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On the Banach algebra $A(\Gamma)$ for smooth sets $\Gamma \subset R^n$

Yngve Domar

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<u>On the Banach algebra</u> $A(\Gamma)$ <u>for smooth sets</u> $\Gamma \subset \mathbb{R}^n$

by

Yngve Domar

0. The main part of this investigation is devoted to the problem of estimating $\|e^{itf}\|_{A(\Gamma)}$, as $t \to \infty$. Here $t \in \mathbb{R}$, $\Gamma \subset \mathbb{R}^n$, and $f \in A(\Gamma)$ is real-valued. $A(\Gamma)$ is the quotient Banach algebra $A(\mathbb{R}^n)/I(\Gamma)$, where $I(\Gamma)$ is the ideal in $A(\mathbb{R}^n) = \mathcal{F}L^1(\mathbb{R}^n)$ of all functions, vanishing on Γ. We shall discuss only very regular situations. Γ is thus in general a well-behaved compact subset of a smooth manifold in $\mathbb{R}^n$, an interval on $\mathbb{R}$, a curve in $\mathbb{R}^n$, a surface in $\mathbb{R}^3$ etc., and f has high differentiability properties. In order not to complicate the discussion and obscure the principal ideas, we shall be very generous with our regularity assumptions. Thus we assume that all manifolds and functions f involved are infinitely differentiable. It can however be shown that each particular result holds as well, if we only require differentiability up to a certain order. The principal object of our work is to show that very simple, straightforward and seemingly rough methods give very precise estimates. In the concluding section we show how our results can be used to determine, for the sets Γ considered, all those homomorphisms of $A(\mathbb{R}^n)$ into $A(\Gamma)$, which are given by C^∞ mappings of Γ into $\mathbb{R}^n$.

The paper is of a preliminary character, and presents the actual state of the continued work in a direction initiated by [5].

1. Let Γ be a compact interval of $\mathbb{R}$ and let $f \in C^{\infty}(\Gamma)$ be real-valued. The following theorem is well known, even under much weaker differentiability assumptions on f:

Theorem 1.1. *If f is non-linear, there are positive constants C_1 and C_2 such that*

$$C_1 t^{1/2} \leq \|e^{itf}\|_{A(\Gamma)} \leq C_2 t^{1/2}, \tag{1.2}$$

for $t \geq 1$. For f linear, $\|e^{itf}\| = 1$ for every $t \in \mathbb{R}$.

The inequality to the left in (1.2) is due to Leibenzon [9], while the right hand inequality is an easy corollary of the inequality of Carlson [1]. We shall give a proof of Theorem 1.1, not the shortest one, but a proof which can serve as a model for the deduction of estimates in more general situations.

Proof of Theorem 1.1. The only non-trivial part is the proof of (1.2) for non-linear f. Let us start with the inequality to the right. Instead of applying Carlson's inequality, we base the proof on three elementary observations.

Firstly, we observe that a partitioning of the unit in $A(\Gamma)$ shows that it suffices to prove that there exists a constant C such that

$$\|e^{itf}\|_{A(\Gamma_t)} \leq C, \tag{1.3}$$

for every subinterval $\Gamma_t \subset \Gamma$ of length $t^{-1/2}$.

Secondly, the norm of a function in any algebra $A(E)$, $E \subset \mathbb{R}^n$, does not change after multiplying the function with a constant of norm 1 or with a bounded continuous character on $\mathbb{R}^n$ (restricted to E). Hence, with x_o denoting an endpoint of Γ_t and x standing as symbol for the variable, we obtain

$$\|e^{itf}\|_{A(\Gamma_t)} = \|e^{it(f(x)-f(x_o)-(x-x_o)f'(x_o))}\|_{A(\Gamma_t)} = \\ = \|e^{it(x-x_o)^2 g_{t,x_o}(x-x_o)}\|_{A(\Gamma_t)},$$

where $g_{t,x_o} \in C^\infty([0,t^{-1/2}])$, and where g_{t,x_o} and all its derivatives have bounds that are uniform in t and x_o.

Thirdly, the norm of a function h in any Banach algebra $A(E)$, $E \subset \mathbb{R}^n$, is not affected by affine bijections of $\mathbb{R}^n$ and corresponding mappings of E and h. Thus, putting $x = x_o + ut^{-1/2}$, with u as new variable, we obtain

$$\|e^{itf}\|_{A(\Gamma_t)} = \|e^{iu^2 g_{t,x_o}(ut^{-1/2})}\|_{A([0,1])} .$$

But the right hand member is the norm of a function on $[0,1]$, bounded uniformly in x_o and t, for $t \geq 1$, as well as all its derivatives. Hence (1.3) is proved.

To prove the left inequality of (1.2) we observe that the assumed non-linearity of f implies the existence of a subinterval $\Gamma' \subset \Gamma$ of positive length where f'' does not vanish. Let $\psi \in \mathcal{D}(\mathbb{R})$, with $\operatorname{Supp}(\psi) \subset \Gamma'$ and $\int_{\mathbb{R}} \psi \, dx = 1$. We consider the function ψe^{-itf}, defined as 0 outside Γ, as an element in the Banach space $PM(\mathbb{R}) = \mathcal{F}L^\infty(\mathbb{R})$ of pseudomeasures on $\mathbb{R}$. It follows from the definition of the norm in $A(\Gamma)$ that

$$(1.4) \qquad 1 = \int_\Gamma e^{itf(x)} e^{-itf(x)} \psi(x)dx \leq \|e^{itf}\|_{A(\Gamma)} \|e^{itf}\psi\|_{PM(\mathbb{R})} .$$

Thus the left inequality of (1.2) follows with $C_1 = C^{-1}$, if

$$(1.5) \qquad \|e^{-itf}\psi\|_{PM(\mathbb{R})} \leq C t^{-1/2} ,$$

$t \geq 1$, for some C. (1.5) can be deduced from the lemma of van der Corput [2]. We shall, however, apply a more general lemma, Lemma 1.6 below, which is needed in later discussions. It is well known that lemmas of this general type exist. This particular formulation is due to J.-E. Björk (personal communication). We omit the proof of the lemma, since it is fairly close to van der Corput's proof.

Lemma 1.6. _Let_ $[a,b]$ _be a compact interval on_ $\mathbb{R}$, _and_ $\psi \in \mathcal{D}(]a,b[)$, $k \in C^p([a,b])$ _with_

$$0 < C_1 \le |k'(x)| + |k''(x)| + \ldots + |k^{(p)}(x)| \le C_2 ,$$

<u>if</u> $x \in [a,b]$, <u>where</u> C_1 <u>and</u> C_2 <u>are constants and</u> p <u>a positive integer. Then there exists a constant</u> C <u>not depending on</u> k , <u>such that</u>

$$\left|\int_a^b e^{isk(x)}\,\psi(x)dx\right| < C\, s^{-1/p} ,$$

<u>for every</u> $s > 0$.

In order to prove (1.5) we first observe that its left member is, by definition, the L^∞ norm of the function with values

$$\int_{\Gamma'} e^{-itf(x)-iux}\,\psi(x)dx , \quad u \in \mathbb{R} ,$$

with the Fourier transform defined properly. Taking $s = t + |u|$, we can apply Lemma 1.6 with $[a,b] = \Gamma'$, $p = 2$, and

$$k(x) = -\frac{t}{s}\,f(x) - \frac{u}{s}\,x , \quad x \in \Gamma' ,$$

$$C_1 = \min_{a\le x\le b,\, -1\le r\le 1} \left(|(1-|r|)f'(x)+r| + (1-|r|)|f''(x)|\right) ,$$

$$C_2 = 1 + \sup_{x\in\Gamma'} \left(|f'(x)| + |f''(x)|\right) .$$

From this we obtain (1.5).

2. Now we assume that Γ is a curve in $\mathbb{R}^2$, representable as the graph of a real-valued function $g \in C^\infty([a,b])$, where $-\infty < a < b < \infty$. As always we have $f \in C^\infty(\Gamma)$, and f is real-valued.

<u>Theorem 2.1</u>. <u>Let</u> Γ <u>have non-vanishing curvature. If</u> f <u>is not the restriction of a linear function on</u> $\mathbb{R}^2$, <u>there exist positive constants</u> C_1 <u>and</u> C_2 <u>such that</u>

$$(2.2) \qquad C_1 t^{1/3} \le \|e^{itf}\|_{A(\Gamma)} \le C_2 t^{1/3} ,$$

<u>for</u> $t \ge 1$. <u>If</u> f <u>is the restriction of a linear function, then</u> $\|e^{itf}\|_{A(\Gamma)} = 1$, <u>for every</u> $t \in \mathbb{R}$.

<u>Proof of Theorem 2.1</u>. The only non-trivial part is the proof of (2.2). A detailed proof of the right hand inequality has been given in [5], and we shall here give only a brief outline. The proof uses the same

technique as the corresponding proof in Section 1. This time we observe that it suffices to obtain a uniform bound for

$$(2.3) \qquad \|e^{itf}\|_{A(\Gamma_t)} ,$$

where Γ_t is the graph of g, restricted to a subinterval $I_t \subset \subset [a,b]$ of length $t^{-1/3}$. Denoting by x_o an endpoint of I_t, we can now use the assumption $g'' \neq 0$ and the presence of a two-parameter family of bounded characters to obtain

$$\|e^{itf}\|_{A(\Gamma_t)} \leq \|e^{it(x-x_o)^3 g_{t,x_o}(x-x_o)}\|_{A(I_t)} ,$$

where g_{t,x_o} has the differentiability and boundedness properties specified in Section 1. The transformation $x = x_o + ut^{-1/3}$ proves the uniform boundedness of (2.3).

To prove the left inequality, we first observe that there is a subinterval $[a',b'] \subset [a,b]$, of positive length, where the function $h = f \circ g$ satisfies the condition

$$(2.4) \qquad \begin{vmatrix} h'' & g'' \\ h''' & g''' \end{vmatrix} \neq 0 .$$

For otherwise the condition $g'' \neq 0$ implies that h'' and g'' are linearly dependent, i.e.

$$h(x) = Ax + Bg(x) + C , \quad x \in [a,b] ,$$

for some constants A, B and C. But this means that f is the restriction to Γ of the linear function

$$(x,y) \to Ax + By + C , \qquad (x,y) \in \mathbb{R}^2 .$$

Choosing $[a',b']$ as above we can now continue as in Section 1. Let $\psi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{Supp}(\psi) \subset [a',b']$ and with $\int_{\mathbb{R}} \psi(x)dx = 1$. μ is the Borel measure on Γ for which the projection on the x-axis is the Lebesgue measure multiplied with ψ. We consider μ as a pseudomeasure on $\mathbb{R}^2$ and obtain

$$1 = \int_a^b e^{itf(g(x))}\, e^{-itf(g(x))}\psi(x)dx = \int_\Gamma e^{itf}\, e^{-itf}d\mu \leq$$

$$\leq \|e^{itf}\|_{A(\Gamma)} \|e^{-itf}d\mu\|_{PM(\mathbb{R}^2)} .$$

Thus it suffices to show that

$$(2.5) \qquad \|e^{-itf}d\mu\|_{PM(\mathbb{R}^2)} \leq C\, t^{-1/3} ,$$

$t \geq 1$, for some constant C. But the left member is the supremum of the absolute value of

$$\int_\Gamma e^{-itf(x,y)-iux-ivy}d\mu(x,y) = \int_{a'}^{b'} e^{-ith(x)-iux-ivg(x)}\psi(x)dx ,$$

as $(u,v) \in \mathbb{R}^2$. Taking $s = t + |u| + |v|$, and observing the relation (2.4), we can apply Lemma 1.6 with $p = 3$ and

$$k(x) = -\frac{t}{s}\, h(x) - \frac{u}{s}\, x - \frac{v}{s}\, g(x) ,$$

and this gives (2.5).

Remark. Theorem 2.1 has analogues for curves in $\mathbb{R}^n$, $n > 3$, now with $t^{-1/3}$ replaced by $t^{-1/(n+1)}$. As for the right hand inequality we refer to [5]. The inequality to the left can be discussed as in the proof of Theorem 2.2, using Lemma 1.6.

3. In this and the next section we are concerned with cases when the dimension of Γ is two or higher. In order to avoid complications at the boundary of Γ we prefer to change our setup in the following way.

Let Ω be a C^∞ manifold in $\mathbb{R}^n$ and f a real-valued function in $C^\infty(\Omega)$. We say that a positive function M on $[1,\infty[$ is a majorant if, for every compact $K \subset \Omega$, there is a constant $C > 0$ such that

$$(3.1) \qquad \|e^{itf}\|_{A(K)} \leq C\, M(t) ,$$

$t \geq 1$. A positive function m on $[1,\infty[$ is a minorant if there exists a compact $K \subset \Omega$ and a constant $C > 0$ such that

(3.2) $\qquad \|e^{itf}\|_{A(K)} \geq C\, m(t)$,

$t \geq 1$.

In this section we assume that Ω is an open non-empty subset of $\mathbb{R}^2$, and denote by k the maximal rank in Ω of the Hessian

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} .$$

Then the following theorem holds.

Theorem 3.3. The function $t \to t^{k/2}$, $t \geq 1$, is both majorant and minorant.

Proof of Theorem 3.3. We first prove that the function is a majorant. In the case when $k = 0$, all second order derivatives of f vanish, and thus f is linear on every component of Ω . It is obvious that the theorem holds in this case. In the discussion of the case when $k = 2$, it suffices to consider the case when K is a square contained in Ω . We can then argue exactly as in the proof of Theorem 1.1. By a partitioning of the unit it follows that it suffices to show that

$$\|e^{itf}\|_{A(K_t)}$$

is uniformly bounded for the family of squares contained in K , with sides parallell to the sides of K , and with side length $t^{-1/2}$, $t \geq 1$. The proof of this is quite parallell to the corresponding part of the proof of Theorem 1.1, and is omitted here.

The proof that $M(t) = t^{1/2}$, $t \geq 1$, is a majorant when $k = 1$ is similar, but is more complicated. It suffices to show that every given point $P = (x_0, y_0) \in \Omega$ has a compact neighborhood K such that (3.1) holds for some C .

We shall use a result of Hartman and Nirenberg [6] (Theorem A), which states that the surface $z = f(x,y)$ in $\mathbb{R}^3$ is locally developable at P when $k = 1$, in the following sense: There exists an $\varepsilon > 0$ and a continuous real function h on $[-\varepsilon, \varepsilon]$ such that the line segments

$$L_s = \{x_o - s\sin h(s) + v\cos h(s),\ y_o + s\cos h(s) + v\sin h(s)) \mid v \in [-\varepsilon,\varepsilon]\},$$

$s \in [-\varepsilon,\varepsilon]$, are disjoint and have a compact neighborhood K of P as their union, and are such that the tangent plane of the surface is common for all $(x,y,f(x,y))$ with (x,y) on the same segment L_s. The property that the segments L_s are disjoint implies evidently that h is Lipschitz continuous. This implies that if ε is chosen small enough, we have for every $s_o \in [-\varepsilon,\varepsilon]$ and $t \geq 1$, that

$$K_{t,s_o} = \bigcup_{|s-s_o|\leq t^{-1/2}} L_s$$

is contained in the rectangle

$$R_{t,s_o} = \{(x_o - s\sin h(s_o) + v\cos h(s_o),\ y_o + s\cos h(s_o) + v\sin h(s_o)) \mid$$
$$\mid |s - s_o| \leq 2t^{-1/2},\ |v| \leq 2\varepsilon\},$$

whereas

$$\bigcup_{|s-s_o|\geq 5t^{-1/2}} L_s$$

is disjoint from the rectangle

$$S_{t,s_o} = 2R_{t,s_o} - (x_o - s_o\sin h(s_o),\ y_o + s_o\cos h(s_o)) .$$

Now there exists a constant C_o such that we can find, for every choice of $t \geq 1$ and $s_o \in [-\varepsilon,\varepsilon]$, a function $\varphi \in A(\mathbb{R}^2)$, with $\varphi(x,y) = 1$ on R_{t,s_o}, $\varphi(x,y) = 0$ outside S_{t,s_o}, and $\|\varphi\|_{A(\mathbb{R}^2)} \leq$ $\leq C_o$. These functions can be used for a partitioning of the unit, and this shows that

$$\|e^{itf}\|_{A(K)} \leq C\, t^{-1/2}, \qquad t \geq 1,$$

for some C, follows if we can find a constant C_1 such that

$$(3.4) \qquad \|e^{itf}\|_{A(K_{t,s_o})} \leq \|e^{itf}\|_{A(R_{t,s_o})} \leq C_1,$$

for $t \geq 1$, $s_o \in [-\varepsilon,\varepsilon]$.

Using the properties of the tangent plane of $z = f(x,y)$, we obtain for $(x,y) \in R_{t,s_o}$,

$$\begin{aligned} d(x,y) &= f(x,y) - f(x_o - s_o\sin h(s_o),\ y_o + s_o\cos h(s_o)) - \\ &- (x - x_o + s_o\sin h(s_o))f'_x(x_o - s_o\sin h(s_o),\ y_o + s_o\cos h(s_o)) - \\ &- (y - y_o - s_o\cos h(s_o))f'_y(x_o - s_o\sin h(s_o),\ y_o + s_o\sin h(s_o)) = \\ &= ((x - x_o)\sin h(s_o) - (y - y_o)\cos h(s_o) + s_o)^2 g_{t,s_o}(x - x_o + s_o\sin h(s_o), \\ &\qquad\qquad y - y_o + s_o\cos h(s_o)), \end{aligned}$$

where g_{t,s_o} is bounded uniformly in t and s_o as well as all its partial derivatives. Hence by the affine transformation

$$\xi = ((x - x_o)\sin h(s) - (y - y_o)\cos h(s) + s_o)t^{1/2}$$
$$\eta = (x - x_o)\cos h(s) + (y - y_o)\sin h(s),$$

we obtain as in the earlier sections

$$\|e^{itf}\|_{A(R_{t,s_o})} = \|e^{itd}\|_{A(R_{t,s_o})} = \left\|e^{i\xi^2 h_{t,s_o}(\xi t^{-1/2},\eta)}\right\|_{A(S)},$$

where $S = [-2,2] \times [-2\varepsilon,2\varepsilon]$, and where h_{t,s_o} is uniformly bounded in t and s_o as well as all its partial derivatives. Hence (3.4) holds.

In the proof of our claim that $t \to t^{k/2}$, $t \geq 1$, is a minorant, it suffices to consider the case when $k = 1$ or 2. We first observe that there is a non-empty open subset Ω_1 of Ω where the maximal rank of the Hessian is attained at every point. We form a function $\psi \in \mathcal{D}(\mathbb{R}^2)$ with support contained in Ω_1 and satisfying

$$\iint_{\mathbb{R}^2} \psi \, dxdy = 1 .$$

ψe^{-itf} is considered as a pseudomeasure on $\mathbb{R}^2$, vanishing outside Ω. Arguing as in Section 1 we find that (3.2) holds with $K = \mathrm{supp}(\psi)$, $m(t) = t^{k/2}$, if for some C

$$(3.5) \qquad \|\psi e^{-itf}\|_{PM(\mathbb{R}^2)} \leq C\, t^{-k/2},$$

for $t \geq 1$. But on Supp (ψ), the <u>minimal</u> rank of f is k, and hence (3.5) follows from the following lemma, easily deducible from the results in Littman [10] (cf. [11] and [4] p. 25).

Lemma 3.6. Let K be a compact subset of an open set $U \subset \mathbb{R}^n$. $\psi \in \mathcal{D}(K)$, and $h \in C^\infty(U)$. For some $\delta > 0$ we assume at every point of U that at least k eigenvalues of the Hessian of ψ have absolute value $\geq \delta$. Then there exists a constant C such that, if ψe^{-ith} is defined as 0 outside K,

$$\|\psi e^{-ith}\|_{PM(\mathbb{R}^n)} \leq C\, t^{-k/2}.$$

The constant C depends on K, U, φ, k, δ, and of the bounds of the partial derivatives of h of all orders.

Remark. Theorem 3.5 has extension possibilities to the case when Ω is an open non-empty subset of $\mathbb{R}^n$ and k is the maximal rank of the Hessian of the real-valued function $f \in C^\infty(\Omega)$. By the same arguments as in the later part of the proof of Theorem 3.3, we find from Lemma 3.6 that $t \to t^{k/2}$, $t \geq 1$, is a minorant. The function is also a majorant in all cases when the first part of the proof can be copied, i.e. if we have a local representation corresponding to the local developability, now by a k-parameter family of $(n-k)$-dimensional affine manifolds.

4. In this section we study the case when Ω is a C^∞ surface in $\mathbb{R}^3$, of non-vanishing Gaussian curvature. We restrict ourselves to the situation when Ω is the graph of a real-valued function g, defined and infinitely differentiable on some open subset U of $\mathbb{R}^2$. Then the Hessian of g does not vanish. f is a real-valued C^∞ function on Ω. We can thus think of f as a C^∞ function on U. For every $(x,y) \in U$ and $\lambda \in \mathbb{R}$, $R_\lambda(x,y)$ denotes the rank of the Hessian of the function $f - \lambda g$ on U, and

$$(4.1) \qquad k = \operatorname*{Max}_{(x,y)\in U} \ \operatorname*{Min}_{\lambda\in\mathbb{R}} R_\lambda(x,y).$$

Since the Hessian of g has rank 2, the subset of U where the maximum in (4.1) is attained is an open set U. It is easy to see that $k \leq 1$ if Ω has positive Gaussian curvature. $k = 2$ for instance if $U = \mathbb{R}^2$, $g(x,y) = x^2 - y^2$, $f(x,y) = xy$.

Theorem 4.2. In the sense precised in Section 3 $t \to t^{k/2}$, $t \geq 1$, is both majorant and minorant, if $k = 0$ or 2. If $k = 1$, $t \to t^{5/6}$, $t \geq 1$, is a majorant and $t \to t^{1/2}$, $t \geq 1$ is a minorant, and there are examples when the first function is a minorant and other examples when the second function is a majorant.

Proof of Theorem 4.2. Let us first consider the case when $k = 0$. Then there exists a function λ on U such that

$$\begin{cases} f_{xx} = \lambda g_{xx} \\ f_{xy} = \lambda g_{xy} \\ f_{yy} = \lambda g_{yy} \end{cases}$$

for every (x,y) U. Since the Hessian of g has rank 2, $\lambda \in C^\infty(U)$. Taking the partial derivatives, we obtain

$$\begin{cases} \lambda_y g_{xx} = \lambda_x g_{xy} \\ \lambda_y g_{xy} = \lambda_x g_{yy} \end{cases}$$

and since the system has non-vanishing determinant, we find that λ is constant on every component of U. Hence

$$f(x,y) = \lambda g(x,y) + Ax + By + C,$$

on every component, for properly chosen constants A, B and C, which shows that f is, on every component, restriction of a linear function on $\mathbb{R}^3$. It follows from this that the function with constant value 1 is a majorant, and it is trivially a minorant.

We continue with the cases $k = 1, 2$. Let C be a compact subset of U, and K the corresponding compact subset of Ω. We have

$$\|e^{itf}\|_{A(K)} \leq \|e^{itf}\|_{A(C)}, \tag{4.3}$$

where in the left hand member f is considered as function on $K \subset \mathbb{R}^3$, and in the right hand member f is considered as function on $C \subset \mathbb{R}^2$.

This is seen by choosing extrapolations of f to the left, which only depend on (x,y). Hence it follows from Theorem 3.3 that $t \to t$ is always a majorant. Furthermore, choosing $U = \mathbb{R}^2$, $f(x,y) = x^2$, $g(x,y) = x^2 + y^2$, we have a case when $k = 1$, and since the maximal rank of f is 1, it follows as above from Theorem 3.3 that $t \to t^{1/2}$ is a majorant.

As for the majorant properties claimed in the theorem, it only remains to prove that $t \to t^{5/6}$, $t \geq 1$, is a majorant when $k = 1$. It suffices to show that

$$t^{-5/6} \|e^{itf}\|_{A(S)}, \quad t \geq 1, \tag{4.4}$$

is bounded for every fixed square $S \subset U$ with sides parallel to the coordinate axis. Here the norm in (4.4) is interpreted in the same sense as the right hand member of (4.3). By a partitioning of the unit we find that it suffices to show the existence of a constant C such that

$$\|e^{itf}\|_{A(S_{t,x_o,y_o})} \leq C\, t^{1/6}, \tag{4.5}$$

for every square $S_{t,x_o,y_o} \subset S$ with center x_o, y_o, with side length $t^{-1/3}$, and with sides parallel to the coordinate axis. At every (x_o,y_o) there exists a λ_o such that the Hessian of $f - \lambda_o g$ has rank ≤ 1. By the assumptions on g, the values of λ_o are uniformly bounded in S. Thus, for every (x_o,y_o) we have a representation

$$f(x,y) - \lambda_o g(x,y) = A + Bx + Cy + D(E(x - x_o) + F(y - y_o))^2 + \\ + G_{t,x_o,y_o}(x - x_o,\ y - y_o), \quad (x,y) \in S_{t,x_o,y_o},$$

where A, B, C, D, E, F are uniformly bounded, and where G_{t,x_o,y_o} has uniformly bounded partial derivatives of all orders, and where

$$G_{t,x_o,y_o} = ((\xi^2 + \eta^2)^{3/2}),$$

as $(\xi,\eta) \to 0$, uniformly. Thus, by the same arguments as in the earlier proofs,

$$\|e^{itf}\|_{A(S_{t,x_o,y_o})} = \|e^{it(D(E(x-x_o)+F(y-y_o))^2+G_{t,x_o,y_o}(x-x_o,y-y_o)}\|_{A(S_{t,x_o,y_o})}$$

$$= \|e^{it^{1/3}(D(E\xi+F\eta)^2}e^{itG_{t,x_o,y_o}(t^{-1/3}\xi,t^{-1/3}\eta)}\|_{A(S_o)},$$

where S_o is the square with corners $(\pm 1/2, \pm 1/2)$. By the submultiplicativity of the norm in $A(S_o)$,

$$\|e^{itf}\|_{A(S_{t,x_o,y_o})} \leq \|e^{it^{1/3}D(E\xi+F\eta)^2}\|_{A(S_o)} \cdot$$

$$\cdot \|e^{itG_{t,x_o,y_o}(t^{-1/3}\xi,t^{-1/3}\eta)}\|_{A(S_o)} .$$

The first factor is $\leq C\ t^{1/6}$, for some constant C. This is seen from Theorem 3.3, or from Theorem 1.1, or by a direct estimate. The function in the exponent of the second factor is uniformly bounded and so are all its partial derivatives, hence the second factor is bounded. Thus (4.5) is proved, and we have shown that $t \to t^{5/6}$ is a majorant, if $k = 1$.

The discussion of the minorant properties can be performed as the corresponding parts of the proofs of Theorems 2.1 and 3.3. We fix $\psi \in \mathcal{D}(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}^2} \psi(x,y)dxdy = 1,$$

and Supp (ψ) included in the set U_o (the open set where k is attained). μ is the measure on Ω for which the projection into the xy-plane is the Lebesgue measure multiplied by ψ. Arguing as before we find that it suffices to show that

$$(4.6) \qquad \|e^{-itf}d\mu\|_{PM(\mathbb{R}^3)} \leq C\ t^{-k/2},$$

$t \geq 1$, for some constant C. But the left hand member is the supremum of the absolute value of

$$\int_{\mathbb{R}^2} e^{-itf(x,y)-iux-ivy-iwg(x,y)}\ \psi(x,y)dxdy,$$

as $(u,v,w) \in \mathbb{R}^3$. The rank of the Hessian of the exponent is $\geq k$ on U_o, and the rank of the Hessian of g is 2. Using Lemma 3.6 one sees directly that (4.6) holds.

Now it only remains to give an example when $k = 1$, and $t \to t^{5/6}$, $t \geq 1$, is a minorant. We take $U = \mathbb{R}^2$, $f(x,y) = x^2 + x^3 - y^2 + y^3$, $g(x,y) = x^2 + y^2$. It suffices to prove that

$$t^{-5/6} \| e^{itf} \|_{A(K)}, \qquad t \geq 1,$$

has a positive lower bound, if S is the closed square with corners $(\pm 1, \pm 1)$, and

$$K = \{(x,y),\ g(x,y)) \,|\, (x,y) \in S\}.$$

By the usual arguments it suffices to show that for some $\psi \in \mathcal{D}(\mathbb{R}^2)$ supported by S and with

$$\int_{\mathbb{R}^2} \psi(x,y)dxdy = 1,$$

$$\left| \int_{\mathbb{R}^2} e^{-itf(x,y)-iux-ivy-iwg(x,y)} \psi(x,y)dxdy \right| \leq C\ t^{-5/6},$$

for some C, when $(u,v,w) \in \mathbb{R}^3$, $t \geq 1$. We choose $\psi(x,y)$ of the form $\varphi(x)\ \varphi(y)$, where $\varphi \in \mathcal{D}(\mathbb{R})$, and find that we have to prove that the product of

$$A(t,u,w) = \left| \int_{\mathbb{R}} e^{-it(x^2+x^3)-iux-iwx^2} \varphi(x)dx \right|$$

and

$$B(t,v,w) = \left| \int_{\mathbb{R}} e^{-it(-y^2+y^3)-ivy-iwy^2} \varphi(y)dy \right|$$

is $\leq C\ t^{-5/6}$. By Lemma 1.6 there exists a constant C_o such that

$$A(t,u,w) \leq C_o t^{-1/3}, \quad B(t,v,w) \leq C_o t^{-1/3}.$$

By the same lemma, we have, for some constant C_1

$$A(t,u,w) \leq C_1 t^{-1/2}, \quad \text{if } tw \geq 0,$$

and

$$B(t,v,w) \leq C_1 t^{-1/2}, \quad \text{if } tw \leq 0,$$

and hence

$$A(t,u,w) \cdot B(t,v,w) \leq C_o C_1 t^{-5/6}$$

is proved. This concludes the proof of Theorem 4.2.

5. Here we collect some minor observations, which may illuminate the earlier theorems.

A. Let Γ be the graph of the function g, defined by

$$g(x) = \begin{cases} e^{-(x-1)^{-1}}, & x > 1 \\ 0, & -1 \leq x \leq 1 \\ e^{(x+1)^{-1}}, & x < 1 . \end{cases}$$

If x is considered as parameter on Γ, and f on Γ is defined by

$$f(x) = \begin{cases} 2e^{-(x-1)^{-1}}, & x > 1 \\ 0, & -1 \leq x \leq 1 \\ e^{(x+1)^{-1}}, & x < 1 , \end{cases}$$

then f is locally at each point of Γ the restriction of a linear function on $\mathbb{R}^2$. Hence there exists a constant C such that

(5.1) $\quad \|e^{itf}\|_{A(\Gamma)} \leq C , \quad t \geq 1 ,$

although f itself is not the restriction of a linear function on $\mathbb{R}^2$.

B. We shall now give a set $\Gamma \subset \mathbb{R}^2$ and a function f on Γ such that (5.1) holds while f is not even locally a restriction of a linear function on $\mathbb{R}^2$.

Let $\Gamma = G \cup H$, where

$$G = \{(x,y) \mid |x| \leq 1, \ y = g(x)\} ,$$

where g is real, $g \in C^\infty([-1,1])$, $g(0) = 0$, g' positive, $g''(0) \neq 0$, and where

$$H = \{(x,0) \mid |x| \leq 1\} .$$

Let

$$f(x,y) = \begin{cases} e^{ix}, & (x,y) \in G \\ 1 , & (x,y) \in H . \end{cases}$$

Then there is no neighborhood of $(0,0)$ where f is the restriction of a linear function. But it is easy to prove that

$$\|e^{itf}\|_{A(\Gamma)} \leq 3 \quad \text{for every } t \in \mathbb{R} .$$

C. Let us now change the setup of example B so that H instead is defined by

$$H = \{(x,y) \,|\, |x| \leq 1,\ y = h(x)\},$$

where h is real, $h \in C^{\infty}([-1,1])$, $h(0) = h'(0) = 0$, $h''(0) \neq 0$. Then we have instead

$$(5.2) \qquad \|e^{itf}\|_{A(\Gamma)} \to \infty,$$

as $t \to \infty$.

We shall show this by an indirect proof. If the norms in $A(\Gamma)$ of e^{itf} are bounded, as $t \to \infty$, we can find a sequence $(t_\nu)_1^\infty$, tending to infinity, and extensions $(g_\nu)_1^\infty$ of $e^{it_\nu f}$ to $\mathbb{R}^2$ such that $(g_\nu)_1^\infty$ converges weakly * in $B(\mathbb{R}^2)$, where $B(\mathbb{R}^2)$ is considered as the dual of the Banach space X of Fourier transforms of functions in $C_0(\mathbb{R}^2)$. We denote the limit function by F.

Let $\psi \in \mathcal{D}(\mathbb{R})$ have support in the set where $g'' \neq 0$, and let μ be the measure on G for which the projection on the x-axis has density function ψ. Then, by Lemma 1.6, $\mu \in X$, for its Fourier-Stieltjes transform $\hat{\mu}$ is given by

$$\hat{\mu}(t,u) = \int_{\mathbb{R}} e^{-itx-iug(x)}\,\psi(x)dx, \quad t \in \mathbb{R}.$$

Thus

$$0 = \lim_{\nu\to\infty} \hat{\mu}(t_\nu, 0) = \lim_{\nu\to\infty} \langle g_\nu, \mu\rangle = \langle F,\mu\rangle = \int_{\mathbb{R}} F(x,\ g(x))\,\psi(x)dx.$$

Varying ψ, we find that F vanishes in a neighborhood of $(0,0)$ on G. By a similar argument we find that F takes the value 1 on H in a neighborhood of $(0,0)$. The continuity of F gives a contradiction.

6. Let $\Gamma \subset \mathbb{R}^n$ be compact, and let α be a C^∞ function from Γ to $\mathbb{R}^m$. We are interested in the problem to determine those functions α which give a homomorphism of $A(\mathbb{R}^m)$ into $A(\Gamma)$ in the sense that $g \in A(\mathbb{R}^m)$ implies that $g \circ \alpha \in A(\Gamma)$. Let $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ be the representation of α by its realvalued components. Then the following theorem holds.

Theorem 6.1. α gives a homomorphism of $A(\mathbb{R}^m)$ onto $A(\Gamma)$ if and only if

(6.2) $\qquad \|e^{it\alpha_i}\|_{A(\Gamma)}, \qquad t \in \mathbb{R},$

is bounded as $t \to \infty$, for every $i = 1, 2, \ldots, m$.

Proof of Theorem 6.1. If α gives a homomorphism, the closed graph theorem shows that

$$\|g \circ \alpha\|_{A(\Gamma)} \leq C\|g\|_{A(\mathbb{R}^m)},$$

for some constant C. Choosing g such that $g \in A(\mathbb{R}^m)$, $g(x) = e^{itx_i}$, $x \in \Gamma$, where x_i is the i-th coordinate of x, we find that (6.2) is bounded. Conversely, if (6.2) is bounded for every i, $\|g \circ \alpha\|_{A(\Gamma)}$ is uniformly bounded for all g which are bounded continuous characters on $\mathbb{R}^m$, and it follows from this directly that α gives a homomorphism.

Theorem 6.1 shows that, with arbitrary m, and with Γ chosen as in Theorem 1.1 or 2.1, or as in Example C of Section 5, then only linear functions α give homomorphisms. The same holds if Γ is a compact subset of the manifold Ω of Theorem 3.3 or 4.2, but now assumed that α can be extended to a C^∞ function on Ω. On the other hand, in Example A of Section 5, all locally linear functions α give homomorphisms, and in Example B of the same section, α need not even be locally linear.

Results can be obtained, in a similar way, concerning homomorphisms of spaces $A_q(\mathbb{R}^m)$ into $A(\Gamma)$. Here $q > 0$, and $A_q(\mathbb{R}^m)$ is the Banach space of Fourier transforms of functions $g \in \mathbb{R}^m$ with norm

$$\int_{\mathbb{R}^m} (1 + |y|)^q |g(y)| dy .$$

By duality we can also find results on the Fourier coefficients of $\alpha^*(\nu)$, where ν is a pseudo-measure in the dual of $A(\Gamma)$, and α^* is the adjoint of a homomorphism $A_q(\mathbb{R}^m) \to A(\Gamma)$, given by α. It should be observed that the dual of $A(\Gamma)$ coincides with the space of pseudo-measures supported by Γ, if Γ is of spectral synthesis. This is the case for instance if Γ is given as in Theorem 2.1 (cf. [3] and [4]).

We conclude by some remarks and state a few open problems.

§ 1. Precise estimates for $\|e^{itf}\|_{A(\Gamma)}$, when Γ is an interval and f has weak differentiability properties, have been given by Leblanc [7], [8].

§ 2. Although it is not known whether curves in $\mathbb{R}^n$, $n \geq 3$, are of spectral synthesis, we can also in this case obtain precise information on the Fourier coefficients of the map of pseudo-measures supported by such a curve. This follows from the fact that our estimates of the norm in $A(\Gamma)$ hold as well for the norm in $A(\mathbb{R})/I_o(\Gamma)$, if Γ is a smooth curve and $I_o(\Gamma)$ the closure of the ideal of functions vanishing in a neighborhood of Γ ([5], p. 188).

§ 3. It would be of interest to determine the differentiability conditions needed to have the conclusion of Lemma 3.6. The extensions of Theorem 3.3 to higher dimensions deserves to be explored. At present it is not known whether the theorem holds without change for higher dimensions.

§ 4. In Theorem 4.2, the gap between $t^{1/2}$ and $t^{5/6}$ is not yet explored. Nor is the possibility of high-dimensional generalisations.

§ 5. In Example C, the exact rate of growth of (5.2) is not known. It has connections with the following problem: For positive weight functions ω on $\mathbb{R}^2$ such that $\omega(x) \leq 1 + |x|^{1/2}$, $x \in \mathbb{R}^2$, we have in the class of measurable functions g with $g/\omega \in L^1$ a natural way of defining its Fourier transform $\hat{g}$ to vanish (or take a constant value) along a given curve with positive curvature (simply by applying smooth measures on the curve, and observing that their transforms are $o(|x|^{-1/2})$). Then the problem is to decide for which ω the class contains an element g with $\hat{g}$ taking the value 0 on G and 1 on H, if G and H in the example have $(0,0)$ as only common point.

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