1(L^1) \cap \mathcal{L}_1(L^q)$ and $USf = \chi_{[0,a')}\, Sf + \chi_{[a',c_N)}\, g$. $\chi_{[0,a')}\, Sf$ is a non-increasing simple function taking no more than $N-1$ different non-zero values (by (iv) in Lemma 2A) and from (4) and (5),

$$\int_t^\infty [\chi_{[0,a')} g]^q\, ds \leq \int_t^\infty [\chi_{[0,a')}\, Sf]^q\, ds \qquad \text{for all} \quad t \geq 0.$$

By the inductive hypothesis there exists an operator $V \in \mathcal{L}_1(L^1) \cap \mathcal{L}_1(L^q)$ with $V(\chi_{[0,a')}\, Sf) = \chi_{[0,a')}\, g$.

Let T be the operator

$$Th = \chi_{[0,a')}\, V[\chi_{[0,a')}\, Sh] + \chi_{[a',c_N)}\, U[\chi_{[a',c_N)}\, Sh]$$

for all $h \in L^1 + L^q$. Then $T \in \mathcal{L}_1(L^1) \cap \mathcal{L}_1(L^q)$ and $Tf = g$, proving Theorem 2.

THEOREM 3. Let $1 \leq p < q < \infty$ and let the number α be given by $1/\alpha = 1/p - 1/q$. Let f and g be non-negative non-increasing simple functions on R_+ such that

$$(6) \qquad \left(\int_0^{t^\alpha} g^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty g^q\, ds\right)^{1/q} \leq \left(\int_0^{t^\alpha} f^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^q\, ds\right)^{1/q}$$

for all positive t. Then there exists an operator $W \in \mathcal{L}_{2^{1/p}}(L^p(R_+)) \cap \mathcal{L}_{2^{1/q}}(L^q(R_+))$ such that $Wf = g$.

Proof: Let $P(t) = \int_0^t f^p - g^p \, ds$ and $Q(t) = \int_t^\infty f^q - g^q \, ds$. Let $A = \{t \in R_+ \mid P(t) \geq 0\}$, $B = \{t \in R_+ \mid Q(t) \geq 0\}$. By (6) $A \cup B = R_+$. A is a union of disjoint intervals A_i $i = 1 \ldots n$ with $P(t) = 0$ at each end point. Similarly $B = \bigcup_{i=1}^m B_i$ where the B_i's are disjoint intervals with $Q(t) = 0$ at the end points.

In the following it will be convenient to use a second copy of R_+ which we shall denote R_+^o. $R_+ \cup R_+^o$ will denote the measure space consisting of the disjoint union of R_+ and R_+^o each equipped with Lebesgue measure. Let φ be the measure preserving map of $R_+ \cup R_+^o$ onto itself which interchanges each point t of R_+ with the corresponding point t^o of R_+^o.

The operator W will be constructed as the composition of three operators $W = W_3 W_2 W_1$, where

(7) $$W_1 \in \mathcal{L}_{2^{1/p}}(L^p(R_+), L^p(R_+ \cup R_+^o)) \cap \mathcal{L}_{2^{1/q}}(L^q(R_+), L^q(R_+ \cup R_+^o))$$

(8) $$W_2 \in \mathcal{L}_1(L^p(R_+ \cup R_+^o), L^p(R_+ \cup R_+^o)) \cap \mathcal{L}_1(L^q(R_+ \cup R_+^o), L^q(R_+ \cup R_+^o))$$

(9) $$W_3 \in \mathcal{L}_1(L^p(R_+ \cup R_+^o), L^p(R_+)) \cap \mathcal{L}_1(L^q(R_+ \cup R_+^o), L^q(R_+)).$$

From this is follows of course that $W \in \mathcal{L}_{2^{1/p}}(L^p(R_+)) \cap \mathcal{L}_{2^{1/q}}(L^q(R_+))$. For each $h \in L^p(R_+) + L^q(R_+)$, W_1 puts a copy of h onto both R_+ and R_+^o, that is :

$$W_1 h(t) = \chi_{R_+}(t)\, h(t) + \chi_{R_+^o}(t)\, h(\varphi t) \quad \text{for all} \quad t \in R_+ \cup R_+^o .$$

Then (7) is obvious.

Since $P(t) = 0$ at the left end point a_i of the interval A_i it follows that $\int_{a_i}^t (\chi_{A_i} g)^p \, ds \leq \int_{a_i}^t (\chi_{A_i} f)^p \, ds$ for all $t \geq a_i$. Thus, using Theorem 1 and an obvious translation, there exists an operator $U_i \in \mathcal{L}_1(L^p(R_+)) \cap \mathcal{L}_1(L^\infty(R_+))$ such that

$$U_i(\chi_{A_i} f) = \chi_{A_i} g.$$

Then the operator U given by

$$Uh = \sum_{i=1}^{n} \chi_{A_i} U_i(\chi_{A_i} h)$$

is also in $\mathcal{L}_1(L^p(R_+)) \cap \mathcal{L}_1(L^\infty(R_+))$ and $U(\chi_A f) = \chi_A g$. Since $Q(t) = 0$ at the right end point of the interval B_i we also have $\int_t^\infty (\chi_{B_i} g)^q \, dx \leq \int_t^\infty (\chi_{B_i} f)^q \, dx$ for all t, and a translation of Theorem 2 gives us an operator $V_i \in \mathcal{L}_1(L^1(R_+)) \cap \mathcal{L}_1(L^q(R_+))$ such that $V_i(\chi_{B_i} f) = \chi_{B_i} g$. Then $Vh = \sum_{i=1}^{m} \chi_{B_i} V_i(\chi_{B_i} h)$ defines an operator in $\mathcal{L}_1(L^1(R_+)) \cap \mathcal{L}_1(L^q(R_+))$.

Let V^o denote the operator which is a copy of V acting on functions defined on R_+^o instead of on R_+. Then W_2 is defined by :

$$W_2 h = U(\chi_A h) + V^o(\chi_{\varphi(B)} h).$$

(8) can readily be deduced with the help of the Riesz-Thorin theorem. ([16] Chapter XII) Finally W_3 collects up pieces of function on R_+ and R_+^o and patches them together on R_+ :

$$W_3 h(t) = \chi_{A \setminus B}(t)\, h(t) + \chi_B(t)\, h(\varphi t) \quad \text{for all} \quad h \in L^p(R_+ \cup R_+^o) + L^q(R_+ \cup R_+^o)$$

and all $t \in R$.

Clearly (9) holds and $Wf = W_3 W_2 W_1 f = g$, completing the proof of the theorem.

REMARK. This proof of theorem 3 does not seem to use the full strength of condition (6). Possibly a more refined proof would enable the sharpened conclusion $W \in \mathcal{L}_1(L^p) \cap \mathcal{L}_1(L^q)$.

THEOREM 4. <u>Let</u> $p, q \in [1, \infty]$ <u>and let</u> f <u>and</u> g <u>be complex valued functions in</u> $L^p + L^q$ <u>on a measure space</u> (X, Σ, μ) <u>such that</u>

(10) $$K(t,g\ ;\ L^p(\mu),L^q(\mu)) \le K(t,f\ ;\ L^p(\mu),L^q(\mu)) \quad \text{for all} \quad t>0.$$

Then there exists an operator $T \in \mathcal{L}_\xi(L^p(\mu)) \cap \mathcal{L}_\eta(L^q(\mu))$, where ξ and η are constants depending only on p and q, such that $Tf = g$.

Proof: The operator Φ, $\Phi h = \varphi h$ where $\|\varphi\|_{L^\infty} \le 1$, is in the class $\mathcal{L}_1(L^p) \cap \mathcal{L}_1(L^q)$ and so it suffices to treat the case where f and g are non-negative. Also, since $K(t,a\ ;\ A_0,A_1) = tK(1/t,a\ ;\ A_1,A_0)$ we can suppose without loss of generality that $p < q$. In view of the estimates given above for $K(t,f\ ;\ L^p,L^q)$ there exists a constant λ depending only on p and q, such that :

(11) $$\left(\int_0^{t^\alpha} (g^*)^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty (g^*)^q\, ds\right)^{1/q} \le \left(\int_0^{t^\alpha} (\lambda f^*)^p\, ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty (\lambda f^*)^q\, ds\right)^{1/q}$$

for all $t>0$, where $1/\alpha = 1/p - 1/q$ and the L^q integrals are understood to be zero if $q = \infty$.

STEP 1. If f and g are simple functions then Theorems 1 and 3 together with (11) give an operator T_2 in $\mathcal{L}_\xi(L^p(R_+)) \cap \mathcal{L}_\eta(L^q(R_+))$ which maps f^* to g^*. ξ and η depend only on p and q (for example $\xi = 2^{1/p}\lambda$, $\eta = 2^{1/q}\lambda$ if $q < \infty$). One can easily find an operator T_1 in $\mathcal{L}_1(L^1(\mu),L^1(R_+)) \cap \mathcal{L}_1(L^\infty(\mu),L^\infty(R_+))$ taking f to f^* and another, T_3 in $\mathcal{L}_1(L^1(R_+),L^1(\mu)) \cap \mathcal{L}_1(L^\infty(R_+),L^\infty(\mu))$ taking g^* to g. (cf. Lemma 2 in [4]). Using the Riesz-Thorin theorem we obtain that $T = T_3T_2T_1 \in \mathcal{L}_\xi(L^p(\mu)) \cap \mathcal{L}_\eta(L^q(\mu))$ and of course $Tf = g$.

STEP 2. If only g is simple then, given any ε, $0<\varepsilon<1$, we shall construct $T \in \mathcal{L}_\xi(L^p) \cap \mathcal{L}_\eta(L^q)$ with $Tf = (1-\varepsilon)g$ where ξ and η are as estimated in step 1. If $q = \infty$ it is easy to see that there exists a simple function $f_\varepsilon \le f$ such that

$\int_0^t \left[(1-\varepsilon)\, g^*\right]^p ds \le \int_0^t (f_\varepsilon^*)^p\, ds$ for all $t>0$. Thus the desired operator is obtained by first multiplying by $(f_\varepsilon/f)\chi_{\{x\,|\,f(x)>0\}}$ and then applying the operator in $\mathscr{L}_\xi(L^p) \cap \mathscr{L}_\eta(L^\infty)$ which maps f_ε to $(1-\varepsilon)g$. For $q<\infty$ more care is needed. We must first examine the behaviour of the function $K(t,g) = K(t,g\,;\,L^p,L^q)$ near $t=0$ and $t=\infty$. For each $t>0$, there exist functions u_t, v_t such that $u_t + v_t = g$, $0 \le u_t \le g$, $0 \le v_t \le g$ and

(12) $$\|u_t\|_{L^p} + t\|v_t\|_{L^q} - \min(t^2, 1/t^2) \le K(t,g) \le \min(\|g\|_{L^p}, t\|g\|_{L^q}).$$

Consequently $\lim_{t\to\infty} \|v_t\|_{L^q} = \lim_{t\to 0} \|u_t\|_{L^p} = 0$. Thus there are subsequences $\{v_{t(n)}\}_{n=1}^\infty$ $\{u_{s(n)}\}_{n=1}^\infty$ which tend to zero almost everywhere. ($\lim_{n\to\infty} t(n) = \infty$, $\lim_{n\to\infty} s(n) = 0$). By dominated convergence $u_{t(n)} \to g$ in L^p and $v_{s(n)} \to g$ in L^q. $K(t,g)$ and $\frac{1}{t}K(t,g) = K(1/t, g\,;\,L^q, L^p)$ are each continuous monotone functions. So using (12) again we deduce that

$$\lim_{t\to\infty} K(t,g) = \|g\|_{L^p} \qquad \text{and} \qquad \lim_{t\to 0} \frac{1}{t}K(t,g) = \|g\|_{L^q}.$$

In particular, given ε, $0<\varepsilon<1$, there exist positive numbers a_0 and a_∞ such that :

$$K(t,(1-\varepsilon)g) < (1-\varepsilon/2)\|g\|_{L^q}\, t < K(t,g) \qquad \text{for all} \quad t \le a_0$$

and

$$K(t\,,\,(1-\varepsilon)g) < (1-\varepsilon/2)\|g\|_{L^p} < K(t,g) \qquad \text{for all} \quad t \ge a_\infty.$$

We seek to construct a continuous piecewise linear function $H(t)$ with finitely many vertices such that $K(t,(1-\varepsilon)g) < H(t) < K(t,g)$ for all $t>0$. From the above estimates we may take $H(t) = (1-\varepsilon/2)\|g\|_{L^q} t$ on $(0,a_0]$ and $H(t) = (1-\varepsilon/2)\|g\|_{L^p}$ on $[a_\infty,\infty)$. Since $K(t,g)$ is continuous and strictly positive on the compact interval $[a_0\,,\,a_\infty]$ it is easy to extend the definition of $H(t)$ to the whole of $(0,\infty)$ using only finitely many linear segments.

Let $(f_n)_{n=1}^{\infty}$ be an increasing sequence of simple functions, $0 \le f_n \le f_{n+1} \le f$ with $\lim_{n\to\infty} f_n = f$ a. e. Since $f \in L^p + L^q$, f_n tends to f in L^p+L^q norm also and thus $\lim_{n\to\infty} K(t,f_n) = K(t,f)$ for each positive t. Also $K(t,f_n) \le K(t,f_{n+1}) \ldots \le K(t,f)$ since multiplication by the function $(f_{n+1}/f_n)\,\chi\{x \mid f_n(x)>0\}$ is an operator in $\mathscr{L}_1(L^p) \cap \mathscr{L}_1(L^q)$. Let $v_1, v_2 \ldots v_M$ be the values of t where $H(t)$ has its vertices. For some sufficiently large n we have $K(v_i,f_n) > H(v_i)$ for $i = 1, 2 \ldots M$. But $K(t,f_n)$ is concave and so for all $t > 0$ $K(t,f_n) > H(t) > K(t,(1-\varepsilon)g)$. It follows that

$$\left(\int_0^{t^\alpha} ((1-\varepsilon)\, g^*)^p ds\right)^{1/p} + t\left(\int_{t^\alpha}^{\infty} ((1-\varepsilon)\, g^*)^q ds\right)^{1/q} \le \left(\int_0^{t^\alpha} (\lambda f_n^*)^p ds\right)^{1/p} + t\left(\int_{t^\alpha}^{\infty} (\lambda f_n^*)^q ds\right)^{1/q}$$

for all positive t, and so, as for $q = \infty$, wa have an operator in $\mathscr{L}_\xi(L^p) \cap \mathscr{L}_\eta(L^q)$ taking f to $(1-\varepsilon)g$.

STEP 3. Proof of the theorem under the assumption that the measure space is σ-finite :

Let $(g_n)_{n=1}^{\infty}$ be a sequence of simple functions which tend monotonically almost everywhere to g from below. Then using step 2, let T_n be an operator in $\mathscr{L}_\xi(L^p) \cap \mathscr{L}_\eta(L^q)$ with $T_n f = (1-1/n)\, g_n$. Let ω be a continuous linear functional of norm one on ℓ^∞ such that $\omega(\{a_n\}) = \lim_{n\to+\infty} a_n$ for every convergent sequence $\{a_n\}$. Define the bilinear functional τ acting on pairs of simple functions, by

$$\tau(\varphi,\psi) = \omega\left(\left\{\int (T_n\varphi)\,\psi\, d\mu\right\}\right).$$

Of course $\tau(\varphi,\psi)$ is defined also for φ and ψ ranging over larger classes of functions In particular

$$|\tau(\varphi,\psi)| \le \xi \|\varphi\|_{L^p} \|\psi\|_{L^{p'}} \quad \text{for all} \quad \varphi \in L^p \,,\ \psi \in L^{p'}$$

and $$|\tau(\varphi,\psi)| \leq \eta \|\varphi\|_{L^q}\|\psi\|_{L^{q'}} \quad \text{for all} \quad \varphi \in L^q \;,\; \psi \in L^{q'}.$$

Thus for a fixed $\varphi \in L^q$ $\tau(\varphi,\psi)$ is a continuous linear functional on $L^{q'}$ and so, since $q > 1$, there exists a function $h_\varphi \in L^q$ determined by φ uniquely to within a set of zero measure, such that $\tau(\varphi,\psi) = \int h_\varphi \psi \, d\mu$ for all $\psi \in L^{q'}$. The above estimates for τ imply that $\|h_\varphi\|_{L^q} \leq \eta\|\varphi\|_{L^q}$ and if $\varphi \in L^p \cap L^q$ we also have $\|h_\varphi\|_{L^p} \leq \xi \|\varphi\|_{L^p}$. The operator T, $T\varphi = h_\varphi$ is thus in $\mathcal{L}_\eta(L^q)$ and its restriction to $L^p \cap L^q$ extends uniquely to an operator in $\mathcal{L}_\xi(L^p)$ which we may also denote by T. If $\psi \in L^p \cap L^q$, $\tau(\varphi,\psi)$ is defined for $\varphi \in L^p + L^q$ and $\tau(\varphi,\psi) = \int (T\varphi)\,\psi \, d\mu$. In particular

$$\int (Tf)\,\psi \, d\mu = \tau(f,\psi) = \omega(\{\int (T_n f)\,\psi \, d\mu\}) = \omega(\{(1-1/n)\int g_n \,\psi \, d\mu\}) = \int g\psi \, d\mu$$

and since this is true for all $\psi \in L^p \cap L^q$ it follows that $Tf = g$.

STEP 4. Proof of the theorem for an arbitrary measure space: If $q < \infty$ then the subset of the measure space where f and g are non zero is σ-finite and the methods of step 3 apply immediately. Thus we need only consider the case $q = \infty$. Given positive functions $f, g \in L^p + L^\infty$ which satisfy (10), it follows that $\int_0^t (g^*)^p \, ds \leq \int_0^t (\lambda f^*)^p \, ds$ for all $t > 0$.

Let $\alpha = \lim_{t\to\infty} g^*(t) = \lim_{t\to\infty} (\frac{1}{t}\int_0^t (g^*)^p \, ds)^{1/p}$. Then $G = \{x \,|\, g(x) > \alpha\}$ is σ-finite and $(g\chi_G)^*(t) \leq g^*(t)$ for all positive t.

Let $\beta = \lim_{t\to\infty} f^*(t) = \lim_{t\to\infty} (\frac{1}{t}\int_0^t (f^*)^p \, ds)^{1/p}$. Then $F_0 = \{x \,|\, f(x) > \beta\}$ is σ-finite.

Case 1. If $\beta = 0$, then $\alpha = 0$ and both f and g have σ-finite support. Step 3 is immediately applicable.

Case 2. $\beta > 0$. Case 2A. If $\mu(F_0) = \infty$ then $(f\chi_{F_0})^*(t) = f^*(t)$ and there exists an operator $T_0 \in \mathcal{L}_\lambda(L^p) \cap \mathcal{L}_\lambda(L^\infty)$ which maps $f\chi_{F_0}$ to $g\chi_G$. Let

$F_n = \{x \mid f(x) > \beta + 1/n\}$ and let ω be the functional introduced in step 3. Define the operator T_1 by

$$T_1 h = \frac{\omega\left(\left\{\frac{1}{\mu(F_n)} \int_{F_n} h \, d\mu\right\}\right)}{\omega\left(\left\{\frac{1}{\mu(F_n)} \int_{F_n} f \, d\mu\right\}\right)} \, g \, \chi_{X \setminus G} .$$

Then T_1 maps L^p to $\{0\}$ and maps L^∞ into itself with norm bounded by $\alpha/\beta \leq \lambda$. The operator T, $Th = \chi_G T_o(\chi_F h) + T_1 h$, is in $\mathscr{L}_\lambda(L^p) \cap \mathscr{L}_\lambda(L^\infty)$ and $Tf = g$.

<u>Case 2B</u>. $\mu(F_o) < \infty$. Then for each n, the set $E_n = \{x \mid \beta \geq f(x) > \beta - 1/n\}$ has infinite measure.

<u>Case 2B (i)</u>. Suppose that each measurable subset E of E_n with $\mu(E) = \infty$ has a subset of finite positive measure. Then each E_n has a subset D_n, $n \leq \mu(D_n) < \infty$. Let $F = F_o \cup \bigcup_{n=1}^{\infty} D_n$. F is σ-finite and $(f \chi_F)^*(t) = f^*(t)$. Much as before we can obtain $T_o \in \mathscr{L}_\lambda(L^p) \cap \mathscr{L}_\lambda(L^\infty)$ which maps $f \chi_F$ to $g \chi_G$, and T_1, given by

$$T_1 h = \frac{\omega\left(\frac{1}{\mu(D_n)} \int_{D_n} h \, d\mu\right)}{\omega\left(\frac{1}{\mu(D_n)} \int_{D_n} f \, d\mu\right)} \, g \, \chi_{X \setminus G}$$

and T, $Th = \chi_G T_o(\chi_F h) + T_1 h$ is the required operator.

<u>Case 2B (ii)</u>. The only remaining possibility is that the above defined sets E_n for each integer bigger than some integer m contain measurable subsets C_n such that every measurable subset of C_n has either zero or infinite measure. Let $L^\infty(C_n)$ be the subspace of $L^\infty(\mu)$ consisting of functions which are a. e. zero on $X \setminus C_n$.

Let ℓ_n be a continuous linear functional of norm 1 on $L^\infty(C_n)$ such that $\ell_n(\chi_{C_n} f) = \| \chi_{C_n} f \|_{L^\infty}$. Let $(Y, \mathcal{S}, \nu)$ be a measure space consisting of the disjoint union of F_o equipped with μ-measure together with a sequence $(R_n)_{n=m}^{\infty}$ of disjoint copies of the real line, each equipped with Lebesgue measure. We define an operator $Q \in \mathscr{L}_1(L^p(\mu),L^p(\nu)) \cap \mathscr{L}_1\ (L^{\infty}(\mu),L^{\infty}(\nu))$ by $Qh = \chi_{F_o} h + \sum_{n=m}^{\infty} \chi_{R_n}\ \ell_n(\chi_{C_n} h)$.

In fact $\chi_{C_n} h = 0$ a. e. for any $h \in L^p(\mu)$. Q has the further property that $(Qf)^*(t) = f^*(t)$, and since $(Y,\mathscr{S},\nu)$ is σ-finite we may use the arguments of case 2B(i) to construct an operator $T \in \mathscr{L}_\lambda(L^p(\nu),L^p(\mu)) \cap \mathscr{L}_\lambda(L^{\infty}(\nu),L^{\infty}(\mu))$ which maps Qf to g. TQ is then the required operator and the proof of theorem 4 is complete.

COROLLARY. $(L^p(\mu)\ ,\ L^q(\mu))$ <u>is a Calderón pair.</u>

III. INTERPOLATION PAIRS WHICH ARE NOT CALDERON.

Define $A_o+\infty.A_1$ to be the space of all elements $a \in A_o+A_1$ for which

$$\|a\|_{A_o+\infty.A_1} = \lim_{t\to\infty} K(t,a\ ;\ A_o,A_1)$$ is finite.

Let $A_1+\infty.A_o$ be defined analogously, so that

$$\|a\|_{A_1+\infty.A_o} = \lim_{t\to\infty} K(t,a\ ;\ A_1,A_o) = \lim_{t\to 0} \tfrac{1}{t} K(t,a\ ;\ A_o,A_1).$$

It is not very difficult to see that $A_o+\infty.A_1$ is a Banach space which contains A_o, and that for each $a \in A_o$ $\|a\|_{A_o+\infty.A_1} \le \|a\|_{A_o}$. In fact $A_o+\infty.A_1$ can be thought of as a sort of closure of A_o with respect to A_1, as the following lemma shows.

LEMMA 1. <u>An element</u> a <u>of</u> A_o+A_1 <u>is in</u> $A_o+\infty.A_1$ <u>if and only if there exists a sequence</u> $(a_n)_{n=1}^{\infty}$ <u>in</u> A_o <u>with</u> $\sup_n \|a_n\|_{A_o} < \infty$ <u>and</u> $\lim_{n\to\infty} \|a-a_n\|_{A_1} = 0$. <u>For each such</u> a, $\|a\|_{A_o+\infty.A_1} = \inf\left\{\sup_n \|a_n\|_{A_o}\right\}$ <u>where the infimum is taken over all sequences</u> (a_n) <u>in</u> A_o <u>for which</u> $\lim_{n\to\infty} \|a-a_n\|_{A_1} = 0$.

Proof: We leave the details to the reader.

LEMMA 2. <u>For all</u> $a \in A_o + A_1$ <u>and all positive</u> t, $K(t,a\ ;\ A_o{+}\infty.A_1, A_1{+}\infty.A_o) = K(t,a\ ;\ A_o,A_1)$.

Proof: Fix a and t, and let $b \in A_o{+}\infty.A_1$ and $c \in A_1{+}\infty.A_o$ be such that $a = b + c$ and

$$\|b\|_{A_o+\infty.A_1} + t\|c\|_{A_1+\infty.A_o} \leq K(t,a\ ;\ A_o{+}\infty.A_1, A_1{+}\infty.A_o) + \varepsilon$$

for some arbitrarily small positive number ε. Let $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ be sequences which approximate b and c in A_1 and A_o norms respectively such that $\sup_n \|b_n\|_{A_o} \leq \|b\|_{A_o+\infty.A_1} + \varepsilon$ and $\sup_n \|c_n\|_{A_1} \leq \|c\|_{A_1+\infty.A_o} + \varepsilon$.

Then
$$\begin{aligned} K(t,a\ ;\ A_o,A_1) &\leq \|b_n + c - c_n\|_{A_o} + t\|c_n + b - b_n\|_{A_1} \\ &\leq \|b\|_{A_o+\infty.A_1} + t\|c\|_{A_1+\infty.A_o} + (1+t)\varepsilon + 0(n). \end{aligned}$$

It follows that $K(t,a;A_o,A_1) \leq K(t,a;A_o{+}\infty.A_1, A_1{+}\infty.A_o)$. The reverse inequality is an immediate consequence of the inequalities $\|a\|_{A_o+\infty.A_1} \leq \|a\|_{A_o}$, $\|a\|_{A_1+\infty.A_o} \leq \|a\|_{A_1}$.

LEMMA 3. <u>If</u> (A_o,A_1) <u>is a Calderón pair, then</u> $A_o = A_o{+}\infty.A_1$ <u>and</u> $A_1 = A_1{+}\infty.A_o$.

Proof: Let $a \in A_o{+}\infty.A_1$ and let $(a_n)_{n=1}^{\infty}$ be a bounded sequence in A_o with $\lim_{n\to\infty} \|a-a_n\|_{A_1} = 0$. For all positive t $K(t,a) \leq \|a\|_{A_o+\infty.A_1}$, and also for any fixed n

$$\begin{aligned} K(t,a) &\leq K(t,a-a_n) + K(t,a_n) \\ &\leq t\,\|a-a_n\|_{A_1} + K(t,a_n). \end{aligned}$$

So $K(t,a) \leq K(t,a_n) + \min(t\|a-a_n\|_{A_1}, \|a\|_{A_o+\infty.A_1})$. $K(t,a)$ is a positive non decreasing concave function and so for a sufficiently large positive number λ, $K(t,a) \leq \lambda K(t,a_n)$. But, by hypothesis, A_o as an interpolation space must be K-monotone and $\lambda a_n \in A_o$.

Thus $a \in A_o$ and $A_o = A_o + \infty . A_1$. Similarly $A_1 = A_1 + \infty . A_o$.

REMARK. It can be seen that if $A_o \neq A_o + \infty . A_1$ then spaces other than the "end point" spaces A_o and A_1 may also fail to be K-monotone.

EXAMPLES. Let the measure space be the real line R equipped with Lebesgue measure. Let $C(R)$ be the space of continuous bounded functions on R with supremum norm, and let $W^{1,1}$ be the Sobolev space of L^1 functions f whose first derivatives f' (in the distribution sense) are also in L^1 with $\|f\|_{W^{1,1}} = \|f\|_{L^1} + \|f'\|_{L^1}$. Then $C(R) + \infty . L^1 = L^\infty$ and for example $C(R)$ and $C(R) \cap L^1$ are not K-monotone. $(L^1, W^{1,1})$ also fails to be a Calderon pair. In fact $W^{1,1} + \infty . L^1 = BV \cap L^1$, the space of functions in L^1 which coincide almost everywhere with functions of bounded variation. $BV \cap L^1$ can be normed by $\|f\|_{BV \cap L^1} = \|f\|_{L^1} + \mathrm{var}(f)$.

The last example will show that Lemma 3 does not have a converse. Let T denote the circle group with Haar measure and $W^{1,p}(T)$ the Sobolev space of functions f in $L^p(T)$ whose (distributional) first derivatives f' are also in $L^p(T)$. As norm take $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|f'\|_{L^p}$. For $1 \leq p < \infty$ we have an estimate of Peetre,

$$K(t,f \,;\, L^p(T), W^{1,p}(T)) \sim \Psi(t,f)$$

where
$$\Psi(t,f) = \sup_{0 \leq |h| \leq t} \|f(x+h) - f(x)\|_{L^p} + t\|f\|_{L^p} \quad \text{for} \quad t < 1$$
$$= \|f\|_{L^p} \qquad \text{for} \quad t \geq 1.$$

See [10], [11], also [1] p. 258. Though these proofs are given for $(L^p(R^n), W^{1,p}(R^n))$ rather than for spaces taken on the circle or n-torus, the result for T or T^n can be readily deduced. (For example construct an operator

$S \in \mathscr{L}(L^p(T^n), L^p(R^n)) \cap \mathscr{L}(W^{1,p}(T^n), W^{1,p}(R^n))$ where Sf is the periodic extension of f multiplied by a suitable C^∞ function of compact support, so that $Sf\Big|_{T^n} = f$ and $\|Sf\|_{L^p(R^n)} \sim \|f\|_{L^p(T^n)}$, $\|Sf\|_{W^{1,p}(R^n)} \sim \|f\|_{W^{1,p}(T^n)}$ and $\Psi(t,Sf) \sim \Psi(t,f)$.)

Let $L^p_\alpha(T)$ be the space of tempered distributions f on T whose Fourier coefficients $\hat{f}(n)$ are given by $\hat{f}(n) = (1 + |n|^2)^{-\alpha/2}\,\hat{\varphi}(n)$, where φ is a function in $L^p(T)$. Let $\|f\|_{L^p_\alpha} = \|\varphi\|_{L^p}$. There are analogous definitions for L^p_α on T^n and R^n. For our purposes it suffices to consider the parameter α in the range $(0,1)$ and in this case $L^p_\alpha = [L^p, W^{1,p}]_\alpha$, as was shown by Calderon. ([2], [3]). However L^p_α is not K-monotone, at least if $2 < p < \infty$ and $1/p < \alpha < 1$. This may be seen with the help of some special functions used by Taibleson [15] to show non-inclusions between L^p_α and certain generalised Lipschitz or Besov spaces $(L^p, W^{1,p})_{\alpha,q}$.

The function $k = k_{\alpha,1/2}$ which has the Fourier series $\sum_1^\infty 2^{-n\alpha}\, n^{-1/2} \cos 2^n x$ does not belong to L^p_α. ([15] p. 473 paragraph (e)). Using the estimate on p. 472 of [15] we see that $\|k(x+h) - k(x)\|_{L^p(T)} \le M_1 |h|^\alpha \log^{-1/2}(1/|h|)$ for some constant M_1. However the function $f = f_{\alpha+1/p',1/p+\varepsilon}$ with Fourier series $\sum_2^\infty n^{-\alpha-1/p'} \log^{-1/p-\varepsilon} n \cos nx$ is in L^p_α for each $\varepsilon > 0$ and furthermore $\|f(x+h) - f(x)\|_{L^p(T)} \ge M_2 |h|^\alpha \log^{-1/p-\varepsilon}(1/|h|)$ for some constant M_2 ([15] pp. 473-474 paragraph (h)). We choose $\varepsilon = 1/2 - 1/p$ and clearly $K(t,k) \le K(t,\lambda f)$ for all t and some constant λ.

(It is easy to deduce that $L^p_\alpha(R^n)$ and $L^p_\alpha(T^n)$ are also not K-monotone for the above ranges of values of p and α using Lemmas 23, 24 and 25 of [15]. Incidentally, by using interpolation methods with an operator of the form S as above, one can give an immediate proof of Lemma 25).

Obviously $L^p +\infty . W^{1,p} = L^p$, and from the weak compactness of the unit ball of L^p for $1 < p \leq \infty$ we may readily deduce that $W^{1,p} +\infty . L^p = W^{1,p}$. Let us summarise the results of this section.

THEOREM. Every Calderón pair (A_o, A_1) has the "mutual closure" property $A_o = A_o +\infty . A_1$, $A_1 = A_1 +\infty . A_o$, but this property is not a sufficient condition for an interpolation pair to be Calderón.

IV. WEAK K-MONOTONICITY.

Having observed that there are at least two different "mechanisms" which may prevent an interpolation space from being K-monotone, we now turn to the study of a monotonicity property weaker than K-monotonicity which holds in all interpolation spaces.

LEMMA 1. Let $w(t)$ be a positive measurable function such that $\int_o^\infty w(t)dt < \infty$, and let (A_o, A_1) be an interpolation pair. Let $f, g \in A_o + A_1$ such that $K(t,g) \leq w(t)\,K(t,f)$ for all positive t. Then there exists an operator $T \in \mathcal{L}_\lambda(A_o) \cap \mathcal{L}_\lambda(A_1)$ such that $Tf = g$. λ may be taken to be any number greater than

$$\min_{\alpha>1} \frac{2\alpha}{\log \alpha} \int_o^\infty w(t)\, dt/t .$$

Proof : Let $r > 1$ be such that $\min_{\alpha>1} 2\alpha/\log \alpha = 2r/\log r$. Choose a number $\varepsilon > 0$. For each $n = 0, \pm 1, \pm 2, \ldots,$ let $g = a_n + b_n$, where $a_n \in A_o$, $b_n \in A_1$, and $\|a_n\|_{A_o} + r^n \|b_n\|_{A_1} \leq (1+\varepsilon)\,K(r^n, g)$. We shall need two estimates :

(1) $\|a_n - a_{n-1}\|_{A_o} \leq (1+\varepsilon)\,\dfrac{(1+r)}{\log r}\left(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\right) K(r^n, f)$

(2) $\|a_n - a_{n-1}\|_{A_1} \le (1+\varepsilon)\dfrac{2r^{-n+1}}{\log r}\left(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\right) K(r^n,f).$

For (1), $\|a_n - a_{n-1}\|_{A_o} \le \|a_n\|_{A_o} + \|a_{n-1}\|_{A_o} \le (1+\varepsilon)(K(r^n,g)+K(r^{n-1},g))$

$\le (1+\varepsilon)(1+r)K(r^{n-1},g)$, since $K(t,g)/t$ is non-increasing,

$$\le (1+\varepsilon)(1+r)\frac{\int_{r^{n-1}}^{r^n} K(t,g)\,dt/t}{\int_{r^{n-1}}^{r^n} dt/t}, \quad \text{since } K(t,g) \text{ is non-decreasing},$$

$$\le (1+\varepsilon)\frac{(1+r)}{\log r}\left(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\right) K(r^n,f).$$

For (2), $\|a_n - a_{n-1}\|_{A_1} = \|b_{n-1} - b_n\|_{A_1} \le (1+\varepsilon)(r^{-n+1}K(r^{n-1},g) + r^{-n}K(r^n,g))$

$\le (1+\varepsilon)(2r^{-n+1}K(r^{n-1},g))$

$$\le (1+\varepsilon)\frac{2r^{-n+1}}{\log r}\left(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\right) K(r^n,f).$$

For each n $\|h\| = K(r^n,h)$ is a norm on A_o+A_1 and thus there exists a continuous linear functional ℓ_n on A_o+A_1 such that $\ell_n(f) = K(r^n,f)$ and $|\ell_n(h)| \le K(r^n,h)$ for all $h \in A_o+A_1$. The operator T will be given by

$$Th = \sum_{n=-\infty}^{\infty} \frac{\ell_n(h)}{K(r^n,f)}(a_n - a_{n-1}) \qquad \text{for all } h \in A_o+A_1.$$

If $h \in A_o$, Th is given by an absolutely convergent A_o – valued series, since

$$\sum_{n=-\infty}^{\infty} \frac{|\ell_n(h)|}{K(r^n,f)}\|a_n - a_{n-1}\|_{A_o}$$

$$\le (1+\varepsilon)\frac{(1+r)}{\log r}\int_o^{\infty} w(t)\,dt/t\ \|h\|_{A_o} \quad \text{from (1).}$$

Similarly if $h \in A_1$

$\displaystyle\sum_{n=-\infty}^{\infty} \frac{|\ell_n(h)|}{K(r^n,f)}\|a_n - a_{n-1}\|_{A_1} \le (1+\varepsilon)\frac{2r}{\log r}\int_o^{\infty} w(t)\,dt/t$ by (2) and so $Th \in A_1$, and indeed $T \in \mathcal{L}_\lambda(A_o) \cap \mathcal{L}_\lambda(A_1)$ for every λ greater than $(1+\varepsilon)\frac{2r}{\log r}\int_o^\infty w(t)\,dt/t$. Since $f \in A_o+A_1$, $Tf = \sum_{-\infty}^{\infty}(a_n - a_{n-1})$ is a series converging absolutely in A_o+A_1 norm.

$$Tf = \sum_{-\infty}^{o}(a_n - a_{n-1}) + \sum_{1}^{\infty}(b_{n-1} - b_n)$$

$$= a_o - \lim_{n\to-\infty} a_{n-1} + b_o - \lim_{n\to\infty} b_n.$$

As in the proof of (1), $K(r^n,g) \le \frac{1}{\log r}\left(\int_{r^n}^{r^{n+1}} w(t)\,dt/t\right) K(r^{n+1},f)$. Thus as $n \to -\infty$ $K(r^n,g) \to 0$ and as $n \to +\infty$ $K(r^n,g)/r^n \to 0$. From $\|a_n\|_{A_o} + r^n\|b_n\|_{A_1} \le (1+\varepsilon)K(r^n,g)$ we have

$$\lim_{n\to-\infty}\|a_n\|_{A_o+A_1} \le \lim_{n\to-\infty}\|a_n\|_{A_o} = 0 \quad , \quad \lim_{n\to\infty}\|b_n\|_{A_o+A_1} \le \lim_{n\to\infty}\|b_n\|_{A_1} = 0.$$

So $Tf = a_o + b_o = g$.

THEOREM 1. <u>Let</u> $w(t)$ <u>be a positive measurable function such that for some positive number</u> ε, $\int_o^\infty \min(\varepsilon, w(t)\,dt/t < \infty$. <u>Let</u> A <u>be an interpolation space for</u> (A_o,A_1). <u>Then if</u> $f \in A$ <u>and</u> $g \in A_o+A_1$ <u>such that</u> $K(t,g) \le w(t)\,K(t,f)$ <u>for all</u> $t > 0$ <u>it follows that</u> $g \in A$.

Proof : We change to a notation in which A_o and A_1 appear more symmetrically. Let $K_*(x,a) = e^{-x/2}K(e^x,a)$ and $w_*(x) = w(e^x)$, so that $K_*(x,g) \le w_*(x)K_*(x,f)$ for all $x \in (-\infty,\infty)$ and $\int_{-\infty}^{\infty}\min(\varepsilon, w_*(x))\,dx < \infty$. For any $a \in A_o+A_1$ and any real x and y we see that $K_*(x+y\ ,\ a) \le e^{|y|/2}K_*(x,a)$. Let $H(x) = K_*(x,g)/K_*(x,f)$. We deduce immediately that $H(x+y) \ge e^{-|y|}H(x)$ for all real x and y. Further, since $H(x) \le w_*(x)$, the set $\{x \mid H(x) > \eta\}$ must have finite Lebesgue measure for any positive η. It follows that $\lim_{|x|\to\infty} H(x) = 0$, and that $\int_{-\infty}^{\infty} H(x)\,dx < \infty$. Let

$w_1(t) = H(\log t)$. $\int_0^\infty w_1(t)\frac{dt}{t} < \infty$ and $K(t,g) \le w_1(t)\, K(t,f)$. Using Lemma 1 we conclude that $g \in A$.

REMARKS. In some particular cases the condition $K(t,g) \le w(t)\, K(t,f)$ for $f \in A$ forces g to be in a class much smaller than A. For example if $f \in (A_0,A_1)_{\theta,\infty}$ g must be in $(A_0,A_1)_{\theta,1}$, and if $f \in A_0$ or A_1 then g must be zero. This seems to suggest that the above theorem is rather crude and that, for example, it should be possible to weaken the conditions imposed on $w(t)$ and still have $g \in A$. Bearing in mind that for some interpolation pairs we only need $w(t)$ to be bounded, we ask if it is possible to weaken the requirement $\int_0^\infty \min(\varepsilon, w(t))\, dt/t < \infty$ to something corresponding to a slower convergence of $w(t)$ to zero as $t \to 0$ and $t \to \infty$, for example $\int_1^\infty \min(\varepsilon, w(t)^p)\, dt/t < \infty$ for some $p > 1$. We shall construct an example which shows that such a sharpening of the theorem is in fact impossible.

Let $\{B_n\}_{n=1}^\infty$ be a sequence of Banach spaces. For $1 \le p \le \infty$ define the space $\ell^p\{B_n\}$ to consist of all vector valued sequences $\{a_n\}_{n=1}^\infty$ satisfying $a_n \in B_n$ for each n, and $\|\{a_n\}\|_{\ell^p\{B_n\}} = \left(\sum_{n=1}^\infty \|a_n\|_{B_n}^p\right)^{1/p} < \infty$. The usual modification is made for $p = \infty$.

LEMMA 2. Let (B_n, C_n) $n = 1, 2, \ldots$ be a sequence of interpolation pairs. Then $(\ell^1\{B_n\}, \ell^1\{C_n\})$ is an interpolation pair and

(i) $K(t, \{a_n\}\,;\, \ell^1\{B_n\}, \ell^1\{C_n\}) = \sum_1^\infty K(t, a_n\,;\, B_n, C_n)$

(ii) $\ell^1\{(B_n,C_n)_{\theta,q}\} \subset (\ell^1\{B_n\},\ \ell^1\{C_n\})_{\theta,q}$ for $0 < \theta < 1$ and $1 \le q \le \infty$.

The inclusion is an equality for $q = 1$.

(iii) $[\ell^1\{B_n\}, \ell^1\{C_n\}]_\theta = \ell^1\{[B_n, C_n]_\theta\}$ for $0 < \theta < 1$.

Proof : It is easy to see that $\ell^1\{B_n\}$ and $\ell^1\{C_n\}$ are each Banach spaces continuously embedded in $\ell^1\{B_n+C_n\}$. The proofs of (i) and (ii) are left to the reader. For (iii) it is convenient to use a different construction for the complex interpolation space $[A_o,A_1]_\theta$. Let $\mathcal{F}_{1,1}(A_o,A_1)$ be the space of A_o+A_1-valued analytic functions $f(z)$ defined in the strip $0 < \operatorname{Re} z < 1$ such that the limits on the boundary $f(j+iy) = \lim_{x\to j} f(x+iy)$ exist in the sense of tempered A_o+A_1 - valued distributions on the line, and satisfy $\int_{-\infty}^{\infty} \|f(j+iy)\|_{A_j}\, dy < \infty$ for $j = 0,1$. Then $[A_o, A_1]_\theta$ consists of all elements $a \in A_o + A_1$ such that $a = f(\theta)$ for some $f(z) \in \mathcal{F}_{1,1}(A_o,A_1)$ and may be normed by $\|a\|_{[A_o,A_1]_\theta} = \inf_{a=f(\theta)} \int_{-\infty}^{\infty} \|f(iy)\|_{A_o} + \|f(1+iy)\|_{A_1}\, dy$. Using ideas implicit in section 9.4 of [3] which are further explained in [12] (Lemma 1.1) it can be seen that this construction gives the same space to within equivalence of norm as that obtained from the original definition.

Let $\{a_n\} \in \ell^1\{[B_n,C_n]_\theta\}$. There exist analytic functions $f_n(z) \in \mathcal{F}_{1,1}(B_n,C_n)$ such that $f_n(\theta) = a_n$ and $\|a_n\|_{[B_n,C_n]_\theta} \geq (1-\varepsilon)\int_{-\infty}^{\infty} \|f_n(iy)\|_{B_n} + \|f_n(1+iy)\|_{C_n}\, dy$. Let $\{f_{n,m}(z)\}$ and $\{a_{n,m}\}$ be truncated sequences, that is $f_{n,m}(z) = f_n(z)$, $a_{n,m} = a_n$ for $n \leq m$ and $f_{n,m}(z) = 0$, $a_{n,m} = 0$ for $n > m$. Noting that $\ell^1\{B_n\} + \ell^1\{C_n\} = \ell^1\{B_n+C_n\}$ has dual space $\ell^\infty\{B_n' \cap C_n'\}$, we see that for each m, $\{f_{n,m}(z)\} \in \mathcal{F}_{1,1}(\ell^1\{B_n\},\ell^1\{C_n\})$ and so $\{a_{n,m}\} \in [\ell^1\{B_n\},\ell^1\{C_n\}]_\theta$ with norm $\|\{a_{n,m}\}\|_{[\ell^1\{B_n\},\ell^1\{C_n\}]_\theta} \leq \sum_{n=1}^{m} \int_{-\infty}^{\infty} \|f_n(iy)\|_{B_n} + \|f_n(1+iy)\|_{C_n}\, dy$. By similar estimates $\{a_{n,m}\}$ is a Cauchy sequence with respect to m in $[\ell^1\{B_n\},\ell^1\{C_n\}]_\theta$. Thus its limit $\{a_n\}$ in $\ell^1\{B_n\} + \ell^1\{C_n\}$ must also be in $[\ell^1\{B_n\},\ell^1\{C_n\}]_\theta$. This shows that $\ell^1\{[B_n,C_n]_\theta\} \subset [\ell^1\{B_n\}, \ell^1\{C_n\}]_\theta$. We leave the proof of the reverse inclusion to the reader.

Let $\{r_n\}_{n=1}^{\infty}$ be a sequence including all the rational numbers in $(1,\infty)$. Let us take $B_n = L^{r_n}(R_+)$ and $C_n = L^{\infty}(R_+)$ for $n = 1, 2, \ldots$ and let $A_o = \ell^1\{L^{r_n}(R_+)\}$, $A_1 = \ell^1\{L^{\infty}(R_+)\}$. Then $[A_o,A_1]_\theta = \ell^1\{[L^{r_n},L^{\infty}]_\theta\} = \ell^1\{L^{r_n/(1-\theta)}\}$ ([3], 13.5, 13.6). We next observe that

(3) $$(A_o\ ,\ A_1)_{\theta,q} \not\subset [A_o\ ,\ A_1]_\theta \qquad \text{for all } q > 1/(1-\theta).$$

The space $(\ell^1\{L^{r_n}\},\ell^1\{L^{\infty}\})_{\theta,q}$ includes sequences $\{a_n\}$ such that $a_n = 0$ for all $n \neq m$ and $a_m \in (L^{r_m},L^{\infty})_{\theta,q} = L(r_m/(1-\theta),q) \not\subset L^{r_m/(1-\theta)}$ if m is such that $q > r_m/(1-\theta)$. (See [1], p. 187 and [7] p. 255.)

Now let us suppose that there exists a number $p > 1$ such that the conclusion of Theorem 1 holds when $w(t)$ satisfies the weakened integrability condition $\int_0^{\infty} \min(\varepsilon,w(t)^p)\,dt/t < \infty$. We shall see that this contradicts (3). Let us choose θ sufficiently small so that $p > 1/(1-\theta)$. We also introduce a second positive number α chosen to ensure that

(4) (i) $p > 1/(1-\theta)(1-\alpha) > 1/(1-\theta)$

(ii) $r = p(1-\theta)(1-\alpha)/\alpha$ is a rational number greater than 1.

Let $g = \{g_n\} \in (A_o,A_1)_{\theta,p(1-\alpha)}$. Then $w(t) = (t^{-\theta}K(t,g\ ;\ A_o,A_1))^{1-\alpha}$ satisfies $\int_0^{\infty} w(t)^p\,dt/t < \infty$ and $K(t,g) = w(t)\,t^{\theta(1-\alpha)}(K(t,g))^{\alpha}$.

Our next step will be to show that $t^{\theta(1-\alpha)}(K(t,g))^{\alpha} \leq K(t,f)$ for some $f \in [A_o,A_1]_\theta$. On the assumption that the sharpened version of Theorem 1 is true, $K(t,g) \leq w(t)\,K(t,f)$ then implies that $g \in [A_o,A_1]_\theta$. But g is an arbitrary element of $(A_o,A_1)_{\theta,p(1-\alpha)}$ and so (3) will be contradicted.

As a non-decreasing concave function of t, $K(t,g)$ must be absolutely continuous on every compact subinterval of $(0,\infty)$. Thus it is differentiable almost everywhere and the derivative $K'(t,g)$ must coincide almost everywhere with a non-increasing non-negative function. We introduce the function $h(t)$,

$$h(t) = \left[\theta(1-\alpha)\, t^{\theta(1-\alpha)-1}(K(t^{1/r},g))^{\alpha r} + \alpha t^{\theta(1-\alpha)+1/r-1}(K'(t^{1/r},g))^{\alpha r-1}\right]^{1/r}.$$

From (4) and the fact that $K(t,g)/t$ is non increasing we see that $h(t)$ is a non-increasing function such that $h(t)^r = \frac{d}{dt}\left[t^{\theta(1-\alpha)}(K(t^{1/r},g))^{\alpha r}\right]$ almost everywhere. But $t^{\theta(1-\alpha)}(K(t^{1/r},g))^{\alpha r}$ is also absolutely continuous on every compact subinterval of $(0,\infty)$ and tends to zero as t tends to zero. It follows that

$t^{\theta(1-\alpha)}(K(t^{1/r},g))^{\alpha r} = \int_0^t h(s)^r\, ds$ and so

$$t^{\theta(1-\alpha)}(K(t,g))^{\alpha} = \left(\int_0^{t^r} h(s)^r\, ds\right)^{1/r} \le K(t,h\ ;\ L^r(R_+),L^\infty(R_+))$$

(as in [8] p. 159). Since r is rational $r = r_m$ for some m and if $f = \{f_n\}$ is a sequence in $A_0 + A_1$ which is zero for all $n \neq m$ and has $f_m = h$, then $K(t,f\ ;\ A_0,A_1) = K(t,h\ ;\ L^r,L^\infty)$. It remains only to show that $f \in [A_0,A_1]_\theta$ which amounts to showing that $h \in L^{r/(1-\theta)}$. But

$h(t)^r \le \frac{1}{t}\int_0^t h(s)^r\, ds = t^{\theta(1-\alpha)-1}(K(t^{1/r},g))^{\alpha r}$, and so

$$\int_0^\infty h(t)^{r/(1-\theta)}\, ds \le \int_0^\infty \left[t^{-\theta/r} K(t^{1/r},g)\right]^{p(1-\alpha)} dt/t$$

$$= (r\|g\|_{(A_0,A_1)_{\theta,p(1-\alpha)}})^{p(1-\alpha)} < \infty.$$

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