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by

TAKAAKI NISHIDA

PREFACE

This note corresponds to the lectures given by the author during 1976-1977 at the UNIVERSITE de PARIS-SUD, ORSAY. He would like to express his sincere gratitude to Professor Roger TEMAM and Professor Luc TARTAR for their kind hospitality and the encouraging discussions at the University, and also he would like to express his hearty thanks to Madame MAYNARD for her kind help.

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CHAPTER I - HYPERBOLIC SYSTEM OF CONSERVATION LAWS

1. Introduction

Many physical laws are written in the conservation laws and if we ignore the mechanisms of dissipation such as viscous stresses, heat conductions etc, then we have the following first order system of conservation laws in the one space-dimensional case :

$$(1.1) \qquad u_t + f(u)_x = 0 \quad ,$$

where t is the time, x is the space variable, u is a n-vector of the physical state variables and f is a n-vector smooth function of u. We consider the initial value problem for the system (1.1). When the smooth solutions to (1.1) are considered, the differentiation of (1.1) gives a quasilinear system of first order equations :

$$(1.2) \qquad u_t + A(u)\, u_x = 0 \quad , \qquad A = \text{grad } f$$

Definition 1.1 - The system (1.1) is called hyperbolic if the system (1.2) is hyperbolic, i.e., if the matrix A = grad f has real and distinct eigenvalues $\lambda_k = \lambda_k(u)$, $k = 1,\dots,n$, for all values of $u \in \Omega \subset \mathbb{R}^n$ which are arranged in the increasing order

$$(1.3) \qquad \lambda_1 < \lambda_2 < \dots\dots < \lambda_n$$

The corresponding right and left eigenvectors of A are denoted by $r_k = r_k(u)$, $\ell_k = \ell_k(u)$, $k = 1,2,\dots,n$.

Definition 1.2 - The k-th characteristic field $\lambda = \lambda_k(u)$ of the system (1.1) is called genuinely nonlinear if for all $u \in \Omega$

$$(1.4) \qquad r_k \,.\, \text{grad } \lambda_k \neq 0 \,.$$

Definition 1.3 - The k-th characteristic field $\lambda = \lambda_k(u)$ of the system (1.1) is called linearly degenerate if for all $u \in \Omega$

$$(1.5) \qquad r_k \,.\, \text{grad } \lambda_k \equiv 0$$

Definition 1.4 - A function $z = z(u)$ is a k-Riemann invariant of the system (1.1) if it satisfies the condition

$$r_k \cdot \text{grad } z = 0 \tag{1.6}$$

for all values of $u \in \Omega$.

These definitions and the following proposition are due to Lax (1957).

Proposition 1.1 - There exist n-1 independent k-Riemann invariants for each k. Here the independence of functions means that their gradients are linearly independent. The proof is given by the classical theory of single first order homogeneous partial differential equation (1.6) for z as function of u. (cf. Courant-Hilbert, vol. II, Ch. II.).

Example 1.1 - The equation of ideal compressible flow in the Lagrangien coordinate : (cf. Courant-Friedrichs, Guel'fand)

$$\begin{aligned} & v_t - u_x = 0 \\ & u_t + p_x = 0 \\ & (e + \frac{u^2}{2})_t + (p\,u)_x = 0 \;, \end{aligned} \tag{1.7}$$

where $t \geqslant 0$, x = Lagrangian coordinate, $v = 1/\rho$ = specific volume, u = velocity, p = pressure, e = internal energy and s = entropy.

The equation of state $p = p(v,s)$ depends on the gas and the polytropic gas has the form :

$$p = a^2\, v^{-\gamma} \; \exp\,((\gamma - 1)s/R), \tag{1.8}$$

where a,R are positive constants and $\gamma \geqslant 1$ is the ratio of specific heat. It follows from (1.8) that $e = \frac{p\,v}{\gamma-1}$ + constant and $p = R\rho T$, because $T = e_s$ and $p = -e_v$. Here the energy conservation law of (1.7) may be replaced for the smooth solutions by $s_t = 0$. Then the conservation laws (1.7) gives the following quasilinear system for the unknowns v, u and s.

$$\begin{pmatrix} v \\ u \\ s \end{pmatrix}_t + \begin{pmatrix} 0 & -1 & 0 \\ p_v & 0 & p_s \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ u \\ s \end{pmatrix}_x = 0 \tag{1.9}$$

This is hyperbolic in $\Omega = \{(v,u,s),\ v > 0\}$ if $p_v < 0$. The eigenvalues and the corresponding right eigenvectors and Riemann invariants are given by

$$(1.10)\quad \begin{cases} \lambda_1 = -\sqrt{-p_v}, \quad r_1 = (1, \sqrt{-p_v}, 0), \quad z_1 = s \text{ and} \\ u - \int^v \sqrt{-p_v}\, dv\ ; \quad \lambda_2 = 0, \ r_2 = (p_s, 0, -p_v) \\ z_2 = u \text{ and } p\ ; \ \lambda_3 = \sqrt{-p_v}, \ r_3 = (1, -\sqrt{-p_v}, 0) \\ z_3 = s \ \text{ and } u + \int^v \sqrt{-p_v}\, dv\,. \end{cases}$$

It is genuinely nonlinear for λ_1 and λ_3 if $p_{vv} \neq 0$ and is linearly degenerate for λ_2.

Example 1.2 - Isentropic gas motion (s = constant)

$$(1.11)\qquad v_t - u_x = 0 \ , \ u_t + p(v)_x = 0,$$

where $p(v) = a^2 v^{-\gamma}$ ($a > 0$ and $\gamma \geqslant 1$ are constants) is usually assumed, $\gamma \simeq 1.4$ for the air and $\gamma = 2$ for the shallow water wave equation.

$$(1.12)\quad \begin{cases} \lambda_1 = -\sqrt{-p_v} \ , \ r_1 = (1, \sqrt{-p_v}), \ z = u - \int^v \sqrt{-p_v}\, dv \\ \lambda_2 = \sqrt{-p_v} \ , \ r_2 = (1, -\sqrt{-p_v}), \ w = u + \int^v \sqrt{-p_v}\, dv \end{cases}$$

If $p_v < 0$ in $v > 0$, the system (1.11) is hyperbolic and if $p_{vv} \neq 0$, it is genuinely nonlinear for λ_1 and λ_2.

Example 1.3 - Nonlinear wave equation

$$(1.13)\qquad y_{tt} = \sigma\ (y_x)_x,$$

where $\sigma = \sigma(v)$ is a nonlinear function of v, for example $\sigma = v + a v^2$, $v + av^3$ $(a > 0)$, or $v/\sqrt{1 + v^2}$. If we put $v = y_x$, $u = y_t$, then the equation (1.13) gives

$$(1.14)\qquad v_t - u_x = 0, \qquad u_t - \sigma(v)_x = 0.$$

$\sigma'(v) > 0$ is the hyperbolicity and $\sigma''(v) \neq 0$ is the genuine nonlinearity.

2. Development of Singularities

Here we see in details that the genuinely nonlinear hyperbolic system of two equations developes in general the singularities in finite time. cf. Lax (1964). This means shock wave formations in gas dynamics. For the genuinely nonlinear hyperbolic systems of n equations we refer John (1974). Consider the system

$$(2.1) \qquad \begin{cases} u_t + a(u,v)\, u_x + b(u,v)\, v_x = 0 \\ v_t + c(u,v)\, u_x + d(u,v)\, v_x = 0 \end{cases}$$

Assumption 2.1 - The system (2.1) is hyperbolic in an open set $\Omega \in \mathbb{R}^2$ i.e., the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{has real distinct eigenvalues :}$$

$$(2.2) \qquad \lambda(u,v) \quad < \quad \mu(u,v) \qquad \text{for all } (u,v) \in \Omega .$$

Let (ℓ_1, ℓ_2) be the left eigenvector of A corresponding to the eigenvalue λ. Multiply the first equation of (2.1) by ℓ_1, the second by ℓ_2 and add. We obtain the characteristic equation

$$(2.3) \qquad \ell_1 u' + \ell_2 v' = 0 , \qquad \text{where } ' = \partial/\partial t + \lambda . \partial/\partial x.$$

Let $\phi = \phi(u,v)$ be an integrating factor for (2.3) such that $w_u = \phi \ell_1$, $w_v = \phi \ell_2$ for some function $w = w(u,v)$. Multiplication (2.3) by Φ gives

$$(2.4) \qquad w' = w_t + \lambda w_x = 0.$$

For the other eigenvalue μ we get a similar equation

$$(2.5) \qquad \grave{z} = z_t + \mu z_x = 0$$

$$\text{where} \quad \grave{} = \partial/\partial t + \mu \partial/\partial x$$

The functions z and w are the 1- and 2-Riemann invariants of the system (2.1) respectively. (cf. definition 1.4).

Assumption 2.2 - The map $(u,v) \in \Omega \to (w,z) \in \Omega_1 = \{(w,z) \in \mathbb{R}^2 \; ; \; w = w(u,v), z = z(u,v), (u,v) \in \Omega\}$ is one to one, onto and $\mathcal{C}^1$-class.
The Riemann invariants w and z diagonalize the system (2.1) to the system (2.4) and (2.5), where $\lambda = \lambda(w,z)$ and $\mu = \mu(w,z)$ by assumption 2.2.

Now we consider the $\mathcal{C}^1$- solution for the initial value problem of the system (2.1), i.e., for the system (2.4) (2.5) with the $\mathcal{C}^1$- initial data :

$$(2.6) \qquad w(0,x) = w_0(x), \quad z(0,x) = z_0(x),$$

where $(w_0,z_0)(x) \in \Omega_1$ for any $x \in \mathbb{R}$.

Lemma 2.1 - The $\mathcal{C}^1$- solution to the system (2.4) and (2.5) with the initial data (2.6) has the a priori estimate :

$$(2.7) \qquad |w(t,x)| \leq W_0 \equiv \sup|w_0(x)| \;, \; |z(t,x)| \leq Z_0 \equiv \sup|z_0(x)|$$

Proof : The characteristic equation for (2.4) is given by

$$\frac{dx}{dt} = \lambda(w,z) \;, \qquad \frac{dw}{dt} = 0$$

, i.e.,

$$(2.8) \quad x(t,\beta) = \beta + \int_0^t \lambda(w(s,x(s,\beta)), z(s,x(s,\beta)))ds, \; w(t,\beta) = w_0(\beta)$$

for $\beta \in \mathbb{R}$

For (2.5) we have a similar expression. qed.

Remark : It follows from lemma 2.1 that if there exists the $\mathcal{C}^1$ solution for the system (2.4) (2.5) with the initial data (2.6) which belong to Ω_1, then the solution also belongs to Ω_1 and the hyperbolicity remains to hold.

Assumption 2.3 - The system (2.1) is genuinely nonlinear in Ω, i.e.,

$$(2.9) \qquad \lambda_w > 0 \;, \quad \mu_z > 0 \qquad \text{for all } (w,z) \in \Omega_1$$

This genuine nonlinearity is equivalent to that in definition 1.2. In fact

$$r.\mathrm{grad}\,\lambda = (\lambda_w w_u + \lambda_z z_u)\, r_1 + (\lambda_w w_v + \lambda_z z_v)\, r_2 =$$

$$= \lambda_w (w_u r_1 + w_v r_2) + \lambda_z\, r.\mathrm{grad}\, z = \lambda_w \phi(\ell_1 r_1 + \ell_2 r_2) \neq 0$$

Therefore changing the sign of w or z if necessary, we may assume (2.9).

Theorem 2.1 - We suppose the assumptions 2.1, 2.2 and 2.3. Let the initial data (2.6) $\in \mathcal{C}^1$ and the rectangular $(\pm W_0, \pm Z_0) \subset \Omega_1$, where $W_0 = \sup|w_0(x)|$, $Z_0 = \sup|z_0(x)|$. If $w_{0,x} \geq 0$ and $z_{0,x} \geq 0$ for all $x \in \mathbb{R}$, then the initial value problem (2.4) (2.5) (2.6) has a unique $\mathcal{C}^1$ solution in the large in time. If $w_{0,x} < 0$ or $z_{0,x} < 0$ somewhere $x \in \mathbb{R}$, then the solution for (2.4) (2.5) (2.6) developes the singularities in the first derivative in finite time, i.e., $w_x \to -\infty$ or $z_x \to -\infty$ as $t \to t_0 < +\infty$.

Proof : cf. Lax (1964), Keller and Lu Ting (1966), Yamaguti and Nishida (1968)
Differentiation (2.8) in β gives

$$(2.10) \qquad x_\beta (t,\beta) = 1 + \int_0^t \lambda_\beta \, ds = 1 + \int_0^t (\lambda_w w_{0,\beta} + \lambda_z z_x x_\beta)\, ds$$

Define the function h(w,z) by

$$(2.11) \qquad h_z = \frac{\lambda_z (w,z)}{\lambda - \mu}$$

If we note by (2.5)

$$(2.12) \qquad z_x = \frac{z_t + \lambda z_x}{\lambda - \mu} = \frac{z'}{\lambda - \mu}$$

we have

$$\partial h(s,x(s,\beta))/\partial s = \partial h(w_0(\beta),\, z(s,x(s,\beta)))/\partial s$$

$$= h_z (z_t + \lambda z_x) = h_z z' = \lambda_z z_x .$$

Substitution this into (2.10) gives

$$x_\beta (t,\beta) = 1 + \int_0^t \lambda_w w_{0,\beta} + h_s x_\beta \quad ds .$$

Therefore we can differentiate this in t for fixed β.

$$x_{\beta t} = \lambda_w w_{0,\beta} + h_t x_\beta \qquad , \text{ i.e., } \qquad (e^{-h} x_\beta)_t = \lambda_w w_{0,\beta} e^{-h}$$

The integration in t gives

$$x_\beta(t,\beta) = e^{h(t,x(t,\beta))}(e^{-h(0,\beta)} + \int_0^t \lambda_w w_{0,\beta} e^{-h(s,x(s,\beta))} ds)$$

Thus we arrive at the following expression for the first derivative

$$w_x(t,x) = w_\beta / x_\beta = w_{0,\beta} / x_\beta = 1/e^{h(t,x)} (e^{-h(0,\beta)}/w_{0,\beta} + $$
$$+ \int_0^t \lambda_w \; e^{-h(s,x(s,\beta))} \; ds),$$

where $h(t,x)$ & $h(0,\beta)$ are bounded continuous if w and z are so. Also $\lambda_w \geq \delta > 0$ in the rectangular $(\pm W_0, \pm Z_0)$. Therefore the theorem follows.

3. Weak Solutions for the Initial Value Problem

The initial value problem for the nonlinear hyperbolic system of conservation laws can not be solved generally in the class of smooth functions in the large in time as shown in § 2. Thus in order to construct the solution in the large in time one has to introduce weak solutions to the initial value problem for the system

(3.1) $$u_t + f(u)_x = 0 \qquad \text{in } t \geq 0,\ x \in \mathbb{R},$$

with the initial data

(3.2) $$u(0,x) = u_0(x) \qquad \text{in } x \in \mathbb{R}.$$

Definition 3.1 - A bounded measureable n-vector function $u(t,x)$ is a weak solution of (3.1) (3.2) if it satisfies the following integral identity :

(3.3) $$\iint_{t>0} u.\ \zeta_t + f(u).\ \zeta_x \, dx\, dt + \int_{t=0} u_0(x).\ \zeta(0,x)\, dx = 0$$

for all smooth n-vector functions $\zeta(t,x)$ with compact support in $t \geq 0$, $x \in \mathbb{R}$.

Of course if the solution $u(t,x)$ of (3.1) (3.2) is smooth, it satisfies (3.3). In fact multiply (3.1) by ζ and integrate it in $t \geq 0$, $x \in R$, and the integration by parts with (3.2) gives (3.3).

Our definition of a weak solution implies that the following "jump conditions" must hold across any smooth curve $x = x(t)$ of the discontinuity in solutions :

$$(3.4) \qquad D\,[u_j] = [f_j] , \qquad j = 1, 2, \ldots, n,$$

where $D = dx/dt$ is the velocity of discontinuity at the point in question, and $[u_i]$ denotes the difference in quantity u_i across the discontinuity curve. If the k-th characteristic field $\lambda = \lambda_k(u)$ of the system (3.1) is linearly degenerate, then the corresponding discontinuity in solutions is called a contact discontinuity and it is characterized by (3.4) with

$$(3.5) \qquad \lambda_k(u_\ell) = \lambda_k(u_r) = D ,$$

where u_ℓ and u_r are the left and right hand side quantities of u on the discontinuity respectively. If the k-th characteristic field $\lambda = \lambda_k(u)$ is genuinely nonlinear, then the corresponding discontinuity in solutions is called a shock wave, the relation (3.4) is called Rankine-Hugoniot shock condition and it is also required that

$$(3.6) \begin{cases} \lambda_{k-1}(u_\ell) < D < \lambda_k(u_\ell) \\ \lambda_k(u_r) < D < \lambda_{k+1}(u_r) \end{cases} \qquad \text{(cf. Lax (1957))}.$$

This comes from the stability of shock waves physically and is required for the uniqueness of weak solutions mathematically.

There is a celebrated theorem by Glimm (1965) on the existence of weak solutions in the large in time for the initial value problem of the general system (3.1) with n conservation laws, which may be summarized as follows :

Hypothesis 3.1 - The system (3.1) is considered in a neighbourhood Ω of a constant vector $c=(c_1,\dots,c_n)$ and f is smooth in $u\in\Omega$. The system (3.1) is hyperbolic and its characteristic fields are genuinely nonlinear or linearly degenerate in Ω .

Hypothesis 3.2 - The initial value $u_0(x)$ is given in Ω for any $x\in\mathbb{R}$ and has the finite total variation on $\mathbb{R}$. Put

(3.7) $$d=\|u_0(x)-c\|_\infty + TV\, u_0(.) \quad ,$$

where $\|\,.\,\|_\infty$ is the L^∞- norm and TV means the total variation on $x\in\mathbb{R}$.

Theorem 3.1 - Under the hypotheses 3.1 and 3.2 there are a $K<+\infty$ and a $\delta>0$ with the following property. If the initial data $u_0(x)$ are given so that $d\leqslant\delta$, then there exists a weak solution u(t,x) of (3.1)(3.2) in the large in time such that

(3.8) $$\|u-c\|_\infty \leqslant K\|u_0-c\|_\infty$$

(3.9) $$TV\, u(t,.) \leqslant K\, TV\, u_0(.)$$

(3.10) $$\int_{-\infty}^{\infty} |u(t_2,x)-u(t_1,x)|\,dx \leqslant K|t_2-t_1|\, TV\, u_0(.)$$

The proof (Glimm (1965)) relies on the solutions of the Riemann's initial value problem and onthe use of the Glimm's finite difference scheme with a nonlinear functional on the approximate solutions which enables the uniform estimate of total variation of approximate solutions and gives its convergence to a weak solution. (cf. Kuznecov & Tupčiev (1975) for a generalization).

For the genuinely nonlinear hyperbolic systems of two equations Glimm and Lax (1970) show the existence and decay of weak solutions to the Cauchy problem with the initial data which are bound measurable functions with the small L^∞- norm.

4. Riemann Problem for the System of a Polytropic Gas Motion

The equation of ideal compressible flow of a polytropic gas has the form (cf. example 1.1) :

$$(4.1)\qquad \begin{cases} u_t + p_x = 0 \\ v_t - u_x = 0 \\ (pv + (\gamma-1)u^2/2)_t + (\gamma-1)\ (pu)_x = 0 \end{cases},$$

where the equation of state for gas is assumed polytropic :

$$(4.2)\qquad v^\gamma = a^2\ p^{-1} \exp\ ((\gamma-1)\ s/R),$$

$a, R > 0$ and $\gamma \geqslant 1$ are constants.

Here the unknown variables u, p, s are considered basic ones and its quasilinear form is given by the following :

$$(4.3)\qquad \begin{pmatrix} u \\ p \\ s \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ -1/v_p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ p \\ s \end{pmatrix}_x = 0$$

The characteristics, eigenvectors and Riemann invariants are summarized as follows :

$$(4.4)\qquad \begin{cases} \lambda = -1/\sqrt{-v_p}, & r_1 = \begin{pmatrix} 1 \\ -1/\sqrt{-v_p} \\ 0 \end{pmatrix}, \ z_1 = s \text{ and } u + \int_1^p \sqrt{-v_p}\, dp, \\ \nu = 0, & r_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad z_2 = u \text{ and } p, \\ \mu = 1/\sqrt{-v_p}, & r_3 = \begin{pmatrix} 1 \\ 1/\sqrt{-v_p} \\ 0 \end{pmatrix}, \quad z_3 = s \text{ and } u - \int_1^p \sqrt{-v_p}\, dp \end{cases}$$

where $v = v(p,s)$ is given by (4.2).

Définition 4.1 - Denote the Riemann invariants which are not the unknown variables as follows :

$$(4.5)\quad Z = u+a(s)\ (p^{\frac{\gamma-1}{2\gamma}}-1)/(\gamma-1)\quad ,\quad W = u-a(s)\ (p^{\frac{\gamma-1}{2\gamma}}-1)/(\gamma-1),$$

where $a(s) = 2\, a^{1/\gamma}\gamma^{1/2} \exp\,((\gamma-1)s/2\gamma R)$.

They give a one to one mapping from $\Omega = \{(u,p,s) \in \mathbb{R}^3, p > 0\}$ onto $\Omega_1 = \{(W,Z,s) \in \mathbb{R}^3,\quad Z - W > -2\, a(s)/(\gamma-1)\}$.

The Riemann problem for system (4.1) is an initial value problem for system (4.1) with the special initial data

$$(4.6)\quad (u,p,s)\,(0,x) = \begin{cases} (u_1, p_1, s_1) & \text{in } x < 0 \\ (u_2, p_2, s_2) & \text{in } x > 0 \end{cases} ,$$

where $(u_i, p_i, s_i)_{i=1,2}$ are two constant states in Ω i.e., $p_i > 0$ $(i = 1,2)$. The Riemann problem (4.1) (4.6) is invariant under the similar transformation $x \to \alpha x$, $t \to \alpha t$, it has the selfsimilar solutions which are functions of $\xi = x/t$. In fact the substitution of $(u,p,s)\,(t,x) = (u,p,s)(\xi)$ into (4.1) gives

$$(4.7)\quad \begin{cases} -\xi\quad u_\xi + p_\xi \quad = 0 \\ -\xi\quad v_\xi - u_\xi \quad = 0 \\ -\xi(pv + (\gamma-1)u^2/2)_\xi \ + (\gamma-1)\ (pu)_\xi \ = 0 \end{cases}$$

$$\text{in}\quad \xi \in R$$

The initial data (4.6) turn out to be a boundary condition as follows :

$$(4.8)\qquad (u,p,s)\ (\xi) = \begin{cases} (u_1, p_1, s_1) & \xi \to -\infty \\ (u_2, p_2, s_2) & \xi \to +\infty \end{cases}$$

So we want to solve the ordinary differential system (4.7) with the boundary condition (4.8). It is solved by the following elementary waves (i) ~ (vi). First the system (4.7) has the constant solutions :

(i) $(u,p,s)\ (\xi) = \text{constant vector} \in \Omega$.

Next, if we seek the smooth solution of (4.7), then we can differentiate it in ξ . Remembering the reduction from (4.1) to (4.3) we have the quasilinear system :

$$(4.9) \qquad \left(-\xi I + \begin{pmatrix} 0 & 1 & 0 \\ -1/v_p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \frac{d}{d\xi} \begin{pmatrix} u \\ p \\ s \end{pmatrix} = 0 \quad .$$

Thus for the smooth solution we have

$$(4.10) \qquad s(\xi) = \text{constant},$$

and so the system (4.9) reduces to a system of two equations

$$(4.11) \qquad \left(-\xi I + \begin{pmatrix} 0 & 1 \\ -1/v_p & 0 \end{pmatrix}\right) \frac{d}{d\xi} \begin{pmatrix} u \\ p \end{pmatrix} = 0 \quad ,$$

which can be diagonalized by the Riemann invariants (4.5) with s = constant as follows :

$$(4.12) \qquad \begin{cases} (-\xi + \lambda)\ dW/d\xi = 0 \\ (-\xi + \mu)\ dZ/d\xi = 0 \\ s\ (\xi) = \text{constant} \end{cases}$$

Therefore the diagonal system (4.12) gives the nontrivial smooth solutions of the system (4.9), if

(ii) $\xi = \lambda$, $dW/d\xi > 0$, $Z = \text{constant}$ $s = \text{constant}$ or

(iii) $\xi = \mu$, $dZ/d\xi > 0$, $W = \text{constant}$, $s = \text{constant}$.

Definition 4.2 - Let $(u_i, p_i, s_i) \in \Omega$, $i = 1,2$ be given. The R_1-curve with the initial (u_1, p_1, s_1) is defined by

$$(4.13) \quad R_1(u_1,p_1,s_1) = \{(u,p,s) : Z(u,p,s) = Z(u_1,p_1,s_1),\ s = s_1,\ p < p_1\}$$

The R_2-curve with the initial (u_2,p_2,s_2) is defined by

(4.14) $$R_2(u_2,p_2,s_2) = \{(u,p,s) : W(u,p,s) = W(u_2,p_2,s_2),\ s = s_2,\ p < p_2\}\ .$$

Then any $(u_1,p_1,s_1) \in \Omega$ can be connected to $(u_3,p_3,s_1) \in R_1(u_1,p_1,s_1)$ on the right by the backward rarefaction wave and any $(u_2,p_2,s_2) \in \Omega$ can be connected to $(u_4,p_4,s_2) \in R_2(u_2,p_2,s_2)$ on the left by the forward rarefaction wave as follows :

(ii) R_1 - wave (the backward rarefaction wave)

(4.15) $$(u,p,s)\ (\xi) = \begin{cases} (u_1,p_1,s_1) & \text{in } \xi < \lambda\ (p_1,s_1) \\ (u(\xi),\ p(\xi),\ s_1) & \text{on } \xi = \lambda\ (p(\xi),s_1) \\ (u_3,p_3,s_1) & \text{in } \xi > \lambda\ (p_3,s_1) \end{cases}\ .$$

where $(u(\xi),\ p(\xi),\ s_1) \in R_1(u_1,\ p_1,\ s_1)$ for $\lambda(p_1,s_1) < \xi < \lambda(p_3,s_1)$.

(iii) R_2-wave (the forward rarefaction wave)

(4.16) $$(u,p,s)\ (\xi) = \begin{cases} (u_4,p_4,s_2) & \text{in } \xi < \mu(p_4,s_2) \\ (u(\xi),\ p(\xi),\ s_2) & \text{on } \xi = \mu\ (p(\xi),\ s_2) \\ (u_2,p_2,s_2) & \text{in } \xi > \mu\ (p_2,s_2) \end{cases}\ ,$$

where $(u(\xi),\ p(\xi),\ s_2) \in R_2(u_2,\ p_2,\ s_2)$ for $\mu(p_4,s_2) < \xi < \mu(p_2,s_2)$.

<u>Definition 4.3</u> - Let us denote the Riemann invariants (4.5) with a fixed $s = s_0$ by

(4.17) $$\begin{aligned} z &= u + a_0\ (p^{\frac{\gamma-1}{2\gamma}}-1)\ /\ (\gamma-1) \\ w &= u - a_0\ (p^{\frac{\gamma-1}{2\gamma}}-1)/\ (\gamma-1) \end{aligned}\ ,$$

where $a_0 = a(s_0)$.

This is a one to one mapping from (u,p), $p > 0$ onto (w,z), $z > w - 2\ a_0/(\gamma-1)$.

<u>Lemma 4.1</u> - Let $(u_i,\ p_i,\ s_i) \in \Omega$, $s_i \geqslant s_0$ $(i = 1,2)$.

Then the R_i-curve, $i = 1,2$, is the straight half-line in the plane (w,z) given by the following :

$$(4.18)\quad \begin{cases} R_1(u_1,p_1,s_1) = \{ (w,z,s_1) : z-z_1 = \dfrac{a_1-a_0}{a_1+a_0}(w-w_1), \quad w > w_1 \} \\ R_2\ (u_2,p_2,s_2) = \{ (w,z,s_2) : \ w-w_2 = \dfrac{a_2-a_0}{a_2+a_0}(z-z_2),\ z < z_2 \} \end{cases},$$

where $a_i = a(s_i)$, $(w_i, z_i) = (w(u_i,p_i), z(u_i,p_i))$, $i = 1,2$.

Proof : The R_1-curve with the initial (u_1,p_1,s_1) is defined by (4.13) i.e.,

$$u+a_1(p^{\frac{\gamma+1}{2\gamma}}-1)/(\gamma-1) = u_1+a_1(p_1^{\frac{\gamma+1}{2\gamma}}-1)/(\gamma-1) .$$

Also by the definition 4.3 we have

$$z_1 = u_1 + a_0\ (p_1^{\frac{\gamma-1}{2\gamma}}-1)/(\gamma-1),$$

$$w_1 = u_1 - a_0\ (p_1^{\frac{\gamma-1}{2\gamma}}-1)/(\gamma-1) .$$

If we eliminate the constants u_1 and p_1 in these three relations, we have a relation in u and p

$$u + a_1(p^{\frac{\gamma-1}{2\gamma}}-1)/(\gamma-1) = \frac{1}{2}\{(1-\frac{a_1}{a_0})\ w_1 + (1+\frac{a_1}{a_0})\ z_1\} .$$

Substituting u and p in terms of z and w by (4.17) into this relation we arrive at the first expression in (4.18).

qed.

In addition to the smooth solutions (ii) and (iii) for the system (4.7) the discontinuous transitions are possible, if the jump conditions (3.4) are satisfied across the discontinuity, which are written in our case as follows :

$$(4.19)\quad \begin{cases} D\ [u] = [p], \quad D\ [v] = -\ [u] \quad \text{and} \\ D\ [pv + (\gamma-1)u^2/2] = [(\gamma-1)\ pu] \end{cases},$$

where $D = \xi$ is the velocity of discontinuity and $[\cdot]$ is the difference in quantity . across the discontinuity. The jump conditions (4.19) give the following three discontinuities according to the characteristics fields ν , λ, μ.

(iv) Contact discontinuity ($D = 0$)

$$(4.20) \qquad (u,p,s)\,(\xi) = \begin{cases} (u_3,\ p_3,\ s_3) & \text{in } \xi < 0 \\ (u_3,\ p_3,\ s_4) & \text{in } \xi > 0 \end{cases} ,$$

where $p_3 > 0$, u_3, s_3, s_4 are arbitrary constants.

(v) S_1-wave (the backward shock wave)

$$(4.21) \qquad (u,p,s)\,(\xi) = \begin{cases} (u_1,\ p_1,\ s_1) & \text{in } \xi < D_1 \\ (u_3,\ p_3,\ s_3) & \text{in } \xi > D_1 \end{cases} ,$$

where $(u_1,\ p_1,\ s_1) \in \Omega$ is an arbitrary constant state, $D_1 = -\{((\gamma+1)\ p_3 + (\gamma-1)p_1)\ 2\gamma\ p_1^{1/\gamma}\}^{1/2} / a(s_1)$, and (u_3,p_3,s_3) is any constant state on the S_1-curve with the initial $(u_1,\ p_1,\ s_1)$ defined by the following :

$$(4.22) \qquad S_1(u_1,p_1,s_1) = \{(u,p,s) : s > s_1 \text{ i.e., } p > p_1 \text{ and}$$

$$u - u_1 = \frac{(p - p_1)\ a(s_1)}{\{((\gamma+1)p + (\gamma-1)\ p_1)\ 2\gamma\ p_1^{1/\gamma}\}^{1/2}} ,$$

$$\exp \frac{(\gamma-1)\ (s-s_1)}{R} = \frac{p}{p_1}\left\{ \frac{(\gamma-1)p + (\gamma+1)\ p_1}{(\gamma+1)p + (\gamma-1)p_1} \right\}^{\gamma} \Bigg\} .$$

(vi) S_2 - wave (the forward shock wave).

$$(4.23) \quad (u,p,s)(\xi) = \begin{cases} (u_4,\ p_4,\ s_4) & \text{in } \xi < D_2 \\ (u_2,\ p_2,\ s_2) & \text{in } \xi > D_2 \end{cases} ,$$

where $(u_2,\ p_2,\ s_2) \in \Omega$ is any constant state,

$D_2 = \{((\gamma+1)p_4 + (\gamma-1)p_2)\ 2\gamma\ p_2^{1/\gamma}\}^{1/2} / a(s_2)$, and (u_4,p_4,s_4) is any constant state on the S_2-curve with the initial (u_2,p_2,s_2) defined by the following :

(4.24) $S_2(u_2,p_2,s_2) = \{(u,p,s) : s > s_2$ i.e., $p > p_2$ and

$$u - u_2 = \frac{(p - p_2)\ a(s_2)}{\{((\gamma+1)\ p + (\gamma-1)\ p_2)\ 2\gamma\ p_2^{1/\gamma}\}^{1/2}} ,$$

$$\exp \frac{(\gamma-1)\ (s-s_2)}{R} = \frac{p}{p_2} \left\{ \left\{\frac{(\gamma-1)p + (\gamma+1)\ p_2}{(\gamma+1)p + (\gamma-1)\ p_2}\right\}^{\gamma} \right\} .$$

In both cases (v) and (vi) $s > s_1$ or $s > s_2$ is required by the entropy condition. The global geometry of shock curves S_1 and S_2 are given by the following :

<u>Lemma 4.2</u> : Assume that $1 \leqslant \gamma \leqslant 5/3$ and let $(w_i,z_i) = (w(u_i,p_i), z(u_i,p_i))$, $a_i = a(s_i)$ for any $(u_i,p_i,s_i) \in \Omega$ and $s_i \geqslant s_0$, $i = 1,2$. Then the S_1-curve is expressed in terms of w,z as follows :

(4.25) $$S_1\ (u_1,p_1,s_1) = \{\ (w,z,s) : z_1-z = f(w_1 - w\ ;\ p_1,s_1) ,$$

$$s - s_1 = g(w_1 - w\ ;\ p_1,s_1) \quad , \quad w < w_1 \} \quad .$$

The S_2-curve is expressed as follows :

(4.26) $$S_2\ (u_2,p_2,s_2) = \{\ (w,z,s) : w - w_2 = f(z-z_2;\ p_2,s_2),$$

$$s - s_2 = g(z-z_2\ ;\ p_2,s_2) \quad , \quad z > z_2 \ \} \quad .$$

Here the functions f and g have the properties :

(4.27) $f(0) = g(0) = g'(0) = f''(0) = g''(0) = 0$, $0 < f(y), g(y), g'(y), f''(y), g''(y)$

for $y > 0$, $0 \leqslant f'(0;\ p_i,s_i) = \dfrac{a_i - a_0}{a_i + a_0} < f'(y;\ p_i,s_i) < 1$ for $y > 0$,

$0 \leqslant f'''(0), g'''(0)$.

proof : It follows from (4.17) and (4.22) that

$$\left.\begin{matrix} z_1 - z \\ w_1 - w \end{matrix}\right\} = u_1 - u \pm a_0 \left(p_1^{\frac{\gamma-1}{2\gamma}} - p^{\frac{\gamma-1}{2\gamma}}\right)/(\gamma-1)$$

$$= \frac{(p - p_1)\, a_1}{\{((\gamma+1)p + (\gamma-1)p_1)\, 2\gamma p_1^{1/\gamma}\}^{1/2}} \mp \frac{a_0 \left(p^{\frac{\gamma-1}{2\gamma}} - p_1^{\frac{\gamma-1}{2\gamma}}\right)}{\gamma - 1}$$

$$= \frac{a_1\, p_1^{\frac{\gamma-1}{2\gamma}}}{(2\gamma)^{1/2}} \left\{ \frac{\alpha - 1}{\{(\gamma+1)\alpha + (\gamma-1)\}^{1/2}} \mp \frac{a_0\, (2\gamma)^{1/2}}{a_1\, (\gamma-1)} \left(\alpha^{\frac{\gamma-1}{2\gamma}} - 1\right)\right\},$$

where $\alpha = p/p_1 \geq 1$.

It is easy to see that for $\alpha \geq 1$, $w_1 - w \geq 0$ and $d(w_1 - w)/d\alpha > 0$ and so that $\alpha = \alpha(w_1 - w) \geq 1$ for $w_1 \geq w$, $\alpha(0) = 1$. Thus we have for $y \geq 0$

$$f'(y) \equiv \frac{d(z_1 - z)}{d(w_1 - w)} = \frac{d(z_1 - z)/d\alpha}{d(w_1 - w)/d\alpha}$$

and $\quad f'(0) = (a_1 - a_0)/(a_1 + a_0)$.

Then using the assumption $1 \leq \gamma \leq 5/3$ the direct calculation shows that $f''(0) = 0$ and $f''(y) > 0$ for $y > 0$. Therefore we have for $y > 0$

$$0 \leq f'(0) = \frac{a_1 - a_0}{a_1 + a_0} < f'(y) < 1.$$

The other inequalities in the lemma are shown by the analogous computations.

qed.

Now, we can solve the Riemann problem (4.1) (4.6) i.e., (4.7) (4.8) by these six elementary waves (i)$\sim$(vi). Let us consider the projection of two points $(u_i, p_i, s_i) \in \Omega$, $i = 1,2$ on the plane (w,z) i.e., $(w_i, z_i) = (w(u_i, p_i), z(u_i, p_i))$, $i = 1,2$ defined by (4.17). Draw the four curves $R_1(u_1,p_1,s_1)$, $S_1(u_1,p_1,s_1)$ and $R_2(u_2,p_2,s_2)$, $S_2(u_2,p_2,s_2)$ in the plane (w,z)

which are characterized by lemma 4.1 and 4.2. They intersect exactly at one point (w_3, z_3), which belongs to one of the following five cases : The solution consists of four constant states (u_i, p_i, s_i) $i = 1,2,3,4$, where $(u_3, p_3) = (u_4, p_4) = (u(w_3, z_3), p(w_3, z_3))$ defined by (4.17), which are connected by the elementary waves as follows :

I. $(w_3, z_3) \in R_1 \cap R_2 \cap \{z > w - 2\, a_0/(\gamma-1)\}$

The solution consists of four constant states (u_i, p_i, s_i) $i = 1,2,3,4$ which are connected by the R_1-wave (4.15), by the contact discontinuity (4.20) and by the R_2-wave (4.16), where $s_3 = s_1$ and $s_4 = s_2$.

II. $(w_3, z_3) \in S_1 \cap R_2$

They are connected by the S_1-wave (4.21), by the contact discontinuity (4.20) and by the R_2-wave (4.16), where $s_4 = s_2$.

III. $(w_3, z_3) \in S_1 \cap S_2$

They are connected by the S_1-wave (4.21), by the contact discontinuity (4.20) and by the S_2-wave (4.23).

IV. $(w_3, z_3) \in R_1 \cap S_2$

They are connected by the R_1-wave (4.15), by the contact discontinuity (4.20) and by the S_2-wave (4.23), where $s_3 = s_1$.

V. $(w_3, z_3) \in R_1 \cap R_2 \cap \{z \leqslant w - 2\, a_0/(\gamma-1)\}$

$$(u,p,s)(\xi) \begin{cases} = (u_1, p_1, s_1) & \text{in} \quad \xi < \lambda\,(p_1, s_1) \\ \in R_1(u_1, p_1, s_1) & \text{in} \quad \lambda(p_1, s_1) < \xi < 0 \\ \in R_2\,(u_2, p_2, s_2) & \text{in} \quad 0 < \xi < \mu(p_2, s_2) \\ = (u_2, p_2, s_2) & \text{in} \quad \mu(p_2, s_2) < \xi \end{cases},$$

when the solution attains the vacuum $p = 0$ on $\xi = 0$ and has a discontinuity in u there, which should be considered in the Eulerian coordinate.

5. Glimm's Difference Scheme and Interaction of Elementary Waves

We are going to solve the initial value problem for the equation of the polytropic gas motion :

$$(5.1)\qquad \begin{cases} u_t + p_x = v_t - u_x = 0 \\ (e + u^2/2)_t + (pu)_x = 0 \end{cases} \qquad \text{in } t \geq 0,\ x \in \mathbb{R},$$

where

$$(5.2)\qquad \begin{cases} v^\gamma = a^2\, p^{-1} \exp((\gamma-1)s/R)\ ,\ a, R > 0,\ 1 \leq \gamma \leq 5/3 \\ e = (pv - a^{2/\gamma})/(\gamma-1) = a^{2/\gamma}(p^{\frac{\gamma-1}{\gamma}} \exp((\gamma-1)\, s/R) - 1)/(\gamma-1) \end{cases}$$

with the initial data

$$(5.3)\qquad (u,p,s)\ (0,x) \qquad \text{given in } x \in \mathbb{R}.$$

We seek the weak solution $(u,p,s)(t,x)$ which are bounded functions and satisfy the integral identity :

$$(5.4)\qquad \iint_{t>0} [u\, \phi_t + p\, \phi_x + v\, \psi_t - u\, \psi_x + (e+u^2/2)\, \chi_t + p\, u\, \chi_x]\, dx\, dt + \int_{t=0} u\phi + v\psi + (e+u^2/2)\, \chi\, d\, x = 0.$$

for any smooth functions ϕ, ψ, χ with compact support in $t \geq 0$, $x \in \mathbb{R}$.

The weak solution in the large in time for the initials with finite total variation is obtained as the limit of the approximate solutions constructed by the Glimm's difference scheme. To simplify the argument we restrict ourselves to treat the system (5.1) with $\gamma = 1 + 0$ i.e.,

$$(5.5)\qquad \begin{cases} u_t + p_x = v_t - u_x = 0 \\ (e+u^2/2)_t + (p\, u)_x = 0 \end{cases} ,$$

where

(5.6) $v = a^2/p, \quad e = a^2\,(\log p + s/R).$

In this case the characteristics and the Riemann invariants which are not the unknown variables u,p,s are given by the following :

$$(5.7)\begin{cases} \lambda = -\,p/a, & \nu = 0, \qquad \mu = p/a \\ w = u - a\log p, & z = u + a\log p \end{cases}$$

Since these quantities are independent of s, the entropy s may be considered as a secondary independent variable for the system (5.5) (5.6). Also the Riemann invariants w,z define a mapping :

$$(5.8)\ \Omega = \{(u,p,s),\ p>0\} \longmapsto \Omega_1 = \{(w,z,s) \in \mathbb{R}^3\}\ .$$

The shock curves and the rarefaction curves are given by the following :

$$(5.9)\begin{cases} S_1(u_1,p_1,s_1) = \left\{ u-u_1 = -\dfrac{a(p-p_1)}{(p\,p_1)^{1/2}}\ ,\ s-s_1 = R\left(\log\dfrac{p_1}{p} + \dfrac{p^2-p_1^2}{2\,p\,p_1}\right),\quad p>p_1 \right\}, \\ R_1(u_1,p_1,s_1) = \left\{ u-u_1 = a\log\dfrac{p_1}{p},\ s=s_1,\ p<p_1 \right\}, \\ S_2(u_2,p_2,s_2) = \left\{ u-u_2 = \dfrac{a(p-p_2)}{(p\,p_2)^{1/2}}\ ,\ s-s_2 = R\left(\log\dfrac{p_2}{p} + \dfrac{p^2-p_2^2}{2\,p\,p_2}\right),\quad p>p_2 \right\}, \\ R_2(u_2,p_2,s_2) = \left\{ u-u_2 = a\log\dfrac{p}{p_2}\ ,\ s=s_2,\ p<p_2 \right\}. \end{cases}$$

<u>Lemma 5.1</u> - The Riemann problem (5.5) (5.6) for the initial data

$$(5.10)\qquad (u,p,s)(0,x) = \begin{cases} (u_-,\ p_-,\ s_-) & x<0 \\ (u_+,\ p_+,\ s_+) & x>0 \end{cases},$$

where $p_{\mp} > 0$, $u_{\mp}$, $s_{\mp}$ are constants, has a piecewise continuous and piecewise smooth weak solution in the large in time and satisfies the estimate :

$$(5.11)\quad \begin{cases} w(t,x) \equiv w((u,p,s)\ (t,x)) \geq w_0, \\ z(t,x) \equiv z((u,p,s)\ (t,x)) \leq z_0\ , \end{cases}$$

where

$$(5.12)\quad w_0 = \min w\ ((u,p,s)\ (0,x)),\quad z_0 = \max z((u,p,s)\ (0,x)).$$

Therefore the speed of propagation is bounded as follows :

$$(5.13)\quad |\lambda|\ ,\mu \leq \max\ p/a \leq a^{-1} \exp\ ((z_0 - w_0)/2a)\ .$$

The proof is a special case of that given in § 4. However the case V there does not exist here because of $\Omega_1 = \mathbb{R}^3$ by (5.8), and so the Riemann problem is always solved without the vacuum. If we note that the quantities (5.7) are independent of s, the estimate (5.11) and (5.13) follows from the consideration of each case I, II, III, IV in the (w,z)-plane and from the properties of the rarefaction- and shock-curves (lemma 4.1, 4.2).

Definition 5.1 -

$$(5.14)\quad q = \frac{z - w}{2} = a \log p\ .$$

This define a mapping $(u,p,s) \in \Omega = \{(u,p,s),\ p > 0\} \mapsto (u,q,s) \in \Omega_2 = \mathbb{R}^3$.

Lemma 5.2 - Let $(u_i, q_i, s_i) = (u_i, a \log p_i, s_i)$ for any $(u_i, p_i, s_i) \in \Omega$, $i = 1,2$. The shock-curve $S_i(u_i, p_i, s_i)$ and rarefaction- curve $R_i(u_i, p_i, s_i)$, $i = 1,2$, in terms of (u,q,s) have the same figures respectively and are independent of the initials (u_i, q_i, s_i) i.e.,

$$(5.15)\quad \begin{cases} S_1(u_1,q_1,s_1) = \{(u,q,s) : u - u_1 = -2a\ sh(q-q_1)/2a, \\ s-s_1 = R(-(q-q_1)/a + sh(q-q_1)/a),\ q > q_1\}\ , \\ R_1(u_1,q_1,s_1) = \{u-u_1 = -(q-q_1),\ s = s_1,\ q < q_1\}\ , \\ S_2(u_2,q_2,s_2) = \{u-u_2 = 2\ a\ sh(q-q_2)/2a, \\ s-s_2 = R\ (-(q-q_2)/a + sh(q-q_2)/a),\quad q > q_2\}\ , \\ R_2(u_2,q_2,s_2) = \{u-u_2 = q-q_2\ ,\ s=s_2,\ q < q_2\} \end{cases}$$

The proof is easy if we note that $q-q_i$ = a log p/p_i and the fact that the shock - and rarefaction- curves depend only on p/p_i in (5.9).

Now we introduce the Glimm's difference scheme to get the approximate solutions (Glimm, 1965). Suppose that the initial data (5.3) are bounded and have bounded total variations and define

$$(5.16)\quad \begin{cases} (u_{\mp}, p_{\mp}, s_{\mp}) = \lim\limits_{x\to\mp\infty} (u, p, s)\ (0,x) \\ p_0 = \inf\limits_x p(0,x) > 0 \quad , \quad s_0 = \inf s(0,x) \\ w_0 = \inf\limits_x w\ ((u,p,s)\ (0,x)) \\ z_0 = \sup z\ ((u,p,s)\ (0,x)). \end{cases}$$

The initial data are approximated by the step functions with the mesh length 2 h $(0 < \forall h \leqslant h_0)$

$$(5.17)\qquad U^h(0,x) = U(0,mh) \quad \text{in } (m-1)h < x < (m+1)\,h, \quad m : \text{even},$$

where $\quad U^h(t,x) = (u^h,p^h,s^h)\ (t,x)$ and $U(0,x) = (u,p,s)\ (0,x)$.

Let us define the time mesh length ℓ by

$$(5.18)\qquad \ell/h = a^{-1} \exp\ ((w_0 - z_0)\ /\ 2a)$$

and set

$$(5.19)\qquad Y = \{\ (n,m)\ ;\ n,m \text{ are integers, } n+m \text{ is even and } n \geqslant 1\ \}$$

Definition 5.2 - (Glimm's difference scheme)

We choose any random sequence of equidistributed numbers in (-1, 1) :

$$(5.20)\qquad \alpha = \{\alpha_n\}_{n\geqslant 1} \quad , \quad \alpha_n \in (-1,\ 1),$$

and set the mesh points as

$$(5.21)\qquad \begin{aligned} a^n_m &= (n\ell,\ mh + \alpha_n h) \text{ for } (n,m) \in Y, \\ a^0_m &= (0,\ mh), \quad m : \text{even} . \end{aligned}$$

The approximation $U^h(t,x)$ on the mesh points a^0_m is defined by (5.17). Suppose that our approximation $U^h(t,x)$ has been defined for $(t,x) = a^{n-1}_{m-1}$ and for $(t,x) = a^{n-1}_{m+1}$ (some $(n,m) \in Y$). We define $U^h(a^n_m)$ as follows :

Let $U = (u,p,s)(t,x)$ be the solution of the Riemann problem for (5.5) in $t \geq (n-1)\ell$, $x \in \mathbb{R}$ with the initial data

$$U((n-1)\ell, x) = \begin{cases} U(a^{n-1}_{m-1}) & \text{in } x < mh \\ U(a^{n-1}_{m+1}) & \text{in } x > mh . \end{cases}$$

Set the approximate solution as

$$(5.22)\qquad U^h(t,x) = U(t,x) \text{ in } \begin{cases} (n-1)\ell \leq t < n\ell, \\ (m-1)h \leq x \leq (m+1)h \end{cases}$$

and define the approximation $U^h(t,x)$ on the mesh point $(t,x) = a^n_m$ by

$$(5.23)\qquad U^h(a^n_m) = U(a^n_m) .$$

Since the Riemann problem for (5.5) is always solved by lemma 5.1., our approximate solution is defined for all a^n_m, $(n,m) \in Y$. Furthermore the approximate solution $U^h(t,x)$ is the exact weak solution in each strip $(n-1)\ell \leq t < n\ell$, $x \in \mathbb{R}$. In fact it follows from the estimate (5.13) and from the choice of (5.16) and (5.18) that there never intersects the two waves coming from the neighbouring discontinuity points $(t,x) = ((n-1)\ell, mh)$ and $((n-1)\ell, (m+2)h)$ for any m such that $(n,m) \in Y$. Thus the approximate solutions for any α and for any $h \in (0,h_0)$ have been defined in $t \geq 0$, $x \in \mathbb{R}$.

Before we prove the convergence to a weak solution we need to consider the interactions of elementary waves and to get some preliminary bounds for them. Remember that the solution of the Riemann problem has four constant states $U_i = (u_i, p_i, s_i)$, i = 1, 2, 3, 4, connected by three of the elementary waves : 1-wave (S_1- or R_1-wave), 0-wave (contact discontinuity) and 2-wave (S_2- or R_2-wave). We denote this vector of three elementary waves joining four constant states U_1 (i = 1,2,3,4) by $\beta = (\beta_1, \beta_0, \beta_2)$, γ and so on.

Definition 5.3 - The magnitude of each i-wave β_i, i = 0, 1, 2, in β is measured by the difference of q or s as follows (cf. (5.15)) :

$$(5.24) \begin{cases} \beta_0 = s_3 - s_2 & , \\ \beta_1 = q_2 - q_1 \gtrless 0 & \text{for } S_1\text{- or } R_1\text{-wave respectively,} \\ \beta_2 = q_3 - q_4 \gtrless 0 & \text{for } S_2\text{- or } R_2\text{-wave respectively.} \end{cases}$$

Its absolute value is called the strength of i-wave.

$$(5.25) \begin{cases} |\beta_0| = |s_3 - s_2| & , \\ |\beta_1| = |q_2 - q_1| \quad \text{and} \quad |\beta_2| = |q_3 - q_4| \end{cases}$$

The increase of the entropy in the i-wave i.e., in the S_i-wave (i = 1,2) is denoted by

$$(5.26) \begin{cases} \varepsilon_{\beta_1} = s_2 - s_1 > 0 \\ \varepsilon_{\beta_2} = s_3 - s_4 > 0 \quad . \end{cases}$$

The interaction of elementary waves is considered in the following way :

Suppose that seven constant states U_i(i = 1,2,...,7) are connected by two vectors of three elementary waves $\beta = (\beta_1,\beta_0,\beta_2)$ and $\gamma = (\gamma_1,\gamma_0,\gamma_2)$. The solution of the Riemann problem (5.5) with the initials U_1 and U_7 in $x < 0$ and in $x > 0$ respectively has the four constant states U_1, U_8, U_9, U_7 which are connected by a vector of three elementary waves denoted by $\alpha = (\alpha_1,\alpha_0,\alpha_2)$. Our aim is to estimate α by β and γ in the above interaction denoted by $\beta + \gamma \mapsto \alpha$. There are the following basic interactions of $\beta + \gamma \mapsto \alpha$, into which the others of $\beta + \gamma \mapsto \alpha$ can be reduced :

(I) If $\beta_i, \gamma_i > 0$, $i = 1,2$, there are three cases in α.

a) $\alpha_i > 0$, $i = 1,2$ i.e., $(S_1, C_0, S_2) + (S_1, C_0, S_2) \mapsto (S_1, C_0, S_2)$, where C_0 denotes the contact discontinuity.

b) $\alpha_1 > 0$, $\alpha_2 \leqslant 0$ i.e., $(S_1, C_0, S_2) + (S_1, C_0, S_2) \mapsto (S_1, C_0, R_2)$.

c) $\alpha_1 \leqslant 0$, $\alpha_2 > 0$ i.e., $(S_1, C_0, S_2) + (S_1, C_0, S_2) \mapsto (R_1, C_0, S_2)$, which is symmetric to (b) and can be reduced to (b) for the estimate.

(II) If $\beta_1, \gamma_1 > 0$ and $\beta_2, \gamma_2 \leqslant 0$, then $\alpha_1 > 0$ and $\alpha_2 < 0$ i.e., $(S_1, C_0, R_2) + (S_1, C_0, R_2) \mapsto (S_1, C_0, R_2)$.

(II') If $\beta_1, \gamma_1 \leqslant 0$ and $\beta_2, \gamma_2 > 0$, then $\alpha_1 < 0$ and $\alpha_2 > 0$ i.e. $(R_1, C_0, S_2) + (R_1, C_0, S_2) \mapsto (R_1, C_0, S_2)$, which is symmetric to (II) and can reduced to it for the estimate.

(III) If $\beta_1, \beta_2 > 0$ and $\gamma_1, \gamma_2 \leqslant 0$, there are four cases.

a) $\alpha_1, \alpha_2 \geqslant 0$ i.e., $(S_1, C_0, S_2) + (R_1, C_0, R_2) \mapsto (S_1, C_0, S_2)$.

b) $\alpha_1 \geqslant 0$, $\alpha_2 \leqslant 0$ i.e., " $\mapsto (S_1, C_0, R_2)$.

c) $\alpha_1 \leqslant 0$, $\alpha_2 \geqslant 0$ i.e., " $\mapsto (R_1, C_0, S_2)$.

d) $\alpha_1 \leqslant 0$, $\alpha_2 \leqslant 0$ i.e., " $\mapsto (R_1, C_0, R_2)$.

(III') If $\beta_1, \beta_2 \leqslant 0$ and $\gamma_1, \gamma_2 > 0$, there are four cases which are symmetric to (III) and can be reduced to it for the estimate.

(IV) If $\beta_i, \gamma_i \leqslant 0$, $i = 1,2$, then $\alpha_i \leqslant 0$, $i = 1,2$ i.e. $(R_1, C_0, R_2) + (R_1, C_0, R_2) \mapsto (R_1, C_0, R_2)$.

Lemma 5.3 - Suppose that the interactions of elementary waves of $\beta + \gamma \mapsto \alpha$ occur in a fixed bounded region $\Omega_0 \subset \{(u,q,s) \in \mathbb{R}^3\}$. Then there exists a constant $G > 0$ such that the following estimate holds :

(I-a) There exist $0 < Q < \min(\beta_1 + \gamma_1, \beta_2 + \gamma_2)$ such that

$$
(5.27)\quad \begin{cases}
\alpha_1 = \beta_1 + \gamma_1 - Q > 0 \\
\alpha_2 = \beta_2 + \gamma_2 - Q > 0 \quad , \\
\varepsilon_{\alpha_1} \geqslant \varepsilon_{\beta_1} + \varepsilon_{\gamma_1} \text{ and } \varepsilon_{\alpha_2} \geqslant \varepsilon_{\beta_2} + \varepsilon_{\gamma_2} - GQ \text{ or} \\
\varepsilon_{\alpha_1} \geqslant \varepsilon_{\beta_1} + \varepsilon_{\gamma_1} - GQ \text{ and } \varepsilon_{\alpha_2} \geqslant \varepsilon_{\beta_2} + \varepsilon_{\gamma_2} \quad , \\
|\alpha_0| \leqslant |\beta_0| + |\gamma_0| + |\varepsilon_{\alpha_1} - \varepsilon_{\beta_1} - \varepsilon_{\gamma_1}| + |\varepsilon_{\alpha_2} - \varepsilon_{\beta_2} - \varepsilon_{\gamma_2}| \quad .
\end{cases}
$$

(I-b) There exists $0 < \beta_2 + \gamma_2 < Q < \beta_1 + \gamma_1$ such that

$$
(5.28)\quad \begin{cases}
\alpha_1 = \beta_1 + \gamma_1 - Q > 0 \\
\alpha_2 = \beta_2 + \gamma_2 - Q \leqslant 0 \\
\varepsilon_{\alpha_1} > \varepsilon_{\beta_1} + \varepsilon_{\gamma_1} \\
|\alpha_0| \leqslant |\beta_0| + |\gamma_0| + \varepsilon_{\alpha_1} - \varepsilon_{\beta_1} - \varepsilon_{\gamma_1} + \varepsilon_{\beta_2} + \varepsilon_{\gamma_2}
\end{cases}
$$

(II) There exists $0 < Q < \gamma_1$ such that

$$
(5.29)\quad \begin{cases}
\alpha_1 = \beta_1 + \gamma_1 - Q > 0 \\
\alpha_2 = -|\beta_2| - |\gamma_2| - Q < 0 \\
\varepsilon_{\alpha_1} \geqslant \varepsilon_{\beta_1} + \varepsilon_{\gamma_1} \\
|\alpha_0| \leqslant |\beta_0| + |\gamma_0| + \varepsilon_{\alpha_1} - \varepsilon_{\beta_1} - \varepsilon_{\gamma_1} \quad .
\end{cases}
$$

(III-a) There exists $\max(-\beta_1, -|\gamma_1|) \leqslant Q \leqslant \max(0, |\gamma_2| - |\gamma_1|)$ such that

$$
(5.30)\quad \begin{cases}
\alpha_1 = \beta_1 + Q \geqslant 0 \\
\alpha_2 = \beta_2 - |\gamma_2| + |\gamma_1| + Q \geqslant 0 \\
\varepsilon_{\alpha_1} \geqslant \varepsilon_{\beta_1} + \min(0, GQ) \\
\varepsilon_{\alpha_2} \geqslant \varepsilon_{\beta_2} + \min(0, G(-|\gamma_2| + |\gamma_1| + Q)) \\
|\alpha_0| \leqslant |\beta_0| + |\gamma_0| + |\varepsilon_{\beta_1} - \varepsilon_{\alpha_1}| + |\varepsilon_{\alpha_2} - \varepsilon_{\beta_2}| \quad .
\end{cases}
$$

(III-b) There exists $\max(-\beta_1, -|\gamma_1|) \leqslant Q \leqslant \max(0, |\gamma_2|-|\gamma_1|)$

$$(5.31)\quad \begin{cases} \alpha_1 = \beta_1 + Q > 0 \\ \alpha_2 = \beta_2 - |\gamma_2| + |\gamma_1| + Q \leqslant 0 \\ \varepsilon_{\alpha_1} \geqslant \varepsilon_{\beta_1} + \min(0, GQ) \\ |\alpha_0| \leqslant |\beta_0| + |\gamma_0| + |\varepsilon_{\beta_1} - \varepsilon_{\alpha_1}| + \varepsilon_{\beta_2} \end{cases}$$

(III-c) There exists $\beta_1 \leqslant Q < |\gamma_1|$ such that

$$(5.32)\quad \begin{cases} \alpha_1 = \beta_1 - Q < 0 \\ \alpha_2 = \beta_2 - |\gamma_2| + |\gamma_1| - Q \geqslant 0 \\ \varepsilon_{\alpha_2} \geqslant \varepsilon_{\beta_2} + \min(0, G(-|\gamma_2| + |\gamma_1| - Q)) \\ |\alpha_0| \leqslant |\beta_0| + |\gamma_0| + |\varepsilon_{\alpha_2} - \varepsilon_{\beta_2}| + \varepsilon_{\beta_1} \end{cases}$$

(III-d) There exists $\beta_1 \leqslant Q < |\gamma_1|$ such that

$$(5.33)\quad \begin{cases} \alpha_1 = \beta_1 - Q \leqslant 0 \\ \alpha_2 = \beta_2 - |\gamma_2| + |\gamma_1| - Q \leqslant 0 \\ |\alpha_0| \leqslant |\beta_0| + |\gamma_0| + \varepsilon_{\beta_1} + \varepsilon_{\beta_2} \end{cases}$$

(IV)

$$(5.34)\quad \begin{cases} \alpha_1 = \beta_1 + \gamma_1 \leqslant 0 \\ \alpha_2 = \beta_2 + \gamma_2 \leqslant 0 \\ |\alpha_0| \leqslant |\beta_0| + |\gamma_0| \end{cases}$$

Proof. First we remember that the shock curves (S_i-curve, $i = 1,2$) have the same figures independent of the initial point in the variables (u,q,s) by lemma 5.2, which is essential in the proof. In the case of (I-a) the existence of $Q > 0$ in the first two equalities comes from the convexity of the S_i-curve in (u,q) plane by (5.15). The fifth inequality is easy if we note that for the entropy it holds

$$\varepsilon_{\beta_1} + \beta_0 - \varepsilon_{\beta_2} + \varepsilon_{\gamma_1} + \gamma_0 - \varepsilon_{\gamma_2} = \varepsilon_{\alpha_1} + \alpha_0 - \varepsilon_{\alpha_2} \quad .$$

The third and fourth inequalities for the entropy follows from the following lemma.

<u>Lemma 5.4</u> - Let (u_2, q_2, s_2), $(u_3, q_3, s_3) \in S_1\,(u_1, q_1, s_1)$, $(u_4, q_4, s_4) \in S_1\,(u_2, q_2, s_2)$ and $u_3 = u_4$ with $q_3 > q_2$.

Then there exists a constant G such that

$$(5.35) \qquad s_2 - s_1 \geqslant s_3 - s_1 - G(q_3 - q_2)$$

and also it holds

$$(5.36) \qquad s_3 - s_1 > s_4 - s_2 + (s_2 - s_1).$$

The inequality (5.35) is an easy consequence of the mean value theorem. The second inequality is shown as follows, where we put $q_2 - q_1 = \beta_1$, $q_4 - q_2 = \gamma_1$ and $q_3 - q_1 = \alpha_1$. The representation of S_i-curve in q-variable (5.15) gives the following identities.

$$u_1 - u_2 = 2a \text{ sh } (\beta_1/2a), \quad u_2 - u_4 = 2a \text{ sh } (\gamma_1/2a)$$

$$u_1 - u_3 = 2a \text{ sh } (\alpha_1/2a).$$

Therefore if we use $u_3 = u_4$ here, we have

$$(5.37) \qquad \text{sh } (\alpha_1/2a) = \text{sh } (\beta_1/2a) + \text{sh } (\gamma_1/2a).$$

On the other hand it follows from (5.15) for the entropy

$$s_3 - s_1 - (s_4 - s_2 + (s_2 - s_1))$$

$$= R\{ - \alpha_1/a + \text{sh } (\alpha_1/a) - (- \beta_1/a + \text{sh } (\beta_1/a) - \gamma_1/a$$

$$+ \text{sh } (\gamma_1/a)\} \geqslant R[\, 2 \text{ ch}(\alpha_1/2a) \{ \text{sh } (\alpha_1/2a) -$$

$$- \text{sh } (\beta_1/2a) - \text{sh } (\gamma_1/2a)\} + (\beta_1 + \gamma_1 - \alpha_1)/a \,] > 0,$$

where lemma 5.2 and (5.37) are used.

qed. of lemma 5.4.

We return to the proof of lemma 5.3. (I-b) can be treated analogously to (II), which is proved as follows. The existence of $Q > 0$ comes again from lemma 5.2 and the fact that S_1-curve in (u,q)-plane has the gradient greater than 1 and is convex and that R_2-curve in (u,q)-plane is the half straight line with thegradient -1. The third inequality comes from lemma 5.4 and from that R_2-curve has the gradient -1. The fourth is easily obtained by the entropy equality.

$$\varepsilon_{\beta_1} + \beta_0 + \varepsilon_{\gamma_1} + \gamma_0 = \varepsilon_{\alpha_1} + \alpha_0 \quad .$$

(III-a) The existence of $Q > 0$ follows from lemma 5.2 and if we remember the gradient of the S_i- and R_i-curbe in (u,q)-plane. The third and fourth comes from lemma 5.2 and 5.4. Also we have the entropy equality $\varepsilon_{\beta_1} + \beta_0 - \varepsilon_{\beta_2} + \gamma_0 = \varepsilon_{\alpha_1} + \alpha_0 - \varepsilon_{\alpha_2}$, which give the last inequality.

(III-b) and (III-c) can be treated analogously to (III-a).

(III-d) and (IV) are easy to get.

qed. of lemma 5.3.

6. Bounds for the Approximate Solutions and the Convergence to a Weak Solution

The approximate solutions $U = U^h_\alpha$ by the Glimm's difference scheme will be estimated on the piecewise linear curves called I-curve j.

Definition 6.1 - I-Curve 0 is composed of the all line segments joining a^0_m to a^1_{m+1} and a^1_{m+1} to a^0_{m+2} for all even m. An immediade successor I-curve j_2 of I-curve j_1 is composed of the same line segments except two segments joining a^n_m to a^{n-1}_{m+1} and a^{n-1}_{m+1} to a^n_{m+2}, which are replaced by those joining a^n_m to a^{n+1}_{m+1} and a^{n+1}_{m+1} to a^n_{m+2}. Then all I-curve j are obtained by the successive procedures to take an immediate successor starting from I-curve 0 .

The bounds for the approximate solutions are obtained by means of a functional. The functional $F = F(U^h/j) = F(j)$ is defined on the approximate solutions U^h restricted on each I-curve j. It dominates the total variation of U^h on j and decreases as function of j in the partial order introduced by the immediate successor. Since U^h/j consists of various shock and rarefaction waves and contact discontinuities as seen in § 5, F is defined as a function of these elementary waves as follows :

Definition 6.2 - Let us use the notation for the strength of waves in Definition 5.3.

$$F(j) = \Sigma|\beta_1| + |\beta_2| + M_0\,(|\beta_0| - \varepsilon_{\beta_1} - \varepsilon_{\beta_2}) \qquad (6.1)$$

where the summation is over all vectors of three elementary waves $\beta = (\beta_1, \beta_0, \beta_2)$ in U^h crossing j and a constant $M_0 > 0$ will be chosen later.

Hypothesis 6.1 - The initial data $(u,p,s)(0,x)$ are bounded, have bounded total variation and

$$p_0 = \inf p(0,x) > 0 \qquad (6.2)$$

From (5.14) it is equivalent to that $(u,q,s)(0,x)$ are bounded and have bounded total variation i.e.,

$$(6.3)\quad \begin{cases} T\,V_1 \equiv T\,V(u,q)\,(0,\cdot) & < +\infty \\ T\,V_2 \equiv T\,V\,s(0,\cdot) & < +\infty \end{cases}$$

where TV means the total variation in $x \in \mathbb{R}$.

We define the intervals in q as follows :

$$(6.4)\quad \begin{cases} I_q = \{q \in \mathbb{R}\ ;\ \underline{q} - 2\,TV_1 \leqslant q \leqslant \bar{q} + 2\,TV_1\ \} & , \\ I_{2q} = \{q \in \mathbb{R}\ ;\ \underline{q} - 4\,TV_1 \leqslant q \leqslant \bar{q} + 4\,TV_1\ \} & , \end{cases}$$

where $\underline{q} = \min\{\lim_{x\to\mp\infty} q(0,x)\}$, $\bar{q} = \max\{\lim_{x\to\mp\infty} q\,(0,x)\}$.

Hereafter we excludethe case $TV_1 = 0$, which is not interesting at all, because its solution

$$U(t,x) = U(0,x) \text{ for any } t \geqslant 0.$$

Lemma 6.1 - Under the hypothesis 6.1 we choose M_0 in the functional (6.1) as

$$(6.5)\qquad M_0 = \min\{1\ ,\ 1/2\ G,\ \ TV_1/2\ TV_2\ \}$$

where G is the constant in lemma 5.3 for I_{2q}. Then we have for all U^h

$$(6.6)\qquad F(0) \leqslant 2\ TV_1 \qquad \text{for I-curve } 0$$

$$(6.7)\qquad F(j_2) \leqslant F(j_1) \quad ,$$

where I-curve j_2 is an immediate successor of I-curve j_1. As a consequence it holds

$$(6.8)\qquad \{\,q^h\,|\,j\,\} \subset I_q \qquad \text{for any I-curve } j \ ,$$

where $\{q^h|\,j\} \equiv \{\,q^h\,(t,x)\ ;\ (t,x) \in j\ \}$.

Proof. First lemmas 5.1 and 5.2 give for I-curve 0

$$TV\{q^h \mid 0\} \leq \Sigma\{|\beta_1| + |\beta_2| ; \beta \text{ on } 0\} \leq TV(u^h, q^h)(0,\cdot) = TV_1 ,$$

where Σ denotes the summation taken for all β on 0. Thus

$$\{q^h \mid 0\} \subset I_q .$$

Therefore we can estimate by lemmas 5.1 and 5.2

$$F(0) = \Sigma\{|\beta_1| + |\beta_2| + M_0 (|\beta_0| \varepsilon_{\beta_i} - \varepsilon_{\beta_2}) ; \beta \text{ on } 0\}$$

$$\leq TV(u^h, q^h)(0,\cdot) + M_0 (GTV(u^h,q^h)(0,\cdot) + TV\, s^h(0,\cdot))$$

$$\leq \frac{3}{2} TV_1 + M_0 TV_2 \leq 2\, TV_1 ,$$

where (6.5) is used.

The inequality (6.7) is proved inductively by lemma 5.3. Let

(6.9) $$F(j_1) \leq F(0) \quad \text{and} \quad \{q^h \mid j_1\} \subset I_q .$$

The difference of I-curve j_1 and its immediate successor I-curve j_2 is a diamond composed of four segments joining a^n_m, a^{n-1}_{m+1}, a^n_{m+2} and a^{n+1}_{m+1}. The waves β and γ enter in the diamond and interact there and the wave α goes out of it. All the other waves crossing j_1 and j_2 are common to both of them. If we remember each interaction in lemma 5.3, the second hypothesis of induction (6.9) gives

(6.10) $$\{q^h \mid j_2\} \subset I_{2q}.$$

Therefore lemma 5.3 with the constant G for I_{2q} applies to the interaction of β and γ to α in the diamond. Thus we treat each case of I-IV.

(I-a) $\beta + \gamma \mapsto \alpha$, where $\beta_i, \gamma_i, \alpha_i \geqslant 0$, $i = 1,2$.

$$F(j_2) - F(j_1) = \alpha_1 + \alpha_2 + M_0(|\alpha_0| - \varepsilon_{\alpha_1} - \varepsilon_{\alpha_2})$$

$$-\{ \beta_1 + \beta_2 + M_0 (|\beta_0| - \varepsilon_{\beta_1} - \varepsilon_{\beta_2}) + \gamma_1 + \gamma_2 + M_0 (|\gamma_0| - \varepsilon_{\gamma_1} - \varepsilon_{\gamma_2}) \}$$

$$\leqslant - 2Q + M_0 (| \varepsilon_{\alpha_1} - \varepsilon_{\beta_1} - \varepsilon_{\gamma_1}| + |\varepsilon_{\alpha_2} - \varepsilon_{\beta_2} - \varepsilon_{\gamma_2}| - \varepsilon_{\alpha_1} + \varepsilon_{\beta_1} + \varepsilon_{\gamma_1}$$

$$- \varepsilon_{\alpha_2} + \varepsilon_{\beta_2} + \varepsilon_{\gamma_2}) \leqslant - 2Q + 2 M_0 GQ = -2Q (1-M_0 G) < 0 ,$$

where (5.27) and (6.5) are used.

(I-b) $\beta + \gamma \mapsto \alpha$, where $\beta_i, \gamma_i \geqslant 0$, $i = 1,2$ and $\alpha_1 \geqslant 0$, $\alpha_2 < 0$.

$$F(j_2) - F(j_1) = \alpha_1 + |\alpha_2| + M_0 (| \alpha_0| - \varepsilon_{\alpha_1}) - \{\beta_1 + \beta_2 + M_0(|\beta_0| - \varepsilon_{\beta_1} - \varepsilon_{\beta_2}) +$$

$$\gamma_1 + \gamma_2 + M_0(|\gamma_0| - \varepsilon_{\gamma_1} - \varepsilon_{\gamma_2})\} \leqslant - Q + (Q - \beta_2 - \gamma_2) - \beta_2 - \gamma_2 +$$

$$+ M_0 (\varepsilon_{\alpha_1} - \varepsilon_{\beta_1} - \varepsilon_{\gamma_1} + \varepsilon_{\beta_2} + \varepsilon_{\gamma_2} - \varepsilon_{\alpha_1} + \varepsilon_{\beta_1} + \varepsilon_{\gamma_1} + \varepsilon_{\beta_2} + \varepsilon_{\gamma_2})$$

$$= - 2 (\beta_2 + \gamma_2) (1 - M_0 G) < 0.$$

(II) $\beta + \gamma \mapsto \alpha$, where $\beta_1, \gamma_1, \alpha_1 \geqslant 0$ and $\beta_2, \gamma_2, \alpha_2 \leqslant 0$.

$$F(j_2) - F(j_1) = \alpha_1 + |\alpha_2| + M_0(|\alpha_0| - \varepsilon_{\alpha_1}) -$$

$$- \{\beta_1 + |\beta_2| + M_0(|\beta_0| - \varepsilon_{\beta_1}) + \gamma_1 + |\gamma_2| + M_0(|\gamma_0| - \varepsilon_{\gamma_1})\}$$

$$\leqslant -Q + Q + |\beta_2| + |\gamma_2| - |\beta_2| - |\gamma_2| + M_0 (\varepsilon_{\alpha_1} - \varepsilon_{\beta_1} - \varepsilon_{\gamma_1} - \varepsilon_{\alpha_1} + \varepsilon_{\beta_1} + \varepsilon_{\gamma_1}) = 0.$$

(III-a-1) $\beta_1, \beta_2 \geqslant 0$, $\gamma_1, \gamma_2 \leqslant 0$ and $\alpha_1, \alpha_2 \geqslant 0$, $0 \leqslant Q < |\gamma_2| - |\gamma_1|$.

$$F(j_2) - F(j_1) = \alpha_1 + \alpha_2 + M_0 (|\alpha_0| - \varepsilon_{\alpha_1} - \varepsilon_{\alpha_2}) -$$

$-\{ \beta_1 + \beta_2 + M_0 (|\beta_0| - \varepsilon_{\beta_1} - \varepsilon_{\beta_2}) + |\gamma_1| + |\gamma_2| + M_0 |\gamma_0| \} \leqslant$

$\leqslant Q - |\gamma_1| + (|\gamma_1| - |\gamma_2| + Q)) - |\gamma_2| + M_0 (\varepsilon_{\alpha_1} - \varepsilon_{\beta_1} + |\varepsilon_{\alpha_2} - \varepsilon_{\beta_2}| -$

$- \varepsilon_{\alpha_1} + \varepsilon_{\beta_1} - \varepsilon_{\alpha_2} + \varepsilon_{\beta_2}) \leqslant -2 (|\gamma_2| - |\gamma_1| - Q) - 2|\gamma_1| + 2 M_0(|\gamma_2| - |\gamma_1| - Q)$

$\leqslant - 2(|\gamma_2| - |\gamma_1| - Q)(1 - M_0 G) - 2|\gamma_1| < 0.$

(III-a-2) $|\gamma_2| - |\gamma_1| \leqslant Q < 0.$

$F(j_2) - F(j_1) = Q - |\gamma_1| + (|\gamma_1| - |\gamma_2| + Q) - |\gamma_2| + M_0 (|\varepsilon_{\alpha_1} - \varepsilon_{\beta_1}| +$

$|\varepsilon_{\alpha_2} - \varepsilon_{\beta_2}| - \varepsilon_{\alpha_1} + \varepsilon_{\beta_1} - \varepsilon_{\alpha_2} + \varepsilon_{\beta_2}) \leqslant - 2 |Q| - 2|\gamma_2| +$

$+ 2 M_0 G |Q| = - 2 |Q| (1-M_0 G) - 2|\gamma_2| \leqslant 0 \quad .$

(III-a-3) $\max(-\beta_1, -|\gamma_1|) < Q < |\gamma_2| - |\gamma_1| \leqslant 0.$

$F(j_2) - F(j_1) = - 2 |Q| - 2|\gamma_2| + 2 M_0 (\varepsilon_{\beta_1} - \varepsilon_{\alpha_1} + \varepsilon_{\beta_2} - \varepsilon_{\alpha_2})$

$\leqslant - 2 |Q| - 2|\gamma_2| + 2 M_0 (G|Q| + G(|\gamma_2| - |\gamma_1| + |Q|))$

$= - 2(|\gamma_2| + |Q|)(1-M_0 G) - 2M_0 G(|\gamma_1| - |Q|) < 0.$

(III-b) $\beta_1, \beta_2 \geqslant 0, \gamma_1, \gamma_2 \leqslant 0$ and $\alpha_1 \geqslant 0, \alpha_2 < 0.$

This can be reduced to the case (III-a) with $\alpha_2 = 0$ and then to the case (II).

(III-c-1) $\beta_1, \beta_2 \geqslant 0, \gamma_1, \gamma_2 \leqslant 0$ and $\alpha_1 < 0, \alpha_2 \geqslant 0,$

where $\beta_1 \leqslant Q < |\gamma_1| - |\gamma_2| \quad .$

$F(j_2) - F(j_1) = |\alpha_1| + \alpha_2 + M_0 (|\alpha_0| - \varepsilon_{\alpha_2}) - \{\beta_1 + \beta_2 + M_0(|\beta_0| - \varepsilon_{\beta_1} - \varepsilon_{\beta_2}) +$

$|\gamma_1| + |\gamma_2| + M_0|\gamma_0|\} \leqslant Q - 2\,\beta_1 - |\gamma_1| + (-|\gamma_2| + |\gamma_1| - Q) - |\gamma_2| +$

$+ M_0\,(\,\varepsilon_{\alpha_2} - \varepsilon_{\beta_2} + \varepsilon_{\beta_1} - \varepsilon_{\alpha_2} + \varepsilon_{\beta_1} + \varepsilon_{\beta_2}) \leqslant - 2\,\beta_1(1-M_0G) - 2|\gamma_2| < 0\,.$

(III-c-2) $\beta_1 < |\gamma_1| - |\gamma_2| \leqslant Q < |\gamma_1|$

$F(j_2) - F(j_1) \leqslant - 2\,\beta_1 - 2|\gamma_2| + M_0(|\varepsilon_{\alpha_2} - \varepsilon_{\beta_2}| + \varepsilon_{\beta_1} - \varepsilon_{\alpha_2} + \varepsilon_{\beta_1} + \varepsilon_{\beta_2})$

$\leqslant - 2\,\beta_1(1-M_0\,G) - 2|\gamma_2| + 2M_0G(|\gamma_2| - |\gamma_1| + Q) = - 2(\,\beta_1 + |\gamma_2|)(1-M_0G) -$

$- 2\,M_0G(|\gamma_1| - Q) < 0\,.$

(III-d) $\beta_1,\ \beta_2 \geqslant 0,\ \ \gamma_1,\ \gamma_2 \leqslant 0$ and $\alpha_1 < 0,\ \ \alpha_2 < 0\,.$

This can be reduced to the case (III-c-2) with $\alpha_2 = 0$ and then to the case (IV).

(IV) $\beta_i,\ \gamma_i \leqslant 0$ and $\alpha_i \leqslant 0\,,\quad i = 1,2.$

$F(j_2) - F(j_1) = |\alpha_1| + |\alpha_2| + M_0|\alpha_0| - \{|\beta_1| + |\beta_2| + M_0\,|\beta_0| +$

$+ |\gamma_1| + |\gamma_2| + M_0|\gamma_0|\,\} \leqslant 0.$

Thus we arrive at the key estimate (6.7). At last it follows from (6.10) and (6.7) that for the same G,

$$\text{(6.11)}\qquad \begin{aligned} &TV\,\{q^h \mid j_2\} \leqslant (1 - M_0G)^{-1}\,F(j_2) \leqslant 2\,F(j_2) \\ &\leqslant 2\,F(0) \leqslant 4\,TV_1 \end{aligned}\qquad , \qquad \text{and so}$$

$$\text{(6.12)}\qquad \{q^h \mid j_2\} \subset I_q\,.$$

qed.

<u>Lemma 6.2</u> - For any $h \in (0,h_0)$ and any random sequence $\alpha = \{\alpha_n\}_{n \geqslant 1}$ the approximate solutions $U = (u^h_\alpha,\ q^h_\alpha\ ,\ s^h_\alpha)$ has the following uniform estimates with a constant $K < +\infty$ independent of h and α .

$$(6.13)\qquad \begin{aligned} TV(u,q)(t,\cdot) &\leqslant K\, TV_1 \quad ,\\ TV\, s(t,\cdot) &\leqslant K(TV_1 + TV_2) \quad , \end{aligned}$$

$$(6.14)\qquad \int_{-\infty}^{+\infty} |U(t_2,x) - U(t_1,x)|dx \leqslant K(|t_2-t_1| + 3\ell)\,(TV_1 + TV_2)$$

proof - It follows from lemma 6.1, (6.11) and (6.12) that

$$(6.15)\qquad q^h(t,x) \in I_q \qquad \text{for any } t \geqslant 0, \text{ any } x \in \mathbb{R}.$$

$$TV\, q^h(t,\cdot) = \lim_{X \to +\infty} TV\, \{q^h(t,x) \mid |x| \leqslant X\}$$

$$\leqslant (1 - M_0 G)^{-1} F(j) \leqslant 4\, TV_1 \qquad \text{for any } t \geqslant 0 \; .$$

From lemma 5.2 (5.15) and (6.15) we have

$$TV\, u^h\, (t,\cdot) \leqslant TV\, \{ C\, q^h\, (t,\cdot)\} \leqslant 4\, C\, TV_1$$

where $C = \max_{q_2, q_1 \in I_q} |2\, a\, sh\, (q_2 - q_1)/2a|$.

In the same way we have

$$TV\, s^h\, (t,\cdot) \leqslant \Sigma\, \{|\beta_0| + \varepsilon_{\beta_1} + \varepsilon_{\beta_2}\; ;\; \beta \text{ on } j\}$$

$$\leqslant \Sigma\, \{|\beta_0| + G(|\beta_1| + |\beta_2|)\; ;\; \beta \text{ on } j\}$$

$$\leqslant \frac{1}{M_0} \cdot \Sigma\, \{M_0|\beta_0| + (|\beta_1| + |\beta_2|)/2\; ;\; \beta \text{ on } j\}$$

$$\leqslant F(j)/M_0 \leqslant 2\, TV_1/M_0 + TV_2 \quad .$$

Next let $t_2 > t_1$ in (6.14) and set

$$t_0 = \max\, \{t\; ;\; t \leqslant t_1\, ,\, t = n\,\ell\} \quad \text{and}$$

$$N = \left[\frac{t_2 - t_0}{\ell}\right] + 1 \quad .$$

$U(t_2,x) = U^h_\alpha(t_2,x)$ for any x is completely determined by the data $U(t_1,y)$, $y \in [x-Nh, x+Nh]$. Therefore by lemma 6.1 the same argument for (6.13) gives

$$|U(t_2,x) - U(t_1,x)| \leqslant (K+1) TV \{U(t_0,\cdot) \mid x-Nh \leqslant y \leqslant x+Nh\} .$$

Thus we have by the integration

$$\int_{mh}^{(m+2)h} |U(t_2,x) - U(t_1,x)| \, dx \leqslant 2\,(K+1)\,h \sum_{j=-N}^{N} TV \{U(t_0,\cdot) \mid (m+j)h \leqslant y \leqslant (m+j+2)h\} ,$$

and so

$$\sum_{m=-\infty}^{+\infty} \int_{mh}^{(m+2)h} |U(t_2,x)-U(t_1,x)| \, dx \leqslant 2(K+1)h \sum_m \sum_{j=-N}^{N} TV\{U(t_0,\cdot) \mid (m+j)h \leqslant y \leqslant (m+j+2)h\}$$

$$= 2(K+1)h(2N+1) \sum_m TV \{U(t_0,\cdot) \mid mh \leqslant y \leqslant (m+2)h\}$$

$$\leqslant 2K\,(K+1)(h/\ell)(2N+1)\,\ell(TV_1 + TV_2)$$

$$\leqslant C(\,t_2 - t_1 + 3\ell)\,(TV_1 + TV_2) .$$

qed.

Now we turn to prove the convergence of the Glimm's approximate solutions to a weak solution for the Cauchy problem. Remember that the difference approximation depends on $h \in (0,h_0)$ and also on the random choice of mesh points $a^n_m = (n\ell, (m-1)h + 2h\,\alpha_n)$, $(n,m) \in Y$, where $\alpha = \{\alpha_n\}_{n \geqslant 1}$ is any sequence of equidistributed numbers in $[0,1]$. α is considered as an element of

$$A = \sum_{n=1}^{+\infty} [0,1] \qquad (6.16)$$

which is a probability space as an infinite product of the interval $[0,1]$ with the Lebesgue measure. Let us denote $U^h_\alpha = (u^h_\alpha, p^h_\alpha, s^h_\alpha)$, $v^h_\alpha = v(p^h_\alpha, s^h_\alpha)$

and $e^h_\alpha = e(p^h_\alpha, s^h_\alpha)$. Let $V = (u,v,e+u^2/2)$ and $V^h_\alpha = (u^h_\alpha, v^h_\alpha, e^h_\alpha + (u^h_\alpha)^2/2)$.

Remembering the definition of the weak solution (5.4) we consider the integral quantity for the approximate solutions :

$$(6.17)\qquad \delta(h,\alpha,\Phi) \equiv \int_0^{+\infty}\int_{-\infty}^{\infty}[u^h_\alpha \varphi_t + p^h_\alpha \varphi_x + v^h_\alpha \psi_t - u^h_\alpha \psi_x + (e^h_\alpha + (u^h_\alpha)^2/2)\chi_t + p^h_\alpha u^h_\alpha \chi_x]\,dx\,dt + \int_{t=0} u^h_\alpha \varphi + v^h_\alpha \psi + (e^h_\alpha + (u^h_\alpha)^2/2)\chi\,dx \quad,$$

where $\Phi = (\varphi, \psi, \chi)$ is any smooth function with compact support. Since U^h_α is the exact weak solution in each strip $(n-1)\ell \leq t < n\ell$, $x \in \mathbb{R}$, we can compute

$$(6.18)\qquad \delta(h,\alpha,\Phi) = -\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\Phi(n\ell,x).(V^h_\alpha(n\ell,x) - V^h_\alpha(n\ell-0,x))\,dx \equiv -\sum_{n\geq 1}\delta_n(h,\alpha,\Phi) \quad.$$

Here $\delta(h,\alpha,\Phi)$ and $\delta_n(h,\alpha,\Phi)$, $n \geq 1$, are functions of $\alpha \in A$.

Lemma 6.3 - There is a null set $N \subset A$ and a sequence $h_j \to 0$ such that for any $\alpha \in A \sim N$ and for any test function Φ, we have

$$(6.19)\qquad \delta(h_j,\alpha,\Phi) \to 0 \quad.$$

The lemma is given in Glimm (1965) and Glimm-Lax (1970) and is essential to the convergence of the Glimm's difference scheme, which is valid in general under the uniform estimate on the total variation of the approximate solutions. First we have for any bounded continuous function Φ

$$(6.20)\qquad ||\delta_n(h,\cdot,\Phi)||_\infty \leq Kh\,||\Phi||_\infty (TV_1 + TV_2) \quad.$$

This follows from (6.13) and from the inequality

$$\left| \Sigma \int_{(m-1)h}^{(m+1)h} \Phi(n\ell,x) \,.\, (V_\alpha^h(n\ell,x) - V_\alpha^h(n\ell-0,x))\, dx \right|$$

$$\leq 2h\,||\Phi||_\infty TV\{ V^h_\alpha(n\ell-0,\cdot)\} \;.$$

Now suppose that Φ has compact support and piecewise constant on each segment $\{n\ell\} \times [(m-1)h,(m+1)h]$ for $(n,m) \in Y$ and let $h = 2^{-j}$, $j = 1,2...$ In this case we have the following

$$(6.21) \qquad \delta_n(h,..,\Phi) \perp \delta_k(h,..,\Phi) \qquad n \neq k$$

$$(6.22) \qquad ||\delta\ (h,..,\Phi)||^2 \to 0 \qquad \text{as} \quad h = 2^{-j} \to 0 \;,$$

where the orthogonality in (6.21) is with respect to $L^2(A)$ and the norm in (6.22) is that of $L^2(A)$. Let $k < n$ and let $\hat{A}$, $d\hat{\alpha}$ be the measure space product with a factor corresponding to n omitted. Put

$$\Delta V(n\ell,x) = V^h_\alpha(n\ell,x) - V^h_\alpha(n\ell-0,x) \quad .$$

The inner product of δ_n and δ_k is a sum of terms of the form

$$(6.23) \qquad \int_{\hat{A}} \int_0^1 \left(\int_{(m-1)h}^{(m+1)h} \Phi(n\ell,x).\ \Delta V\ (n\ell,x)\ dx \right).$$

$$\cdot \left(\int_{-\infty}^{\infty} \Phi(k\ell,x).\ \Delta V(k\ell,x)\ dx \right) d\,\alpha_n\ d\alpha \quad .$$

Since $C = \int_{-\infty}^{\infty} \Phi(k\ell,x).\ \Delta V(k\ell,x)\ dx$ is independent of α_n and Φ is constant on

$\{n\ell\} \times [(m-1)_h,(m+1)_h]$, (6.23) is equal to

$$\int_{\hat{A}} C\ \Phi(n\ell,mh) \left(\int_0^1 d\alpha_n \int_{(m-1)h}^{(m+1)h} \Delta V(n\ell,x)dx \right) d\hat{\alpha}$$

$$= \int_{\hat{A}} C\ \Phi(n\ell,mh) \left[\frac{1}{2h} \int_0^{2h} \left\{ \int_{(m-1)h}^{(m+1)h} (V(n\ell-0,(m-1)h+a_n) - \right.\right.$$

$$\left.\left. - V(n\ell-0,x))\ dx \right\} d\,a_n \right] d\,\hat{\alpha} = 0 \quad ,$$

where $a_n = 2h\ \alpha_n$.

Also if we use (6.20) and note that Φ has compact support, we have

$$\|\delta(h,.,\Phi)\|^2 = \sum_n \|\delta_n\ (h,.,\Phi)\|^2 \leqslant \sum_n \|\delta_n\ (h,.,\Phi)\|_\infty^2 = O(h).$$

In the same way we obtain for Φ with compact support

(6.24) $\quad \|\delta(h,.,\Phi)\|_\infty \leqslant \text{const.}\ \|\Phi\|_\infty$.

Thus for each piecewise constant Φ with compact support there is a subsequence $h_j \to 0$ such that $\delta(h_j,.,\Phi) \to 0$ a.e. . By the diagonal process we can achieve this for a dense set $\{\Phi_i\}_{i=1}^{\infty}$. Let N be a null subset of A such that for $i = 1,2,...$

$$\delta(h_j,.,\Phi_i) \to 0 \quad \text{on} \quad A \sim N \quad \text{as} \quad h_j \to 0 \quad .$$

We can apply (6.24) with Φ replaced by $\Phi - \Phi_i$ and conclude that as $j \to +\infty$

(6.25) $\quad \delta(h_j,.,\Phi) \to 0 \quad \text{on} \quad A \sim N \quad \text{for any} \quad \Phi$.

qed. of lemma 6.3.

Let $U_j = U_\alpha^{h_j}$ for any $\alpha \epsilon A \sim N$. By lemma 6.2 U_j is uniformly bounded and has bounded total variation on horizontal lines uniformly in j . By Helly's theorem a subsequence of U_j converges in L^1 on bounded intervals of any given horizontal line. By the diagonal process we can achieve the same result for the countable number of horizontal lines at rational times $t = k/n$. For an arbitrary t we have from (6.14)

$$\int_{|x|\leqslant M} |U_j(t,x) - U_i(t,x)|\,dx \leqslant \int_{|x|\leqslant M} |U_j(t,x)-U_j(k/n,x)|\,dx$$

$$+ \int_{|x|\leqslant M} |U_j(k/n,x)-U_i(k/n,x)|\,dx + \int_{|x|\leqslant M} |U_i(k/n,x)-U_i(t,x)|\,dx$$

$$\leqslant C(|t-k/n| + h_i+h_j) + \int_{|x|\leqslant M} |U_j(k/n,x)-U_i(k/n,x)|\,dx \quad .$$

Thus a subsequence U_{j_k} converges (uniformly for bounded t) to $U=(u,p,s)$ on the intervals $|x| < {}^{\forall}M$ on $t = {}^{\forall}t_0$. Therefore we have

(6.26) $\lim_{k\to+\infty} U_{j_k} = U = (u,p,s)$ a.e. and boundedly,

and the same is true for $V = (u,v,e+u^2/2)$ and pu. Also the limit function is a weak solution of the Cauchy problem (5.1) with (5.6). In fact

$$\int_0^\infty \int_{-\infty}^\infty (u\,\phi_t + p\,\phi_x + v\,\psi_t - u\,\psi_x + (e+u^2/2)\,\chi_t + pu\,\chi_x)\,dx\,dt$$

$$+ \int_{t=0} u\,\phi + v\psi + (e+u^2/2)\chi\,dx$$

$$= \lim_{h_{j_k}\to 0} \{\delta(h_{j_k},\alpha,\Phi) + \int_{t=0} (u-u_{j_k})\,\phi + (v-v_{j_k})\psi +$$

$$+ (e + u^2/2 - e_{j_k} - u_{j_k}{}^2/2)\chi\,dx\} = 0 .$$

Hence we arrive at the following existence theorem in the case $\gamma = 1$, when the assumption (6.27) is trivial.

Theorem 6.1.- Under the hypothesis 6.1 there exist two constants $\varepsilon > 0$ and $K < +\infty$ such that for any adiabatic constant $\gamma \in [1,5/3]$ satisfying

(6.27) $(\gamma-1)\ TV_1\ (1 + TV_2) < \varepsilon$

the Cauchy problem (5.1) with (5.2) has a global weak solution (u,p,s) which has the properties :

(6.28)
$|(u,p,s)\ (t,x)| < K$ in $t \geq 0,\ x \in \mathbb{R}$
$0 < \underline{p} \leq p\ (t,x)$ for a constant $\underline{p}$.

(6.29)
$TV\ (u,p)\ (t,.) < KTV_1$
$TV\ s(t,.) < K(TV_1 + TV_2)$,

where ε, K are independent of $\gamma \in [1,5/3]$.

The theorem is due to Liu (preprint) and in the case $1 < \gamma \leq [5/3]$ it is proved by a kind of perturbation from $\gamma = 1$ under the condition (6.27), which is the same idea as Nishida and Smoller (1973) for the isentropic model equation.

Remark : The above presentation of the weak solution to the hyperbolic conservation laws for the polytropic gas motion is an introduction to it and we have to consult for the full theory and problems on it the works in the references, which are not complete though. Especially in one space-dimension we refer to

(i) Lax (1957), Glimm (1965) and Kuznecov-Tupčiev (1975) for the existence of weak solutions to the general system of n equations,

(ii) Oleinik (1957), Lax (1971) and Dafermos (1973) for the entropy condition and the uniqueness question,

(iii) Guel'fand (1959), Foy (1964), Kruzhkov (1970) and Conley-Smoller (1972) for the relation to the system with the viscosity,

(iv) Lax (1957), Glimm-Lax (1970), DiPerna (1975 and preprint) and Liu (preprints) for the asymptotic behaviors of weak solutions as $t \to +\infty$,

(v) Vol'pert (1967) and Di Perna (1976) for the structure of the weak solutions.

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CHAPTER II - QUASILINEAR WAVE EQUATIONS WITH THE DISSIPATION

1. Introduction

We consider the initial value problem for the second order quasilinear wave equations with the first order dissipation :

(1.1) $$y_{tt} + \alpha y_t = \sigma(y, y_t, y_x)_x \qquad \text{in} \qquad t \geqslant 0,\ x \in \mathbb{R}$$

with the initial data

(1.2) $$y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x) \qquad \text{in} \qquad x \in \mathbb{R}\ .$$

Here α is a positive constant and the equation (1.1) is related to a theory for the heat conduction with the finite speed of propagation. cf. Cattaneo (1958), Vernotte (1958), Lebon and Lambermont (1976), and also Gurtin and Pipkin (1968), MacCamy and Wong (1972), MacCamy (preprint).

When $\alpha = 0$ and $\sigma_y \equiv 0$, the equation (1.1) is a nonlinear wave equation in the conservation form. Cf. example 1.3, Chapter 1. For the model of the heat conduction the equation (1.1) has the lower order term αy_t with not small $\alpha > 0$. In this case Rabinowitz (1969) showed that there exists the time periodic smooth solution (so global in time) for the initial boundary value problem (1.1) with the forcing term which is a time periodic function. In § 2 the simplest case (1.1) with $\sigma = \sigma(y_x)$ is investigated on the global smooth solutions to the Cauchy problem with the initial data (1.2) which is small relative to α and also on the development of singularities in general for not small initial data by the method of § 2, Chapter 1. cf. Nishida (1975). Its application to the integro-differential equation of the heat flow in the materials with memory, which is written in the form

$$u_t(t,x) = \int_0^t a(t-s)\ \sigma(u_x(s,x))_x\, ds + f(t) \qquad ,$$

is considered by MacCamy (1975) on the existence of global smooth solutions and on the asymptotic decay of solutions. In § 3 , the general case (1.1) with $\sigma = \sigma(y, y_t, y_x)$ will be considered on the existence and decay of the global smooth solutions for the initial (-boundary) value problem with the small initial data by the L^2-energy method, which is applicable to the n-space dimensional case as shown by Matsumura (preprint).

2. The Global Smooth Solution

The quasilinear wave equation with the dissipation is considered in the simplest case :

(2.1) $$y_{tt} + 2\alpha\ y_t = \sigma(y_x)_x \qquad \text{in} \quad t \geqslant 0 ,\ x \in \mathbb{R} .$$

Hypothesis 2.1

α is a positive constant.

The equation (2.1) is hyperbolic in a neighbourhood of $y_x = 0$, i.e., for a constant $R > 0$

(2.2) $$d\sigma\ (v)/d\ v > 0 \qquad \text{in} \quad |v| < R \quad ,$$
$$\sigma\ (v) \in \mathscr{C}^2\ (\ |v| < R).$$

The initial data are given by

(2.3) $$y(0,x) = y_1\ (x),\ \ y_t\ (0,x) = y_1\ (x) \qquad \text{in} \quad x \in \mathbb{R} .$$

First we transform the equation (2.1) by

(2.4) $$y_t = u \quad , \qquad y_x = v$$

into the system of two equations

(2.5) $$\begin{cases} v_t - u_x = 0 \\ u_t - \sigma(v)_x + 2\ \alpha\ u\ = 0 \end{cases} .$$

The principal part of the system (2.5) is the same as the system of the nonlinear wave equation (example 1.3, Chapter 1). Therefore the characteristics and Riemann invariants are given by the followings :

(2.6) $$\lambda = -\sqrt{\sigma'\ (v)} \quad , \qquad z = \ \phi\ (v) - u$$
$$\mu = \sqrt{\sigma'\ (v)} \quad , \qquad w = -\ \phi\ (v) - u \ ,$$

where $$\phi(v) = \int_0^v \sqrt{\sigma'\ (v)}\ dv \quad .$$

By the hypothesis 2.1 the system (2.5) is strictly hyperbolic in $\Omega = \{(u,v)\ ;\ u \in \mathbb{R},\ |v| < R\}$, and the Riemann invariants give a one to one smooth mapping from Ω onto

$$\Omega_1 = \{(w,z)\ ;\ \ 2\,\phi(-R) < z-w < 2\,\phi(R)\}$$

The Riemann invariants w,z diagonalize the principal part of the system (2.5) as

(2.7) $$w_t + \lambda\, w_x = -\,\alpha\,(w + z)$$

(2.8) $$z_t + \mu\, z_x = -\,\alpha\,(w + z) \quad ,$$

where

$$\lambda = \lambda\,(z - w) < 0 < \mu = \mu\,(z - w) \quad .$$

The initial data (2.3) reduces by (2.4) and (2.6) to

(2.9) $$\begin{cases} w(0,x) = w\,(y_1(x),\, d\, y_0(x)/dx) = w_0(x)\ , \\ z(0,x) = z\,(y_1(x),\, d\, y_0(x)/dx) = z_0(x) \end{cases}$$

<u>Hypothesis 2.2</u>

$w_0(x),\ z_0(x) \in \mathscr{C}^1(\mathbf{R})$ and $(w_0(x),\ z_0(x)) \in \Omega_1$ for any $x \in \mathbf{R}$ as follows,

(2.10) $$W_0+Z_0 < \min\,\{\,2\,\phi(R),\ -\,2\,\phi(-R)\,\}\ ,$$
where $W_0 = \sup\,|w_0(x)|$, $Z_0 = \sup\,|z_0(x)|$,

(2.11) $$W_1 = \sup\,|d\, w_0(x)/dx|\ ,\ Z_1 = \sup\,|d\, z_0(x)/dx| < +\infty \quad .$$

<u>Lemma 2.1</u> - Under the hypotheses 2.1 and 2.2 the Cauchy problem (2.7) ~(2.9) has the a priori estimate for the $\mathscr{C}^1$-solution :

(2.12) $$\sup_x |w\,(t,x)| + \sup_x |z\,(t,x)| < W_0+Z_0$$

for $t \geqslant 0$ as long as the $\mathscr{C}^1$-solution exists. Therefore the solution remains in the region Ω_1.

<u>Proof</u>.- The characteristic equation of (2.7) is given by $dx_1/dt = \lambda$, $dw/dt = -\,\alpha(w+z)$, i.e.,

$$(2.13)\quad \begin{cases} x_1(t,\beta) = \beta + \int_0^t \lambda(w(s,x_1(s,\beta)),\ z(s,x_1(s,\beta)))\ ds \\ w(t,\beta) \equiv w(t,x_1(t,\beta)) = e^{-\alpha t}\ (w_0(\beta) - \alpha \int_0^t z(s,x_1(s,\beta))\ e^{\alpha s}\ ds), \end{cases}$$

in $\beta \in \mathbb{R}$.

In the same way from (2.8) we have

$$(2.14)\quad \begin{cases} x_2(t,\gamma) = \gamma + \int_0^t \mu(w(s,x_2(s,\gamma)),\ z(s,\ x_2(s,\gamma)))\ ds \quad , \\ z(t,\gamma) \equiv z(t,x_2(t,\gamma)) = e^{-\alpha t}\ (z_0(\gamma) - \alpha \int_0^t w(s,x_2(s,\gamma))\ e^{\alpha s}\ ds)\ , \end{cases}$$

in $\gamma \in \mathbb{R}$.

Let $W(t) = \sup_x |\, w(t,x)\,|$, $Z(t) = \sup_x |\, z(t,x)\,|$.
By (2.13) and (2.14) we have

$$(W(t) + Z(t))e^{\alpha t} \leq W_0 + Z_0 + \alpha \int_0^t (W(s) + Z(s))\ e^{\alpha s}\ ds$$

and so we obtain

$$W(t) + Z(t) \leq W_0 + Z_0 < \min \{ 2\ \phi(R),\ -2\ \phi(-R)\} \quad .$$

qed.

<u>Lemma 2.2</u>- Under the same hypotheses as in lemma 2.1 there exist a $\varepsilon = \varepsilon(\alpha,\sigma) > 0$ and $C = C\ (\alpha,\sigma) < +\infty$ such that if $W_0 + Z_0 + W_1 + Z_1 < \varepsilon$, then any $\mathcal{C}^1$-solution to the Cauchy problem (2.7) $\sim$ (2.9) has the a priori estimate :

$$(2.15)\quad \begin{cases} |w_x\ (t,x)| < C\ W_1 + C(W_0 + Z_0) \\ |z_x\ (t,x)| < C\ Z_1 + C(W_0 + Z_0) \quad . \end{cases}$$

<u>Proof</u> - Since by (2.13)

$$w_x\ (t,\ x_1\ (t,\beta)) = \frac{\partial\ w(t,\beta)\ /\partial\beta}{\partial\ x_1(t,\beta)/\partial\beta}$$

is differentiable in t for fixed β , w_x can be differentiated along the first characteristic curve (2.13), i.e.,

$$(w_x)' \equiv \left(\frac{\partial}{\partial t} + \lambda \frac{\partial}{\partial x}\right) w_x$$

(2.16)

$$= - \alpha \, w_x - \alpha \, z_x - \lambda_w \, w_x^{\,2} - \lambda_z \, z_x \, w_x \quad .$$

By (2.8) we have

(2.17) $$z_x = \frac{z_t + \lambda \, z_x + \alpha(w+z)}{\lambda - \mu} = \frac{z' + \alpha(w+z)}{2\lambda} \quad .$$

Define

$$h = \frac{1}{2} \log \left(- \lambda \, (z-w)\right) \quad .$$

Its differentiation along the characteristic curve gives

(2.18) $$h' = \frac{\lambda_w \, w' + \lambda_z \, z'}{2\lambda} = \frac{-\alpha \, \lambda_w}{2\lambda}(w+z) + \frac{\lambda_z}{2\lambda} z' \quad .$$

The substitution (2.17) and (2.18) into (2.16) gives

$$(w_x)' + (\alpha + \lambda_w \, w_x + h') \, w_x = - \frac{\alpha}{2\lambda} z' - \frac{\alpha^2 \, (w+z)}{2\lambda} \quad , \text{ namely}$$

(2.19) $$(e^h \, w_x)' + (\alpha + \lambda_w \, w_x) \, e^h \, w_x = - \frac{\alpha \, e^h}{2\lambda} \, z' - \frac{\alpha^2 \, e^h \, (w+z)}{2\lambda}$$

Let

$$g = - \alpha \int_0^{z-w} \frac{e^{h(\xi)}}{2\lambda \, (\xi)} \, d\,\xi \quad .$$

Then by (2.7) we compute

$$g' = \frac{\alpha e^h}{2\lambda} \, w' - \frac{\alpha e^h}{2\lambda} z' = - \frac{\alpha^2 \, e^h}{2\lambda} \, (w+z) - \frac{\alpha \, e^h}{2\lambda} z' \quad .$$

Hence (2.19) can be rewritten

(2.20) $$(e^h \, w_x)' + (\alpha + \lambda_w \, w_x) \, e^h \, w_x = g' \quad .$$

Put

(2.21) $$k(s) \equiv \alpha + \lambda_w \, w_x$$

along the first characteristic curve $x = x_1(t,\beta)$. Suppose that $|w_x|$ is so small that

(2.22) $$k(s) \geqslant \alpha/2 \qquad ,$$

which is verified later. Then the integration (2.20) along the first characteristics gives after integration by parts

$$(e^h w_x)(t,x_1(t,\beta)) = ((e^h w_x)(0,\beta) - g(0,\beta))\, e^{-\int_0^t k(s)ds} +$$
$$+ g(t,x_1(t,\beta)) - \int_0^t g(s,x_1(s,\beta))\, k(s)\, e^{-\int_s^t k(\tau)\,d\tau} ds$$

(2.23) $$|(e^h w_x)(t,x)| \leqslant |e^{h(0,\beta)} w_x(0,\beta)| + |g(0,\beta)| + $$
$$+ |g(t,x)| + \sup_{0\leqslant s\leqslant t} |g(s,x_1(s,\beta))|$$

Since h and g are defined in Ω_1 and are bounded in $\Omega_0 = \{(w,z) \in \Omega_1\, ,\ |w| + |z| \leqslant W_0 + Z_0\}$, it follows from lemma 2.1 that

(2.24) $$\begin{cases} |h(t,x)| = |h(w(t,x),\ z(t,x))| \leqslant C \\ |g(t,x)| = |g(w(t,x),\ z(t,x))| \leqslant \alpha C\,(W_0 + Z_0) \end{cases} ,$$

where C depends only on σ.

Therefore by (2.23) and (2.24) we arrive at the estimate

(2.25) $$|w_x(t,x)| \leqslant C\, W_1 + \alpha\, C(W_0 + Z_0) \qquad .$$

In the same way we obtain

(2.26) $$|z_x(t,x)| \leqslant C\, Z_1 + \alpha\, C(W_0 + Z_0) \qquad .$$

Now in order to verify (2.22) we restrict W_0, Z_0, W_1 and Z_1 so small that for any $t > 0$, $x \in \mathbb{R}$

$$|\lambda_w w_x| \quad , \quad |\mu_z z_x| \leqslant \alpha/2 \qquad .$$

In fact since $|\lambda_w|$, $|\mu_z| \leqslant C$ in Ω_0 , we may take by (2.25) and (2.26)

$$W_0 + Z_0 \leqslant 1/4\, C^2 \quad ,$$
$$W_1 \, , Z_1 \leqslant \alpha/4\, C^2 \quad .$$

qed.

Theorem 2.1 - Under the hypotheses 2.1 and 2.2 there exists $\varepsilon > 0$ such that if

$$W_0 + Z_0 + W_1 + Z_1 < \varepsilon \quad ,$$

then the Cauchy problem (2.1) (2.3) has the unique smooth solution in the large in time.

The a priori estimates in lemmas 2.1 and 2.2 and the well known local existence theorem (cf. Douglis (1952) and Hartman-Wintner (1952) for (2.7), (2.8), (2.9) give the theorem.

Remark 2.1 - When the more smoothness of the initial data is assumed in addition, the uniform (in t) bounds for the higher derivatives of solutions are obtained in the same way.

Remark 2.2 - If we do not suppose that W_1 and Z_1 are small, the singularities in the first derivatives w_x, z_x develop in general in finite time. In fact for example we can see it in the genuinely nonlinear case, i.e.,

$$\lambda_w = \mu_z \geqslant \delta = \text{constant} > 0 \qquad \text{in} \quad \Omega_1 \quad . \tag{2.27}$$

Under the hypotheses 2.1 and 2.2 the $\mathscr{C}^1$-solution satisfies (2.12) and (2.24) as long as it is in $\mathscr{C}^1$-class :

$$\begin{cases} \sup\limits_x |w(t,x)| + \sup\limits_x |z(t,x)| \leqslant W_0 + Z_0 \\ |h\ (t,x)| \leqslant C \\ |g\ (t,x)| \leqslant \alpha C\ (W_0 + Z_0) \equiv C_0 \end{cases} \quad . \tag{2.28}$$

Then we suppose that for the derivative of the initials $(e^h\, w_x)(0,\beta) < -(2\, C_0 + \alpha e^C/\delta)$ for some $\beta \in \mathbf{R}$.

The integration of (2.20) along the first characteristics gives

$$(e^h w_x(t,x_1(t,\beta)) = (e^h w_x)(0,\beta) + g(t,x_1(t,\beta)) - g(0,\beta)$$

$$- \int_0^t (\alpha + \lambda_w e^{-h} e^h w_x) e^h w_x \, ds$$

$$\leqslant (e^h w_x)(0,\beta) + 2 C_0 - \int_0^t (\alpha + \delta e^{-C} e^h w_x) e^h w_x \, ds$$

Therefore if we compare this integral inequality for $e^h w_x$ along the first characteristics with the ordinary differential equation

$$dW(t) / dt = - (\alpha + \delta e^{-C} W(t)) W(t)$$

$$W(0) = (e^h w_x) (0,\beta) + 2 C_0 \quad ,$$

we have as $t \to t_0 < + \infty$

$$(e^h w_x)(t, x_1 (t,\beta)) \leqslant W(t) \to - \infty .$$

At last we may note that the weak solution for (2.1) and (2.3) should be considered in the large in time for not small initial data, while the equation (2.5) is not the conservation form.

3. The Energy Method and the Decay of Solutions as $t \to +\infty$.

Although the L^2-energy method applies to the n space-dimensional case, we restrict ourselves here for simplicity to consider the Cauchy problem for the equation (3.1) in the one space-dimension and to describe briefly the idea to get the global smooth solution by the method.

(3.1) $$y_{tt} + \alpha y_t = \sigma(y,y_t,y_x)_x \qquad \text{in } t \geqslant 0, x \in R \quad ,$$

where

(3.2) $$y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x) \qquad \text{in } x \in R \quad .$$

Hypothesis 3.1 - $\alpha = 1$ without loss of generality. $\sigma \in \mathcal{C}^4(\Omega)$ and $\sigma(0,0,0) = 0$, where $\Omega = \{ (y,y_t,y_x) \; ; \; |y| \; , \; |y_t| \; , \; |y_x| < R \}$. Set $\sigma(y,y_t,y_x)_x = a(y,y_t,y_x)y_{xx} + b(y,y_t,y_x)y_{xt} + c(y,y_t\, y_x)y_x$ and suppose the hyperbolicity

(3.3) $\quad a(y,y_t,y_x) \geq a_0 = \text{constant} > 0 \quad \text{in } \Omega .$

Also we make a restriction

(3.4) $$c(0,0,0) = 0 \qquad .$$

We seek the smooth solution $y(t,x) \in \mathcal{C}^2\,(t \geq 0,\; x \in \mathbb{R})$ and

(3.5) $$\|y(t)\|_2 \equiv |y(t,\cdot)|_{\mathcal{C}^2} + |y_t(t,\cdot)|_{\mathcal{C}^1} + |y_{tt}(t,\cdot)|_{\mathcal{C}^0} < +\infty \quad \text{for } t \geq 0 \; ,$$

where

$$|f(\cdot)|_{\mathcal{C}^k} \equiv \sum_{0 \leq j \leq k} \sup | \, d^j f(x)/d\, x^j | \qquad .$$

By a direct calculation or by Sobolev's lemma in the one dimension, we have

(3.6) $$|f(\cdot)|_{\mathcal{C}^k} \leq C\, \|f(\cdot)\|_{H^{k+1}} \qquad ,$$

where H^m is the Sobolev space of L^2-functions in $x \in \mathbb{R}$ with their m-th derivative and

$$\|f(\cdot)\|^2_{H^m} \equiv \sum_{0 \leq j \leq m} \| \, d^j f(\cdot)/d\, x^j \|^2_{L^2} \qquad .$$

Thus using the L^2-energy method (Courant-Friedrichs-Lewy (1928)) we are going to solve the Cauchy problem (3.1) (3.2) in the Banach space X_3, which is defined by

(3.7) $$X_m = \{ y(t) \in L^\infty\,(t;H^m),\; y_t(t) \in L^\infty\,(t;H^{m-1}),$$
$$y_{tt}\,(t) \in L^\infty\,(t;H^{m-2}) \; , \quad 0 \leq t \leq {}^\forall T \} \qquad ,$$

where $L^\infty\,(t;H^k)$ is the space of functions which is bounded in $t \in [0,T]$ with the values in H^k.

Hypothesis 3.2 -

$$(3.8)\qquad \begin{cases} (y_0(x),\ y_1(x),\ y'_0(x)) \in \Omega & \text{for any} \quad x \in \mathbb{R}, \\ y_0\ (x) \in\ H^3\ (\mathbb{R}), \quad y_1\ (x) \in\ H^2\ (\mathbb{R}) & . \end{cases}$$

First we note that the classical local existence theorem gives the solution for the Cauchy problem of the quasilinear wave equation (3.1) in the space X_m ($\forall m \geq 3$) locally in time. Cf. Schauder (1935), Sobolev (1961), Dionne (1962) and so on. In order to get the global smooth solution in $t > 0$ we only need the a priori estimate in the norm (3.5), for which the a priori estimate in the norm of X_3 is sufficient by (3.6) i.e.,

$$(3.9)\qquad |||y\ (t)|||_3^{\ 2} \equiv \|\,y(t)\,\|^2_{H^3} + \|y_t(t)\ \|^2_{H^2} + \|y_{tt}(t)\ \|^2_{H^1} \ < \ +\infty \quad \text{for } t \geq\ 0,$$

It is easy to see that the norm (3.9) is equivalent to

$$(3.10)\qquad E(t) = \sum_{j=0}^{3} E_j(t)$$

for the solution belonging to Ω in each (t,x), where

$$(3.11)\qquad \begin{cases} E_0(t) = \frac{1}{2} \int (y^2/2 + y\ y_t)\ dx \\ E_1(t) = \frac{1}{2} \int (y_t^{\ 2} + a\ y_x^{\ 2})\ dx \\ E_2(t) = \frac{1}{2} \int (y_{tt}^{\ 2} + (1 + a)\ y_{tx}^{\ 2} + a\ y_{xx}^{\ 2})\ dx \\ E_3(t) = \frac{1}{2} \int (y^2_{ttt} + (1{+}a)\ y^2_{ttx} + (1{+}a)\ y^2_{txx} + a\ y^2_{xxx}\ )\ dx & . \end{cases}$$

Hence by (3.6) we note for $y(t,x)$ with $(y,\ y_t,\ y_x) \in \Omega$

$$(3.12)\qquad \|y(t)\ \|_2^{\ 2} \leq\ C|||y(t)|||_3^{\ 2}\ \leq\ C^2\ E(t) \qquad .$$

Lemma 3.1 - Under the hypotheses 3.1 and 3.2 there exists a $\varepsilon = \varepsilon\ (R,\sigma) > 0$ such that if the solution $y(t) \in X_3$ to the Cauchy problem (3.1) (3.2) is small as

$$(3.13)\qquad \|y(t)\ \|_2 \ <\ \varepsilon \qquad \text{in} \quad 0 \leq t \leq T \quad ,$$

then it has the a priori estimate

(3.14) $\qquad E(t) \leq C\,E(0) \qquad$ in $\quad 0 \leq t \leq T,$

where $C \geq 1$ depends only on σ and R.

Proof : First we assume that the solution $y(t)$ belongs to the space X_4 with $y_0(x) \in H^4$ and $y_1(x) \in H^3$. Multiply the equation (3.1) by y and y_t respectively and integrate them in $t \in [s,t]$, $x \in \mathbb{R}$. After the integration by parts we have

$$E_0(t) - E_0(s) + \frac{1}{2} \int_s^t \int \sigma y_x \, dx\, dt = \frac{1}{2} \int_s^t \int y_t^2 \, dx\, dt,$$

$$E_1(t) - E_1(s) + \int_s^t \int y_t^2 \, dx\, dt =$$

$$= \int_s^t \int (a_t y_x^2 /2 - a_x y_x y_t - b_x y_t^2 /2 + c\, y_x y_t) \, dx\, dt \quad .$$

We compute these by (3.3) as follows :

$$E_1(t) + E_0(t) + \frac{1}{4} \int_s^t \int (y_t^2 + \sigma y_x) \, dx\, dt \quad \leq$$

$$\leq E_1(s) + E_0(s) - \frac{1}{2} \int_s^t \int (\frac{1}{2} - |a_x| - |b_x| - |c|)\, y_t^2 +$$

$$+ \frac{\sigma}{2} y_x - (|a_t| + |a_x| + |c|)\, y_x^2 \, dx\, dt$$

$$\leq E_1(s) + E_0(s) - \frac{1}{2} \int_s^t \int (\frac{1}{2} - \delta)(y_t^2 + \sigma y_x) \, dx\, dt \;,$$

where by (3.3) and (3.4)

(3.15) $\qquad \delta = \delta(\| y(t)\|_2, \ \sup |c| \ ) = \delta\, (\|y(t)\|_2) \quad .$

Here there exists a $\varepsilon > 0$ such that if (3.13) is true, then

(3.16) $\qquad \delta \leq 1/2 \qquad$ in $\quad 0 \leq t \leq T.$

Therefore under (3.13) we have in $0 \leq t \leq T$

(3.17) $$E_1(t) + E_0(t) + \frac{1}{4}\int_s^t \int y_t^2 + \sigma\, y_x \, dx\, dt \leq E_1(s) + E_0(s) ,$$

and also

(3.18) $$E_1(t)t \leq \int_0^t E_1(s)\, ds \leq C \int_0^t \int y_t^2 + \sigma\, y_x \; dx\, dt$$

$$\leq C\, (E_1(0) + E_0\,(0)).$$

In the same way we obtain the estimate for $E_2(t)$.

$$E_2(t) - E_2(s) + \frac{1}{2}\int_s^t \int y_{tt}^2 + y_{tx}^2\, dx\, dt \leq \frac{1}{2}\int_s^t \int |c_x|\, (y_t^2 + y_x^2)\, dx\, dt$$

$$- \frac{1}{2}\int_s^t \int (1-|a_t| - |b_x| - |c_x| - |c|)\, (y_{tt}^2 + y_{tx}^2) - |a_t|\, y_{xx}^2\, dx\, dt$$

$$\leq \frac{\delta}{2}\int_s^t \int y_t^2 + y_x^2\, dx\, dt - \frac{1}{2}\int_s^t \int (1-\delta)\, (y_{tt}^2 + y_{tx}^2)\, dx\, dt,$$

where δ is the same as (3.15).

Therefore under the assumptions (3.13), (3.16), we have

$$(3.19) \qquad E_2(t) + \frac{1}{2} \int_s^t \int y_{tt}^2 + y_{tx}^2 \, dx \, dt \leqslant E_2(s) + C(E_1(s) + E_0(s)).$$

Furthermore for $E_3(t)$ in the same way we have

$$E_3(t) - E_3(s) + \frac{1}{2} \int_s^t \int y_{ttt}^2 + y_{ttx}^2 + y_{txx}^2 \quad dx \, dt$$

$$\leqslant - \frac{1}{2} \int_s^t \int (1 - \delta) \, (y_{ttt}^2 + y_{ttx}^2 + y_{txx}^2) \, dx \, dt + C(E_2(s) + E_1(s) + E_0(s)),$$

where δ is the same as (3.15).

Therefore under the assumption (3.13) we have

$$(3.20) \qquad E_3(t) + \frac{1}{2} \int_s^t \int y_{ttt}^2 + y_{ttx}^2 + y_{txx}^2 \, dx \, dt \leqslant$$

$$\leqslant E_3 (s) + C \, (E_2(s) + E_1(s) + E_0(s)).$$

Thus we arrive at the a priori estimate (3.14) by (3.17) (3.19) and (3.20) under the assumption (3.13). This a priori estimate is also valid for the solution $y(t)$ in X_3 by use of the Friedrichs'mollifier (cf. Friedrichs (1954), Mizohata (1973), Matsumura (preprint) under the same assumption (3.13).

qed.

<u>Theorem 3.1</u> - Under the hypotheses 3.1 and 3.2 there exists a constant $\varepsilon > 0$ such that if the initial data are small as $E(0) < \varepsilon$, then the Cauchy problem (3.1) and (3.2) has a unique smooth solution in the large in time. The solution $y(t)$ decays to zero in the L^∞ - norm as $t \to + \infty$.

The existence of the solution in the large in time is a consequence of the local existence theorem and the a priori estimate in lemma 3.1. cf. Matsumura (preprint). In fact we choose the initial data so small that

$$(3.21) \qquad E(0) < \varepsilon^2 / 2 \, C^3 \quad ,$$

where ε is the same as in lemma 3.1.

First by the local existence theorem there exists $t_0 > 0$ such that the solution $y(t) \in X_3$ exists in $0 \leqslant t \leqslant t_0$ and satisfies the estimate

$$E(t) \leqslant 2\, E(0) \qquad \text{in} \quad 0 \leqslant t \leqslant t_0 \quad .$$

Then by (3.12) we have in $0 \leqslant t \leqslant t_0$

$$(3.22) \qquad \|y(t)\|_2^{\,2} \leqslant C^2\, E(t) \leqslant 2\, C^2\, E(0) \leqslant 2\, C^3\, E(0) < \varepsilon^2 \quad .$$

Thus by lemma 3.1 we have

$$(3.23) \qquad E(t) \leqslant C\, E(0) \qquad \text{in} \quad 0 \leqslant t \leqslant t_0 \quad .$$

Next by the local existence theorem for $t \geqslant t_0$ again there exists $\tau = \tau(CE(0)) > 0$ such that the solution $y(t)$ exists in $0 \leqslant t \leqslant t_0 + \tau$ and satisfies

$$(3.24) \qquad E(t) \leqslant 2\, E(t_0) \qquad \text{in} \quad t_0 \leqslant t \leqslant t_0 + \tau \quad .$$

By (3.12), (3.21), (3.23) and (3.24) we have in $t_0 \leqslant t \leqslant t_0 + \tau$

$$(3.25) \qquad \|y(t)\|_2^{\,2} \leqslant C^2\, E(t) \leqslant 2\, C^2\, E(t_0) \leqslant 2\, C^3\, E(0) < \varepsilon^2 \,.$$

Therefore (3.22), (3.25) and lemma 3.1 give

$$(3.26) \qquad E(t) \leqslant C\, E(0) \qquad \text{in} \quad 0 \leqslant t \leqslant t_0 + \tau \quad .$$

Repeating the same procedure with the same time interval $\tau > 0$, we complete the proof of the global existence of small solution.

The decay of solution is an easy consequence of the inequalities (3.17) and (3.18). In fact

$$y^2(t,x) \leqslant 2 \int_{-\infty}^{x} |y\, y_x|\, dx \leqslant 2\, \left(\int y^2\, dx \int y_x^{\,2} dx \right)^{1/2}$$

$$\leqslant C\, (E_1(t) + E_0(t))^{1/2}\, E_1(t)^{1/2} \leqslant C/t^{1/2} \quad .$$

$$\left.\begin{array}{l} y_t^{\,2}\ (t,x) \\ y_x^{\,2}\ (t,x) \end{array}\right\} \leqslant C\ (E_2(t) + E_1(t) + E_0(t))^{1/2}\ E_1(t)^{1/2} \leqslant C/t^{1/2} \quad .$$

qed.

On the other hand the solution in $0 \leqslant t,\ 0 \leqslant x \leqslant 1$ for the mixed problem (3.1) (3.2) with the zero Dirichlet boundary data decays to zero exponentially as $t \to +\infty$. Because by the Poincaré inequality we have in this case

$$E_1\ (t) + E_0(t) + \beta \int_s^t E_1\ (\tau) + E_0\ (\tau)\ d\tau \leqslant E_1\ (s) + E_0\ (s)$$

for some β = constant > 0.

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CHAPTER III - AN ABSTRACT NONLINEAR CAUCHY-KOWALEWSKI THEOREM AND ITS APPLICATIONS

1. Introduction

It is wellknown that the initial value problem for functions $u = (u_1(t,x),\ldots, u_N(t,x))$

$$(1.1)\quad \begin{cases} \frac{\partial u}{\partial t} = f(t,x,u,\partial u) & \text{in } t\in\mathbb{R},\ x\in\mathbb{R}^n \\ u(0,x) = g(x) & \text{in } x\in\mathbb{R}^n \end{cases},$$

where ∂u denotes the first partial derivatives in x, is always solved by the Cauchy-Kowalewski theorem uniquely in the class of analytic functions in a neighbourhood of any point $(0,x_0)$ under the assumptions that $f = (f_1,\ldots,f_N)$ and $g = (g_1,\ldots,g_N)$ are analytic in all its arguments.

Nagumo (1941) pointed out that it is not necessary to assume analyticity in t, i.e., if f is continuous in t with values as an analytic function of the other variables, then there exists a unique solution $u(t,x)$ continuously differentiable in t with values in analytic functions of x.

Ovsjannikov (1971) and Nirenberg (1972) generalized this result into an abstract form of the initial value problem in a scale of Banach spaces in a little different formulation, where f is not necessarily a differential operator on u and it may be a non-local "quasi-differential" operator on u.

In § 2, we improve the Nirenberg's formulation to get an abstract nonlinear Cauchy-Kowalewski theorem in a scale of Banach spaces which includes both theorems of Nirenberg and Ovsjannikov. The first application of it is of course to derive the Nagumo's theorem. The second application concerns the nonstationary problem of the water waves (the incompressible inviscid fluid flows under gravity) with free surface. Although there are many approximate theories to the nonstationary problem (cf. Korteweg and de Vries (1895), Stoker (1957), Benjamin (1974) and so on), the precise results were not known until recently, i.e., the initial value problem with full nonlinearity is solved in the class of analytic functions locally in time by Nalimov (1969) and by Ovsjannikov (1971) using the abstract Cauchy-

Kowalewski theorem, and by Shinbrot (1975), Reeder and Shinbrot (1976). Recently Nalimov (1974) solved this Cauchy problem in the class of functions with finite smoothness locally in time and justified the linear water wave approximation in the 2 space-dimension. In § 3 by the abstract Cauchy-Kowalewski theorem a justification of the shallow water approximation is shown locally in time for the initial value problem of the water waves with free surface in the two-space dimension in the class of analytic functions, which is an extension of Ovsjannikov's theorem (1976) for the periodic initial data. (cf. Kano and Nishida (preprint).

2. An Abstract Cauchy-Kowalewski Theorem in a Scale of Banach Spaces

Définition 2.1. Let $S = \{B_\rho\}_{\rho \geqslant 0}$ be a scale of Banach spaces and all B_ρ for $\rho > 0$ be linear subspaces of B_0. It is assumed that

$$B_\rho \subset B_{\rho'} \quad , \quad \|\cdot\|_{\rho'} \leqslant \|\cdot\|_\rho \qquad \text{for any } \rho' \leqslant \rho$$

where $\|\cdot\|_\rho$ denotes the norm in B_ρ .

Consider in S the initial value problem of the form

$$(2.1) \qquad \frac{du}{dt} = F(t, u(t)) \quad , \qquad |t| < \tau \quad ,$$

$$(2.2) \qquad u(0) = 0 \quad .$$

Hypothesis 2.1

(i) For some numbers $R > 0$, $\tau > 0$, $\rho_0 > 0$ and every pair of numbers ρ, ρ' such that $0 \leqslant \rho' < \rho < \rho_0$, $(t,u) \mapsto F(t,u)$ is a continuous mapping of

$$(2.3) \qquad \{t \ ; \ |t| < \tau\} \times \{u \in B_\rho \ ; \ \|u\|_\rho < R\} \quad \text{into } B_{\rho'} \ .$$

(ii) For any $\rho' < \rho < \rho_0$ and all $u, v \in B_\rho$ with $\|u\|_\rho < R$, $\|v\|_\rho < R$, and for any t, $|t| < \tau$, F satisfies the following

$$(2.4) \qquad \|F(t,u) - F(t,v)\|_{\rho'} \leqslant \frac{C\,\|u - v\|_\rho}{\rho - \rho'} \ ,$$

where C is a constant independent of t,u,v,ρ or ρ'.

(iii) $F(t,0)$ is a continuous function of t, $|t| < \tau$ with values in B_ρ for every $\rho < \rho_0$ and satisfies with a fixed constant K,

$$(2.5) \qquad \| F(t,0) \|_\rho \leqslant \frac{K}{\rho_0 - \rho} \quad , \quad 0 \leqslant \rho < \rho_0 \quad .$$

Theorem 2.1. Under the hypothesis 3.1 there is a constant $a > 0$ such that there exists a unique function $u(t)$ which, for every positive $\rho < \rho_0$ and $|t| < a(\rho_0 - \rho)$, is a continuously differentiable function of t with values in B_ρ, $\|u(t)\|_\rho < R$ and satisfies (1) (2).

Remark 2.1. The assumption (ii) on F is simpler than those of Ovsjannikov (1971) and Nirenberg (1972). The scale of Banach spaces is also less restrictive than that of Ovsjannikov.

Remark 2.2. When t is a complex variable, Hypothesis 2.1 (i), (iii) must be strengthened as follows :

(i)' If $0 \leqslant \rho' < \rho < \rho_0$ and $u(t)$ is a holomorphic function of t, $|t| < \tau$, valued in B_ρ such that $\|u(t)\|_\rho < R$ for all t, $|t| < \tau$, then

(2.6) $F(t,u(t))$ is a holomorphic function of t, $|t| < \tau$, valued in $B_{\rho'}$.

(iii)' $F(t,0)$ is a holomorphic function of t, $|t| < \tau$ with values in B_ρ for every $\rho < \rho_0$ and satisfies (2.5).

Then theorem 2.1 holds for complex variable t, i.e., the solution $u(t)$ is holomorphic in t with values in B_ρ .

Proof - cf. Nirenberg (1972) and Nishida (preprint)
Let B be the Banach space of functions $u(t)$ which, for every $0 \leqslant \rho < \rho_0$ and $|t| < a(\rho_0 - \rho)$, are continuous functions of t with values in B_ρ , and have the norm

$$(2.7) \qquad M[u] = \sup_{\substack{0 \leqslant \rho < \rho_0 \\ |t| < a(\rho_0 - \rho)}} \|u(t)\|_\rho \left(\frac{a(\rho_0 - \rho)}{|t|} - 1 \right) < +\infty \quad .$$

We seek a solution of

$$(2.8) \qquad u(t) = \int_0^t F(s,u(s))\,ds$$

with finite norm $M[u]$ with a suitably small. Our solution will be obtained as the limit of a sequence u_k defined recursively by

$$(2.9) \qquad u_0(t) = 0\ , \quad u_{k+1}(t) = \int_0^t F(s,u_k(s))\,ds \quad ,$$

where $k = 0, 1, 2, \ldots\ldots$ and

$$(2.10) \qquad \|u_k(t)\|_\rho < R \qquad \text{for} \quad |t| < a_k\,(\rho_0 - \rho) \quad .$$

Set for $k = 0, 1, 2, \ldots$

$$(2.11) \qquad v_k(t) = u_{k+1}(t) - u_k(t) \quad .$$

Here, for every $\rho < \rho_0$ and $|t| < a_k\,(\rho_0 - \rho)$, $u_k(t)$ and $v_k(t)$ are continuous functions of t with values in B_ρ for which $M_k\,[v_k]$ are finite, where

$$(2.12) \qquad M_k[v] = \sup_{\substack{0 \leq \rho < \rho_0 \\ |t| < a_k(\rho_0 - \rho)}} \|v(t)\|_\rho \left(\frac{a_k\,(\rho_0 - \rho)}{|t|} - 1\right) \quad .$$

The numbers a_k are defined by

$$(2.13) \qquad a_{k+1} = a_k\,(1 - (k+2)^{-2})\ , \quad k = 0, 1, 2, \ldots,$$

so that

$$(2.14) \qquad a = a_0 \prod_0^{+\infty} (1 - (k+2)^{-2}) > 0 \quad ,$$

and a_0 will be chosen suitably small later.

Let us imagine that u_i are determined for $i \leq k$ with $\|u_i(t)\|_\rho < R/2$ in $|t| < a_i\,(\rho_0 - \rho)$. Then by the assumption (i) $v_k(t)$ is well defined. Let

(2.15) $$\lambda_k = M_k [v_k] < +\infty \quad .$$

Then

$$\|v_k(t)\|_\rho \leqslant \frac{\lambda_k}{a_k/a_{k+1} - 1} \quad \text{for } |t| < a_{k+1}(\rho_0 - \rho)$$

and it follows for $|t| < a_{k+1}(\rho_0 - \rho)$

$$\|u_{k+1}(t)\|_\rho \leqslant \frac{\lambda_k}{a_k/a_{k+1} - 1} + \|u_k(t)\|_\rho$$

and so, by recursion,

(2.16) $$\|u_{k+1}(t)\|_\rho \leqslant \sum_0^k \frac{\lambda_j}{a_j/a_{j+1} - 1} \quad .$$

We will require that

(2.17) $$\sum_0^k \frac{\lambda_j}{a_j/a_{j+1} - 1} < \frac{R}{2} \quad .$$

Then for $|t| < a_{k+1}(\rho_0 - \rho)$ we have $\|u_{k+1}(t)\|_\rho < R/2$ and so $F(t, u_{k+1}(t))$ is defined.

Our aim is to estimate λ_k so that $\lambda_k \to 0$ as $k \to +\infty$ and (2.17) holds for any $k \geqslant 0$. By the definitions (2.9) and (2.11) we have

$$v_{k+1}(t) = \int_0^t F(s, u_{k+1}(s)) - F(s, u_k(s))\, ds$$

Thus for $|t| < a_{k+1}(\rho_0 - \rho)$, we see from the assumption (ii) that

$$\|v_{k+1}(t)\|_\rho \leqslant C \left| \int_0^t \frac{\|v_k(s)\|_{\rho(s)}}{\rho(s) - \rho}\, ds \right|$$

for some choice of $\rho < \rho(s) < \rho_0 - |s|/a_{k+1}$. We may set

$$\rho(s) = (\rho_0 - |s|/a_{k+1} + \rho)/2.$$

Then we find by virtue of (2.15) (assuming, say, $t > 0$)

$$\|v_{k+1}(t)\|_\rho \leqslant C\lambda_k \int_0^t ds / \frac{1}{2s}(a_{k+1}(\rho_0 - \rho) - s) . \frac{1}{2a_{k+1}}(a_{k+1}(\rho_0 - \rho) - s)$$

$$= 4\ C\ a_{k+1}\ \lambda_k \int_0^t s\ ds/(a_{k+1}(\rho_0 - \rho) - s)^2 \leqslant 4\ C\ a_{k+1}\ \lambda_k\ t \int_0^t ds/(a_{k+1}(\rho_0 - \rho) - s)^2$$

$$= 4\ C\ a_{k+1}\ \lambda_k \frac{t}{a_{k+1}(\rho_0 - \rho)} \Big/ \left(\frac{a_{k+1}(\rho_0 - \rho)}{t} - 1\right) .$$

Consequently

$$\lambda_{k+1} = M_{k+1}[v_{k+1}] \leqslant 4\ C\ a_{k+1}\ \lambda_k \sup_{\substack{0 \leqslant \rho \leqslant \rho_0 \\ |t| < a_{k+1}(\rho_0 - \rho)}} \left| \frac{t}{a_{k+1}(\rho_0 - \rho)} \right|$$

$$\leqslant 4\ C\ a_{k+1}\ \lambda_k \leqslant 4\ C\ a_0\ \lambda_k .$$

Hence for $k = 0, 1, 2, \ldots$

(2.18) $$\lambda_{k+1} \leqslant 4\ C\ a_0\ \lambda_k .$$

Now we can choose a_0. Using the assumption (iii) we know that

$$\lambda_0 = M_0 \left[\int_0^t F(s,0)\ ds\right] \leqslant K \sup_{\substack{0 \leqslant \rho < \rho_0 \\ |t| < a_0(\rho_0 - \rho)}} \frac{|t|}{\rho_0 - \rho}\left(\frac{a_0(\rho_0 - \rho)}{t} - 1\right) \leqslant a_0\ K.$$

We shall require that for $j = 0, 1, 2, \ldots$

(2.19) $$\lambda_j \leqslant 2^4\ a_0\ K\ (j+2)^{-4} .$$

Assuming that this is true for λ_k we find from (2.13) and (2.18)

$$\lambda_{k+1} \leqslant 4\ C\ a_0\ 2^4\ a_0\ K(k+2)^{-4} \leqslant 2^4\ a_0\ K(k+3)^{-4}\ \left(4\ C\ a_0 \left(\frac{k+3}{k+2}\right)^4\right) \leqslant 2^4\ a_0\ K(k+3)^{-4} ,$$

provided $a_0 \leqslant a'$ independent of k.

We have to verify (2.17). From (2.13) and (2.19)

$$\sum_0^k \frac{\lambda_j}{a_j/a_{j+1} - 1} \leqslant \sum_0^k \frac{\lambda_j}{1 - a_{j+1}/a_j} = \sum_0^k \lambda_j\ (j+2)^2$$

$$\leqslant 2^4\ a_0\ K \sum_0^k (j+2)^{-2} < 2^4\ a_0\ K \sum_0^\infty (j+2)^{-2} < R/2$$

provided $a_0 \leqslant a''$. If we choose $a_0 \leqslant a'$, $a_0 \leqslant a''$, we find the functions u_k defined for all k, with

(2.20) $\quad \|u_k(t)\|_\rho < R/2 \qquad$ for $\qquad |t| < a_k\ (\rho_0 - \rho)$.

Furthermore we have from (2.15) for $|t| < a\ (\rho_0 - \rho) < a_k\ (\rho_0 - \rho)$

$$\|u_{k+1}\ (t) - u_k(t)\|_\rho \leqslant \lambda_k/(a_k\ (\rho_0 - \rho)/|t| - 1) < \lambda_k/(a(\rho_0 - \rho)/\ |t| - 1),$$

$$M\,[u_{k+1} - u_k] \leqslant \lambda_k \quad .$$

Since $\Sigma\,\lambda_k < +\infty$, it follows that u_k converges to some $u(t)$ in B. From (2.20)

$$\|u(t)\|_\rho \leqslant R/2 \qquad \text{for} \quad |t| < a(\rho_0 - \rho) \quad .$$

$u(t)$ is a solution of (2.8). In fact we have for $|t| < a\ (\rho_0 - \rho')$ and $\rho' < \rho$

$$\|\int_0^t F(s,u(s))\ ds - u\ (t)\|_{\rho'}$$

$$\leqslant \int_0^t \|F(s,u(s)) - F(s,u_k(s))\|_{\rho'}\ d\ s + \|u(t) - u_{k+1}\ (t)\|_{\rho'}$$

$$\leqslant \frac{C}{\rho - \rho'} \int_0^t \|u(s) - u_k(s)\|_\rho\, ds + \|u(t) - u_{k+1}\ (t)\|_{\rho'}$$

by (ii). All the terms on the right go to zero as $k \to \infty$, and it follows that $u(t)$ is a solution of (2.8) and is also a solution of (2.1) (2.2).

The uniqueness of the solution is proved as follows. Suppose $v(t)$ is also a solution. Then $w(t) = u(t) - v(t)$ satisfies

$$w(t) = \int_0^t F(s,u(s)) - F(s,v(s))\ ds \qquad .$$

For any fixed $\rho_1 < \rho_0$, the functions u and v have finite M^1 norm, where

$$M^1\ [u] = \sup_{\substack{0 \leqslant \rho < \rho_1 \\ |t| < a(\rho_1 - \rho)}} \|u(t)\|_\rho\ (a(\rho_1 - \rho)/|t| - 1) \qquad .$$

Hence for $|t| < a\ (\rho_1 - \rho)$ we find from (ii)

$$\|w(t)\|_\rho \leqslant C \left| \int_0^t \frac{\|w(s)\|_{\rho(s)}}{\rho(s) - \rho} \, ds \right|$$

for some choice of $\rho(s) < \rho_1 - |s|/a$.

The same argument to get the estimate (2.18) gives the inequality

$$\|w(t)\|_\rho \leqslant 4\, C\, a\, M^1 [w] \,/(a\ (\rho_1 - \rho)/\ |t| - 1)$$

and so we obtain

$$M^1 [w] \leqslant 4\, C\, a\, M^1 [w] \quad .$$

Hence we conclude that $M^1 [w] = 0$ provided $4\, C\, a < 1$ which can be always assumed by decreasing a if necessary. Thus

$$\|w(t)\|_\rho = 0 \qquad \text{for} \quad |t| < a\ (\rho_1 - \rho).$$

Since this is true for every ρ_1 we conclude that $w \equiv 0$, and the theorem 2.1 is proved.

Remark 2.3- Instead of (2.1) and (2.2) or (2.8) we can consider the integral equation in the form :

(2.21) $$u(t) = u_0(t) + \int_0^t F(t-s,\ s,\ u(s))\, ds \quad \text{for} \quad 0 \leqslant t < \tau .$$

Here $u_0(t)$ is a continuous function of t, $0 \leqslant t < \tau$ with values in B_ρ for every $\rho < \rho_0$ and satisfies with a constant R_0

(2.22) $$\|u_0(t)\|_\rho \leqslant R_0 \quad , \quad 0 \leqslant \rho < \rho_0, \quad 0 \leqslant t < a_0\ (\rho_0 - \rho) .$$

$F(t,s,u)$ satisfies the analogues to hypothesis 2.1, i.e.,

(i) For some numbers $R > R_0 > 0$, $\tau > 0$, $\rho_0 > 0$ and any $0 \leqslant \rho' < \rho < \rho_0$, $(t,s,u) \longrightarrow F(t,s,u)$ is a continuous mapping of

(2.23) $$\{0 \leqslant t < \tau\} \times \{0 \leqslant s < \tau\} \times \{u \in B_\rho\ ;\ \|u\|_\rho < R\} \quad \text{into} \quad B_{\rho'}$$

and satisfies with a fixed constant K,

$$(2.24) \qquad \|F(t,s,0)\|_{\rho'} \leqslant \frac{K\,\|K\|_\rho}{\rho-\rho'}, \qquad 0 \leqslant \rho' < \rho < \rho_0 \quad .$$

(ii) For any $\rho' < \rho < \rho_0$ and all $u,v \in B_\rho$ with $\|u\|_\rho < R$, $\|v\|_\rho < R$, and for any $0 \leqslant t < \tau$ and $0 \leqslant s < \tau$, F satisfies the following

$$(2.25) \qquad \|F(t,s,u) - F(t,s,v)\|_{\rho'} \leqslant \frac{C\,\|u-v\|_\rho}{\rho-\rho'},$$

where C is a constant independent of t,s,u,v,ρ or ρ'.

Under these assumptions there exists a constant $a > 0$ such that the integral equation (2.21) has a unique function $u(t)$ which, for any $\rho < \rho_0$ and $0 \leqslant t < a(\rho_0 - \rho)$, is a continuous function of t with values in B_ρ, $\|u(t)\|_\rho < R$ and satisfies (2.21).

The proof of this statement is an analogue to that of theorem 2.1, but this formulation is a little more general than theorem 2.1 and it will be used to get the fluid dynamical limit of Boltzmann equation in the level of compressible Euler equation in Chapter 4.

Now as the first application of our abstract theorem we rederive the Nagumo's theorem as a generalization of Cauchy-Kowalewski theorem for the initial value problem

$$(2.26) \qquad \partial u/\partial t = f(t,x,u,u_{x_1},\dots,u_{x_n}), \ |t| < \tau, \ x \in D \subset \mathbb{R}^n$$

$$(2.27) \qquad u(0,x) = g(x) \quad , \quad x \in D ,$$

where $\qquad D = \prod_{j=1}^{n} \{ |x_j| < \rho_0 \} \quad , \quad u = u(t,x) \quad ,$

f and g are N-vector functions.

Here f is continuous in t with values in the space of holomorphic N-vector functions of the other variables for $x \in D$, $|u_i| < R$, $|u_{x_i}| < R$. g is holomorphic in D and may be assumed identically zero by a suitable substraction. If f is also analytic in t, the same proof for a complex neighbourhood of $|t| < \tau$ gives the classical Cauchy-Kowalewski theorem.

First let us reduce the initial value problem (2.26) (2.27) with $g = 0$ to a quasilinear form by introducing $p_i = u_{x_i}$, $i = 1, 2, \dots, n$. We have an equivalent system to (2.26)

$$\partial u/\partial t = f(t,x,u,p)$$
$$\partial p_i/\partial t = f_{x_i}(t,x,u,p) + f_u(t,x,u,p)\, u_{x_i} + f_p(t,x,u,p)\,{}'p_{x_i} \quad ,$$
$$u(0,x) = p_i(0,x) = 0 \quad , \qquad i = 1,2,\dots,n \quad .$$

Thus it is sufficient to consider the initial value problem for a quasilinear system of the form

$$\partial u/\partial t = \Sigma\, a_j(t,x,u)\, u_{x_j} + b(t,x,u) \tag{2.28}$$

$$u(0,x) = 0 \quad , \tag{2.29}$$

where u is a N-vector, a_j is a $N \times N$ matrix and b is a N-vector. We suppose that the components of a_j and b are continuous in t for $|t| \leqslant \tau$ with values in the space of functions which are holomorphic in a neighbourhood of

$$\Omega = \prod_{j=1}^{n} \{\, |x_j| \leqslant \rho_0 \} \times \prod_{i=1}^{N} \{ |u_i| \leqslant R \,\} \quad ,$$

where x_j and the components u_i are complex values. Then a_j and b and their first derivatives with respect to x_k and u_i are bounded by a constant C on $\{ |t| \leqslant \tau \} \times \Omega$.

For $0 \leqslant \rho < \rho_0$ let B_ρ denote the space of vector functions $u(x)$ which are holomorphic and bounded in $D_\rho = \prod_j \{ |x_j| < \rho \}$, and set

$$\|u\|_\rho = \sup_{D_\rho} |u(x)| \quad . \tag{2.30}$$

By the Cauchy's integral formula for the holomorphic functions we have

$$\|u_{x_j}\|_{\rho'} \leqslant \frac{\|u\|_\rho}{\rho - \rho'} \quad \text{for} \quad 0 \leqslant \rho' < \rho \quad . \tag{2.31}$$

Denote $\Sigma\, a_j(t,x,u) u_{x_j} + b(t,x,u) = F(t,u(t))$, where $u(t) = u(t,x)$, and let us check the assumptions (i) (ii) (iii) on F. By the assumptions on a_j and b above and by (2.30) (2.31) F satisfies (i). $F(t,0) = b(t,x,0)$ is bounded by C and so satisfies (iii). Last by the mean value theorem we see that if

$$|u(x)| < R \qquad \text{in } D_\rho \quad ,$$

then in $D_{\rho'}$, $\rho' < \rho$

$$|F(t,u) - F(t,v)| = |\sum_j a_j(t,x,v)\,(u_{x_j} - v_{x_j}) +$$

$$+ \sum_i a_{j,u_i}\,(t,x,v + \theta(u-v))\,(u_i - v_i)\,u_{x_j} + b_{u_i}\,(t,x,v+\theta\,(u-v))\,(u_i - v_i)|$$

$$\leqslant \Sigma\, C\, |u_{x_j} - v_{x_j}| + \Sigma\, C\, (|u_{x_j}| + 1)\,|u_i - v_i|$$

$$\leqslant \quad C\,\frac{\|u - v\|_\rho}{\rho-\rho'} + C\,\|u-v\|_{\rho'}\,\left(\frac{\|u\|_\rho}{\rho-\rho'} + 1\right) \qquad \text{by (2.31)}$$

$$\leqslant \quad C\,\frac{\|u - v\|_\rho}{\rho-\rho'} \quad .$$

Thus (ii) is satisfied with C independent of t,u,v,ρ or ρ' . Therefore theorem 2.1 applies to (2.28) (2.29) and gives the local existence of unique solution which is analytic in x near the origin.

3. Water Waves with Free Surface and the Formulation by a Conformal Mapping

The nonstationary water waves with free surface under gravity in the 2 space-dimension can be described in the Eulerian coordinate by the following : (cf. Lamb)

(3.1) $$\partial^2 \Phi / \partial x^2 + \partial^2 \Phi / \partial y^2 = 0 \qquad \text{in} \quad (x,y) \in D(t)$$

(3.2) $$\partial\Phi/\partial y = 0 \qquad \text{on } y = 0$$

(3.3) $$\partial\Phi/\partial t + ((\partial\Phi/\partial x)^2 + (\partial\Phi/\partial y)^2)/2 + g\,y = 0 \qquad \text{on } y = \Gamma(t,x),$$

(3.4) $$\partial\Gamma/\partial t + \partial\Phi/\partial x . \partial\Gamma/\partial x - \partial\Phi/\partial y = 0 \qquad \text{on } y = \Gamma(t,x) \; ,$$

where $y = 0$ is the bottom, $y = \Gamma(t,x)$ is the free surface, on which the pressure is constant, $D(t) = \{x \in R,\ 0 < y < \Gamma(t,x)$ is fullfilled by the water (incompressible and inviscid fluid), the fluid motion is assumed a potential flow and $\Phi = \Phi(t,x,y)$ is the velocity potential, g is the gravity acting downward and will be assumed to be 1 hereafter.

The initial value problem of the water waves with free surface under gravity is to determine Γ and Φ satisfying (3.1) $\sim$ (3.4) for $t \geq 0$ and the initial data :

$$\text{(3.5)} \qquad \begin{cases} \Gamma(0,x) = \Gamma_0(x) > 0 & \text{in } x \in R \\ \Phi(0,x,y) = \Phi_0\ (x,y) & \text{in } (x,y) \in D(0) \end{cases},$$

where the potential Φ should satisfy (3.1) in $D(t)$, $t \geq 0$ and so it is determined by the equation (3.1) with the boundary condition (3.2) and with

$$\text{(3.6)} \qquad \bar{\Phi}(t,x) = \Phi(t,x,\Gamma(t,x)) \quad \text{in} \quad t \geq 0,\ x \in R \ .$$

The nonlinear shallow water approximation assumes that the depth of the water has the order of ε and the initial data (3.5) are so small that

$$\text{(3.7)} \qquad \begin{cases} \Gamma(0,x) = \varepsilon\ \Gamma'\ (0,x\ ;\ \varepsilon) > 0 \\ \Phi(0,x,y) = \varepsilon^{1/2}\ \ \Phi'\ (0,x,y\ ;\ \varepsilon) \end{cases}$$

for a small parameter $\varepsilon > 0$, where $\Gamma' > 0$ and Φ' remains finite as $\varepsilon \to 0$. If we rescale the variables as follows :

$$\text{(3.8)} \qquad \begin{array}{llll} x \mapsto x' , & y \mapsto \varepsilon y' , & t \mapsto \varepsilon^{-1/2}\, t' & , \\ \Gamma \mapsto \varepsilon\Gamma' , & \Phi \mapsto \varepsilon^{1/2}\, \Phi' & , \bar{\Phi} \mapsto \varepsilon^{1/2}\, \bar{\Phi}' & , \end{array}$$

then the equations (3.1) (3.4) are transformed to the following

$$\text{(3.9)} \qquad \varepsilon^2\ \Phi_{xx} + \Phi_{yy} = 0 \quad \text{in } D(t) = \{x \in R,\ 0 < y < \Gamma(t,x)\} \ .$$

$$\text{(3.10)} \qquad \Phi_y = 0 \qquad \text{on } y = 0$$

$$\text{(3.11)} \qquad \varepsilon^2\ (\Phi_t + \Phi_x{}^2/2 + \Gamma) + \Phi_y{}^2/2 = 0 \quad \text{on } y = \Gamma(t,x) \ ,$$

$$\text{(3.12)} \qquad \varepsilon^2\ (\Gamma_t + \Phi_x\ \Gamma_x) - \Phi_y = 0 \qquad \text{on } y = \Gamma(t,x) \ ,$$

where the primes are abbreviated and the subscription means the differentiation with respect to the variable. Now following Friedrichs (1948) we assume the expansion of Φ and Γ in $\sigma = \varepsilon^2$

$$\left\{\begin{aligned} \Phi &= \Phi^0 + \sigma \Phi^1 + \sigma^2 \Phi^2 + \dots \\ \Gamma &= \Gamma^0 + \sigma \Gamma^1 + \sigma^2 \Gamma^2 + \dots \end{aligned}\right. \qquad (3.13)$$

Equating the same order of σ in the equations (3.9)~(3.12) we have the equations for Φ^k, Γ^k, $k = 0, 1, 2, \dots$ successively. As the lowest order approximation we have by (3.9) and (3.13)

$$\Phi^0_{yy} = 0$$

and by use of (3.10) we have

$$\Phi^0_y = 0 \quad \text{and} \quad \Phi^0 \text{ is independent of } y \quad . \qquad (3.14)$$

The first order in σ of (3.9) gives

$$\Phi^0_{xx} + \Phi^1_{yy} = 0$$

and by (3.10) we have

$$\Phi^1_y = - \int_0^y \Phi^0_{xx}\, d y = - y\, \Phi^0_{xx} \quad . \qquad (3.15)$$

The first order in σ of (3.11) and (3.14) give

$$\Phi^0_t + (\Phi^0_x)^2/2 + \Gamma^0 = 0 \quad . \qquad (3.16)$$

The first order in σ of (3.12) and (3.15) give

$$\Gamma^0_t + \Phi^0_x\, \Gamma^0_x - \Gamma^0\, \Phi^0_{xx} = 0 \quad , \quad \text{i.e.,}$$

$$\Gamma^0_t + (\Phi^0_x\, \Gamma^0)_x = 0 \quad . \qquad (3.17)$$

Thus the lowest order approximation (3.16) (3.17) is the nonlinear shallow water equation, which is a nonlinear hyperbolic conservation laws and is the same equation as that describing the isentropic gas motion with the adiabatic gas constant $\gamma = 2$. cf. Stoker (1957).

Here instead of considering the full expansion (3.13) we solve the initial value problem (3.9)~ (3.12) for the unknown functions $\Phi^{\varepsilon}(t,x,y)$ and $\Gamma^{\varepsilon}(t,x)$ with the initial data

$$(3.18) \qquad \begin{cases} \Gamma^{\varepsilon}(0,x) = \Gamma_0(x) > 0 & \text{in } x \in R, \\ \Phi^{\varepsilon}(0,x,y) = \Phi_0(x,y) & \text{in } D(0) \end{cases} .$$

for all $\varepsilon \in (0, \varepsilon_0)$, locally in time $t \in (0,t_0)$, t_0 independent of ε, in the class of analytic functions and will show that there exists

$$\lim_{\varepsilon \to 0} (\Gamma^{\varepsilon}(t,x), \Phi^{\varepsilon}(t,x,y)) = (\Gamma^0(t,x), \Phi^0(t,x,y))$$

and that the limit functions Γ^0, Φ^0 satisfy the nonlinear shallow water equation (3.16), (3.17) with the initials (3.18).

At first in order to avoid the difficulty that the domain filled by the water $D(t)$ depends on t we use a conformal mapping of $D(t)$ onto a fixed strip independent of t. Let

$$(3.19) \qquad z = z(t,\zeta) = x + i\,y, \qquad \text{where} \quad \zeta = \xi + i\eta$$

give the conformal mapping of the strip

$$(3.20) \qquad D_{\delta} = \{ \zeta = \xi + i\,\eta \;;\; \xi \in \mathbb{R},\; 0 < \eta < \delta \}$$

onto the domain $D(t) = \{ z = x + i\,y \;;\; x \in \mathbb{R},\; 0 < y < \Gamma(t,x) \}$,

where $\eta = 0 \mapsto y = 0$ and $\eta = \delta \mapsto y = \Gamma$.

Set the complex velocity potential $F = \Phi + i\,\Psi$, where Ψ is the complex conjugate of the harmonic function Φ, and let the complex velocity be $W = U - iV = F_z$. Then we define the functions in the variable ζ by

$$(3.21) \qquad f = f(t,\zeta) = F(t, z = z(t,\zeta)) = \phi(t,\zeta) + i\,\psi(t,\zeta) \quad ,$$

and

(3.22) $$w = w(t,\zeta) = W(t,z = z(t,\zeta)) = F_z\,(t,z = z(t,\zeta)) = f_\zeta \,/\, z_\zeta \quad .$$

The equation of the water surface (3.4) may be written in the variables t,ζ by the following

$$\frac{- x_t\, y_\xi + y_t\, x_\xi}{|z_\zeta|} = \frac{- U\, y_\xi + V\, x_\xi}{|z_\zeta|} \qquad \text{on } \eta = \delta$$

and so it has the form

(3.23) $$\mathrm{Im}\, \frac{z_t}{z_\zeta} = - \,\mathrm{Im}\, \frac{f_\zeta}{|z_\zeta|^2} \qquad \text{on } \eta = \delta \quad .$$

Also by $\Phi_t = \mathrm{Re}\, F_t$ and $|F_z|^2 = \Phi_x{}^2 + \Phi_y{}^2$ the equation (3.3) can be transformed into

(3.24) $$\mathrm{Re}(f_t - \frac{f_\xi}{z_\zeta}\, z_t) = - \frac{1}{2}\, \left|\frac{f_\zeta}{z_\zeta}\right|^2 - y \qquad \text{on } \eta = \delta \quad .$$

The boundary condition (3.2) on the bottom is given by

(3.25) $$\mathrm{Im}\, \frac{z_t}{z_\zeta} = \frac{\partial}{\partial\eta}\; \mathrm{Re}\, (f_t - f\, \frac{z_t}{z_\zeta}) = 0 \qquad \text{on } \eta = 0.$$

Here z, f and z_t/z_ζ , $f_t - z_t\, f_\zeta/z_\zeta$ are considered analytic in D_δ and they can be constructed by the boundary values, which are given on the right hand side of (3.23)$\sim$(3.25) . Especially we take the following construction. (cf. Woods (1961)). Let

(3.26) $$w(\xi,\eta) = u(\xi,\eta) + i\, v(\xi,\eta) \qquad \text{be analytic in } D_\delta \ ,$$

continuous on $\overline{D}_\delta$ and $v(\xi,0) = 0$ on $\eta = 0$, and let $u(\xi) \equiv u(\xi,\delta)$, $v(\xi) \equiv v(\xi,\delta)$ be Hölder-continuous in $\xi \in \mathbb{R}$. Then we have for any $\xi_0 \in \mathbb{R}$.

(3.27) $$v(\xi_0) = A_\delta\, u(\xi_0) = \frac{1}{2\delta} \int_{-\infty}^{+\infty} \frac{u(\xi)\, d\xi}{\mathrm{sh}\,\frac{\pi}{2\delta}\,(\xi-\xi_0)} \qquad ,$$

where the integral means the principal value. The inverse operator of A_δ which gives $u(\xi_0)$ from $v(\xi)$ is given by

(3.28) $$u(\xi_0) = -\frac{1}{2\delta}\int_{-\infty}^{+\infty} \frac{v(\xi)\,d\xi}{\mathrm{th}\frac{\pi}{2\delta}(\xi-\xi_0)} + \frac{1}{2}\,(u_{+\infty}+u_{-\infty})\ ,$$

where

(3.29) $$\frac{1}{2}\,(u_{+\infty}-u_{-\infty}) = \frac{1}{2\delta}\int_{-\infty}^{+\infty} v(\xi)\,d\xi \quad .$$

Therefore if $v(\xi)$ is summable in $\xi \in \mathbb{R}$ and Hölder-continuous, (3.28) has the meaning as the principal value and is rewritten in the form

$$u(\xi_0) = -\frac{1}{2\delta}\int_{-\infty}^{+\infty} \frac{v(\xi)\,d\xi}{\mathrm{sh}\frac{\pi}{2\delta}(\xi-\xi_0)} + \frac{1}{2}\,(u_{+\infty}+u_{-\infty})$$

$$-\frac{1}{2\delta}\int_{-\infty}^{+\infty} v(\xi)\ \mathrm{th}\frac{\pi}{4\delta}(\xi-\xi_0)\,d\xi \qquad \text{by (3.29)}$$

$$= -A_\delta\, v(\xi_0) + \frac{1}{2\delta}\int_{-\infty}^{+\infty} v(\xi)(1-\mathrm{th}\frac{\pi}{4\delta}(\xi-\xi_0))\,d\xi + u_{-\infty} \quad ,$$

where $u_{-\infty}$ is an arbitrary constant.

Thus

(3.30) $$u(\xi_0) = B_\delta\, v(\xi_0) = -A_\delta\, v(\xi_0) + C_\delta\, v(\xi_0) \quad ,$$

where

(3.31) $$C_\delta\, v(\xi_0) = \frac{1}{2\delta}\int_{-\infty}^{+\infty} v(\xi)\,(1-\mathrm{th}\frac{\pi}{4\delta}(\xi-\xi_0))d\xi + u_{-\infty} \quad .$$

By the analyticity of z and f and by the definition (3.26) and (3.27) we have on $\eta=\delta$

(3.32) $y = A_\delta\, x,\ \ \psi = A_\delta\, \varphi$ and the same for their derivatives.

By these operators for (3.23) (3.24) (3.25) we have the equations on $\eta=\delta$

(3.33) $$\begin{cases} z_t/z_\xi = -B_\delta\,(\psi_\xi/\,|z_\xi|^2) - i\ \psi_\xi/|z_\xi|^2 \ , \\ f_t - z_t\, f_\xi/z_\xi = -\frac{1}{2}\,|f_\xi/z_\xi|^2 - y - i\ A_\delta\,(\frac{1}{2}\,|f_\xi\,/z_\xi|^2 + y), \end{cases}$$

where $\Psi_\xi/|z_\xi|^2 \in L^1(\mathbb{R})$ will be verified later. The separation of (3.33) into the real and imaginary part, using (3.32), leads to the system of two equations for $x = x(t,\xi,\delta)$ and $\phi = \phi(t,\xi,\delta)$

$$(3.34) \quad \begin{cases} x_t = \dfrac{A_\delta \; x_\xi \cdot A_\delta \; \phi_\xi}{x_\xi^2 + (A_\delta \; x_\xi)^2} - x_\xi \; B_\delta\left(\dfrac{A_\delta \; \phi_\xi}{x_\xi^2 + (A_\delta \; x_\xi)^2}\right) , \\[2ex] \phi_t = - A_\delta \; x + \dfrac{1}{2} \; \dfrac{(A_\delta \; \phi_\xi)^2 - \xi^2}{x_\xi^2 + (A_\delta \; x_\xi)^2} - \phi_\xi \; B_\delta\left(\dfrac{A_\delta \; \phi_\xi}{x_\xi^2 + (A_\delta \; x_\xi)^2}\right) . \end{cases}$$

Thus the problem (3.1)~(3.5) is reduced to the equation (3.34) with (3.32) to be solved in $t \geq 0$, $\xi \in \mathbb{R}$ with the initial data

$$(3.35) \quad \begin{cases} x\,(0,\xi,\delta) = \mathrm{Re}\; z(0,\, \xi + i\delta) \\ \phi\,(0,\xi,\delta) = \Phi(0,\, z(0,\, \xi + i\delta)) \end{cases} .$$

Before we consider the shallow water limit in the formulation (3.34) with (3.32) we note the following properties of the operators A_δ and $B_\delta = - A_\delta + C_\delta$, $\delta \in (0, \delta_0]$.

Definition 3.1.- Let $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}^\sigma(\mathbb{R})$, $0<\sigma<1$ be the space of functions which are bounded continuous and bounded Hölder-continuous with the Hölder-exponent σ , in $\xi \in \mathbb{R}$ respectively. The norm is given by

$$(3.36) \quad \begin{cases} |u|_0 = \sup\limits_{\mathbb{R}} |u(\xi)| \\[1ex] |u|_\sigma = |u|_0 + \sup\limits_{\xi_1 \neq \xi_2} \dfrac{|u(\xi_1) - u(\xi_2)|}{|\xi_1 - \xi_2|^\sigma} \end{cases} .$$

Let L^1 be the Lebesque space of summable functions in $\xi \in \mathbb{R}$ with the norm

$$(3.37) \quad |u|_{L^1} = \int_{-\infty}^{+\infty} |u(\xi)| \; d\xi .$$

Lemma 3.1

(i) If $u \in \mathcal{B}^\sigma$, $0 < \sigma < 1$, then

$$(3.38) \quad |A_\delta \, u|_\sigma \leq C \; |u|_\sigma ,$$

where C is independent of $\delta \in (0,\delta_0]$.

(ii) If u is continuous and $u_\xi \in L^1 \cap \mathcal{B}^\sigma$, $0 \leq \sigma < 1$, then for a constant C independent of $\delta \in (0,\delta_0]$

(3.39) $$\left| \frac{A_\delta u}{\delta} \right|_\sigma \leq C \, |u_\xi|_\sigma$$, and especially

(3.40) $$\frac{A_\delta u}{\delta}(\xi) \to u_\xi(\xi) \quad \text{as} \quad \delta \to 0 .$$

(3.41) $$\left| \frac{A_\delta u}{\delta} \right|_{L^1} \leq C \, |u_\xi|_{L^1} .$$

(iii) If $v \in L^1 \cap \mathcal{B}^\sigma$, $0 \leq \sigma < 1$, then

(3.42) $$|\delta C_\delta v|_0 + \left| \frac{\partial}{\partial \xi} \delta C_\delta v \right|_{L^1} \leq C |v|_{L^1}$$

and specially

(3.43) $$\delta C_\delta v(\xi) \to \int_{-\infty}^{\xi} v(\xi)\, d\xi \quad \text{as} \quad \delta \to 0 ,$$

$\delta B_\delta v$ has the same limit as $\delta \to 0$ by (3.38).

(3.44) $$\left| \frac{\partial}{\partial \xi} \delta C_\delta v \right|_\sigma \leq C \, |v|_\sigma .$$

Proof - (i) The proof is standard for the principal value. Set $\xi_2 - \xi_1 = d > 0$.

$$A_\delta u(\xi_2) - A_\delta u(\xi_1) = \int_{\xi_1 - d}^{\xi_2 + d} \frac{u(\xi) - u(\xi_2)}{2\delta \, \mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_2)} - \frac{u(\xi) - u(\xi_1)}{2\delta \, \mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_1)} \, d\xi$$

$$+ \int_{\xi_2 + d}^{+\infty} \frac{1}{2\delta} \left(\frac{1}{\mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_2)} - \frac{1}{\mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_1)} \right) (u(\xi) - u(\xi_2)) - \frac{u(\xi_2) - u(\xi_1)}{2\delta \, \mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_1)} \, d\xi$$

$$+ \int_{-\infty}^{\xi_1 - d} \frac{1}{2\delta} \left(\frac{1}{\mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_2)} - \frac{1}{\mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_1)} \right) (u(\xi) - u(\xi_1)) - \frac{u(\xi_2) - u(\xi_1)}{2\delta \, \mathrm{sh} \frac{\pi}{2\delta}(\xi - \xi_2)} \, d\xi$$

$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 .$$

$$|I_1| \leqslant \int_{\xi_1 - d}^{\xi_2 + d} \frac{|\xi - \xi_2|^\sigma \, |u|_\sigma}{2\delta \, |sh \frac{\pi}{2\delta}(\xi - \xi_2)|} \, d\xi \leqslant C \, |u|_\sigma \int_{|\xi| < \frac{\pi d}{\delta}} \frac{|\xi|^\sigma \, d\xi}{|sh\xi|} \, \delta^\sigma$$

$$\leqslant \begin{cases} C \, \delta^\sigma |u|_\sigma \displaystyle\int_{0 \leqslant \xi \leqslant \frac{\pi d}{\delta}} \xi^{\sigma - 1} \, d\xi = \frac{C}{\sigma} \, |u|_\sigma \, d^\sigma & \text{for } \frac{\pi d}{\delta} \leqslant 1 \\ C \, \delta^\sigma |u|_\sigma \, \Big(\displaystyle\int_{0 \leqslant \xi \leqslant 1} \xi^{\sigma - 1} \, d\xi + \left(\frac{\pi d}{\delta}\right)^\sigma \int_{1 \leqslant \xi < \frac{\pi d}{\delta}} \frac{d\xi}{sh \, \xi} \leqslant \end{cases}$$

$$\leqslant \frac{C}{\sigma} \, |u|_\sigma \, d^\sigma \qquad \text{for} \qquad \frac{\pi d}{\delta} > 1 .$$

I_2 can be treated in the same way as I_1 .

$$|I_3| \leqslant \int_{\frac{\pi d}{2\delta}}^{+\infty} \frac{1}{\pi} \left(\frac{1}{sh \, \xi} - \frac{1}{sh(\xi + \frac{\pi d}{2\delta})} \right) \left| u\left(\xi_2 + \frac{2\delta}{\pi}\xi\right) - u(\xi_2) \right| d\xi$$

$$\leqslant \frac{1}{\pi} \int_{\frac{\pi d}{2\delta}}^{+\infty} \frac{\frac{\pi d}{2\delta}}{sh^2 \xi} \, |u|_\sigma \left(\frac{2\delta}{\pi}\xi\right)^\sigma d\xi \leqslant C \, \delta^{\sigma - 1} \, |u|_\sigma \, d \, . \int_{\frac{\pi d}{2\delta}}^{+\infty} \delta^{\sigma - 2} \, d\xi = \frac{C}{1 - \sigma} \, |u|_\sigma \, d^\sigma .$$

$I_4 + I_6 = 0$ and I_5 has the same estimate as I_3.

(ii) Since

$$\frac{A_\delta u}{\delta}(\xi_1) = \frac{1}{\delta\pi} \int_{-\infty}^{+\infty} \frac{u(\xi_1 + \frac{2\delta}{\pi}\xi) - u(\xi_1)}{sh \, \xi} \, d\xi$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\delta} \left(\int_0^{\frac{2\delta}{\pi}\xi} u_\xi(\xi_1 + \eta) \, d\eta \right) \frac{d\xi}{sh\xi} ,$$

$$\left| \frac{A_\delta u}{\delta}(\xi_2) - \frac{A_\delta u}{\delta}(\xi_1) \right| = \left| \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\delta} \int_0^{\frac{2\delta}{\pi}\xi} (u_\xi(\xi_2 + \eta) - u_\xi(\xi_1 + \eta)) \, d\eta \, \frac{d\xi}{sh\xi} \right|$$

$$\leqslant \frac{|u_\xi|_\sigma \, |\xi_2 - \xi_1|^\sigma}{\pi} \int_{-\infty}^{+\infty} \frac{1}{\delta} \, \frac{2\delta}{\pi} \, \frac{\xi \, d\xi}{sh\xi} \leqslant C \, |u_\xi|_\sigma |\xi_2 - \xi_1|^\sigma .$$

For any $\varepsilon > 0$ we have

$$\left| \frac{A_\delta u}{\delta}(\xi_0) - u_\xi(\xi_0) \right| = \frac{1}{\pi} \left| \int_{-\infty}^{+\infty} \frac{1}{\delta} \int_{\infty}^{\frac{2\delta}{\pi}\xi} (u_\xi(\xi_0+\eta) - u_\xi(\xi_0))\, d\eta \frac{d\xi}{sh\xi} \right|$$

$$\leq \frac{1}{\pi} \sup_{|\eta| < \frac{2\delta}{\pi}M} |u_\xi(\xi_0+\eta) - u_\xi(\xi_0)| \cdot \int_{|\xi|<M} \frac{2}{\pi}\frac{\xi d\xi}{sh\xi} + \frac{4}{\pi^2} |u_\xi|_0 \int_{|\xi|>M} \frac{\xi d\xi}{sh\xi} < \varepsilon$$

for sufficiently large M and sufficiently small δ.

Thus $\frac{A_\delta u}{\delta}(\xi_0) \to u_\xi(\xi_0)$ as $\delta \to 0$.

For (3.41) we have by Fubini's theorem

$$\int_{-\infty}^{+\infty} \left| \frac{A_\delta u}{\delta}(\xi_0) \right| d\xi_0 \leq \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\delta} \left| \int_0^{\frac{2\delta}{\pi}\xi} |u_\xi(\xi_0+\eta)| |d\eta| \right| \frac{d\xi}{|sh\,\xi|} \quad d\xi_0$$

$$= \frac{1}{\pi} \int \frac{d\xi}{|sh\xi|} \frac{1}{\delta} \left| \int_0^{\frac{2\delta|\xi|}{\pi}} \int |u_\xi(\xi_0+\eta)|\, d\xi_0 \right| d\eta$$

$$\leq \frac{1}{\pi} \int \frac{d\xi}{|sh\xi|} \left(|u_\xi|_{L^1} \frac{|2\xi|}{\pi} \right) \leq \frac{2}{\pi^2} |u_\xi|_{L^1} \int \frac{\xi d\xi}{sh\xi} \quad .$$

(iii)

$$|\delta C_\delta v(\xi_0)| = \left| \frac{1}{2} \int v(\xi) \left(1 - th \frac{\pi}{4\delta}(\xi-\xi_0)\right) d\xi \right| \leq |v|_{L^1} \quad .$$

For (3.43) we have

$$\left| \delta C_\delta v(\xi_0) - \int_{-\infty}^{\xi_0} v(\xi)\, d\xi \right| = \left| \frac{1}{2\delta} \int_{-\infty}^{\xi_0} v(\xi) \left(-1 - th \frac{\pi}{4\delta}(\xi-\xi_0)\right) d\xi \right.$$

$$\left. + \frac{1}{2\delta} \int_{\xi_0}^{+\infty} v(\xi) \left(1 - th \frac{\pi}{4\delta}(\xi-\xi_0)\right) d\xi \right| \to 0 \quad \text{as} \quad \delta \to 0$$

by the Levesgue's theorem.

For (3.42) we have by the definition (3.31)

$$\int_{-\infty}^{+\infty} \left| \frac{\partial}{\partial\xi_0} \delta\, C_\delta v(\xi_0) \right| d\xi_0 = \frac{1}{2} \int d\xi_0 \left| \int v(\xi_0 + \frac{4\delta}{\pi}\xi) \frac{d\xi}{ch^2\xi} \right|$$

$$\leq \quad \frac{1}{2} \int \frac{d\xi}{ch^2\xi} \int \left| v(\xi_0 + \frac{4\delta}{\pi}\xi) \right| d\xi_0 \leq C\, |v|_{L^1} \quad .$$

If we define the functions Γ^0 and Φ^0 by

$$(3.49)\qquad \begin{cases} \Gamma^0\,(t,x^0\,(t,\xi)) = y^0\,(t,\xi) \\ \Phi^0\,(t,x^0(t,\xi)) = \phi^0\,(t,\xi) \end{cases} ,$$

then the shallow water wave equation (3.16) (3.17) follows from (3.48). Inversely if we introduce an auxiliary variable ξ by the first equation of (3.48), then the other equations of (3.48) follows from (3.16) (3.17). Therefore for a justification of the shallow water equation we are going to solve (3.46) (3.47) for $\delta \in (0, \delta_0]$ and to get the limit (3.48) of (3.46) as $\delta \to 0$. They are accomplished in the space of analytic functions locally in time.

cf. Ovsjannikov (1974) for the periodic initial data and Kano and Nishida (preprint) for the pure initial data.

Supplements to definition 3.1. and lemma 3.1. Let L^σ $(0 \leq \sigma < 1)$ be the space of summable functions which have the σ-Hölder continuous integral, i.e.,

$$(3.50)\qquad |u|_{L^\sigma} = \int_{-\infty}^{+\infty} |u\,(\xi)|\,d\xi + \sup_{d>0} \frac{1}{d^\sigma} \int_{-\infty}^{+\infty} |u\,(\xi + d) - u\,(\xi)|\,d\xi < +\infty .$$

It is easy to see that if $u_\xi \in L^\sigma$, then for any given $u(-\infty)$

$$(3.51)\qquad u = u\,(-\infty) + \int_{-\infty}^{\xi} u_\xi\,d\xi$$

is well defined and has the estimate

$$(3.52)\qquad |u - u\,(-\infty)|_\sigma \leq |u_\xi|_{L^\sigma} .$$

Furthermore the functions in L^σ have analogous properties to those in $\mathcal{B}^\sigma$ as follows :

i) If $u \in L^\sigma$, $0<\sigma<1$, then

$$(3.53)\qquad |A_\delta u|_{L^\sigma} \leq C\,|u|_{L^\sigma} ,$$

where and in (ii) and (iii) C is independent of $\delta \in (0, \delta_0]$.

ii) If $u_\xi \in L^\sigma$, $0 \leqslant \sigma < 1$, then

(3.54) $$\left| \frac{A_\delta u}{\delta} \right|_{L^\sigma} \leqslant C \, |u_\xi|_{L^\sigma}$$ and

(3.55) $$\frac{A_\delta u}{\delta} \rightarrow u_\xi(\xi) \quad \text{a.e} \quad \text{as } \delta \rightarrow 0 .$$

iii) If $v \in L^\sigma$, $0 \leqslant \sigma < 1$ then

(3.56) $$|\delta C_\delta v|_\sigma + \left| \frac{\partial}{\partial \xi} \delta C_\delta v \right|_{L^\sigma} \leqslant C \, |v|_{L^\sigma}$$ and

(3.57) $$\delta C_\delta v(\xi) \rightarrow \int_{-\infty}^{\xi} v(\xi) \, d\xi \quad \text{as } \delta \rightarrow 0 .$$

The proofs of these properties are the same as those for the functions in $\mathcal{B}^\sigma$ except that the integration in ξ and the integration by parts are needed.

4 - THE CAUCHY PROBLEM OF THE WATER WAVES AND THE SHALLOW WATER APPROXIMATION

The equation of the water waves with free surface for the unknown functions $x_\xi = v$ and $\phi_\xi = u$ is given by (3.34) after differentiation in ξ

$$
(4.1)\quad \begin{cases} v_t = \dfrac{\partial}{\partial \xi}\left\{ w\, A_\delta\, v\, A_\delta\, u + v\, A_\delta\, (w\, A_\delta\, u) - v\, C_\delta\, (w\, A_\delta\, u)\right\} \\ u_t = -\dfrac{1}{\delta} A_\delta\, v - \dfrac{\partial}{\partial \xi}\left\{ \dfrac{w}{2}\, (u^2 - (A_\delta\, u)^2) - u\, A_\delta (w\, A_\delta\, u) + u\, C_\delta\, (w\, A_\delta\, u)\right\} \end{cases}
$$

where $t \geq 0$, $\xi \in \mathbb{R}$ and

$$
(4.2)\qquad w = 1/\left\{ v^2 + (A_\delta\, v)^2 \right\},
$$

and A_δ, $B_\delta = -A_\delta + C_\delta$ are the linear operators defined in (3.27) and (3.30) for $0 < \delta \leq \delta_0$=constant. The initial data are given by

$$
(4.3)\qquad v(0,\xi) = v_0(\xi),\quad u(0,\xi) = u_0(\xi) \qquad \text{in } \xi \in \mathbb{R}.
$$

<u>Définition 4.1</u>.- Fix $\sigma \in (0,1)$. For any $\rho > 0$ we consider the analytic functions $u(\xi)$ in the complex neighbourhood of the real axis

$$
D_\rho = \left\{ \xi + i\eta \;;\; \xi \in \mathbb{R},\ |\eta| < \rho \right\}
$$

with the norms :

$$
(4.4)\quad \begin{aligned} |u|_{\sigma,\rho} &\equiv \sup_{|\eta|<\rho} |u(\xi + i\eta)|_\sigma = \sup_{D_\rho} |u| + \sup_{|\eta|<\rho}\ \sup_{\xi_2 \neq \xi_1} \frac{|u(\xi_2 + i\eta) - u(\xi_1 + i\eta)|}{|\xi_2 - \xi_1|^\sigma} \\ |u|_{L^\sigma,\rho} &\equiv \sup_{|\eta|<\rho} |u(\xi + i\eta)|_{L^\sigma} = \sup_{|\eta|<\rho} \int_{-\infty}^{+\infty} |u(\xi + i\eta)|\, d\xi + \sup_{|\eta|<\rho}\ \sup_{d>0} \int_{-\infty}^{+\infty} |u(\xi + d + i\eta) \\ &\qquad - u(\xi + i\eta)|\, d\xi \end{aligned}
$$

$\mathcal{B}_\rho^\sigma = \left\{ u(\xi)\ ;\ \text{analytic in } D_\rho,\ |u|_{\sigma,\rho} < +\infty \right\}$

$L_\rho^\sigma = \left\{ u(\xi)\ ;\ \text{analytic in } D_\rho,\ |u|_{L^\sigma,\rho} < +\infty \right\}$

$X_\rho = \left\{ u \in \mathcal{B}_\rho^\sigma \right.$ and $u_\xi \in L_\rho^\sigma$ with the norm

(4.5) $\qquad \|u\|_\rho = \max\{|u|_{\sigma,\rho}, |u_\xi|_{L^\sigma,\rho}\} < +\infty\}$

$S = \bigcup_{\rho>0} \{(v,u)\ ;\ v,\ u \in X_\rho\}$ will be our scale of Banach spaces.

It follows from (3.51) and (4.4) that if $u_\xi \in L^\sigma_\rho$, then for any $u(-\infty)$

$$u(\xi) = u(-\infty) + \int_{-\infty}^{\xi} u_\xi(\xi)\, d\xi$$

is well-defined and

(4.6) $$|u - u(-\infty)|_{\sigma,\rho} \leqslant |u_\xi|_{L^\sigma,\rho}$$

It is easy to see that for $\rho > 0$

(4.7) $$\begin{cases} |uv|_{\sigma,\rho} \leqslant |u|_{\sigma,\rho}\, |v|_{\sigma,\rho} & \text{for any } u,v \in \mathcal{B}^\sigma_\rho, \\ |uv|_{L^\sigma,\rho} \leqslant |u|_{\sigma,\rho}\, |v|_{L^\sigma,\rho} & \text{for any } u \in \mathcal{B}^\sigma_\rho, v \in L^\sigma_\rho. \end{cases}$$

It follows from the Cauchy's integral formula for the holomorphic functions that for any $0<\rho'<\rho$

(4.8) $$\begin{aligned} |u_\xi|_{\sigma,\rho'} &\leqslant \frac{|u|_{\sigma,\rho}}{\rho-\rho'} && \text{for any } u \in \mathcal{B}^\sigma_\rho, \\ |v_\xi|_{L^\sigma,\rho'} &\leqslant \frac{|v|_{L^\sigma,\rho'}}{\rho-\rho'} && \text{for any } v \in L^\sigma_\rho. \end{aligned}$$

The properties of the operators A_δ and $B_\delta = -A_\delta + C_\delta$ in lemma 3.1 and in (3.52) (3.53) and (3.55) can be extended to the spaces of analytic functions $\mathcal{B}^\sigma_\rho$ and L^σ_ρ.

<u>Lemma 4.1.</u>- Fix $\sigma \in (0,1)$. The operators A_δ and C_δ, $0<\delta\leqslant\delta_0$ = constant have the following estimates in $\mathcal{B}^\sigma_\rho$ and L^σ_ρ, $\rho>0$, with a constant C independent of $\delta \in (0,\delta_0]$ and of $\rho>0$.

(i) If $u \in \mathcal{B}^\sigma_\rho$ and $v \in L^\sigma_\rho$ for $\rho > 0$, then

(4.9) $$|A_\delta u|_{\sigma,\rho} \leqslant C|u|_{\sigma,\rho}, \quad |A_\delta v|_{L^\sigma,\rho} \leqslant C|v|_{L^\sigma,\rho}$$

(ii) If $u_\xi \in L^\sigma_\rho$ for $\rho>0$, then

(4.10) $$\left|\frac{A_\delta u}{\delta}\right|_{L^\sigma,\rho} \leqslant C|u_\xi|_{L^\sigma,\rho}.$$

(iii) If $v \in L^{\sigma}_{\rho}$ for $\rho > 0$, then

(4.11) $$\| \delta \, C_{\delta} \, v \|_{\rho} \leq C \, |v|_{L^{\sigma},\rho}$$

This lemma is a consequence of lemma 3.1 and the supplements to it (3.52) (3.53) and (3.55) and of the definition of the norms (4.4). Here we note that (ii) and (iii) in lemma are valid also for $\delta = 0$ because of (3.54) and (3.56) with (4.6).

Now we consider the initial value problem (4.1) (4.3) in the scale of Banach spaces S. First we assume that the initial data $(v_0,u_0)(\xi) \in S$, i.e.

(4.12) $$v_0,\ u_0 \in X_{\rho_0} \quad \text{for some} \quad \rho_0 > 0.$$

Then $(v_{\pm}, u_{\pm}) = \lim\limits_{\xi \to \pm\infty} (v_0,u_0)(\xi)$ exist and it may be assumed that

(4.13) $$v_+,\ v_- > 0 \quad \text{and} \quad v_0(\xi) > 0 \quad \text{for any} \quad \xi \in \mathbb{R}$$

to avoid the dried bottom and the singularities of the free surface. The solutions $v(t,.) > 0$, $u(t,.)$ are sought in S for $0 < \rho < \rho_0$, $0 \leq t < a(\rho_0 - \rho)$ with some $a > 0$. They satisfy $\lim\limits_{\xi \to \pm\infty} (v,u)(t,\xi) = (v_{\pm}, u_{\pm})$. In fact for the solution $(v,u)(t,.) \in S$ with $\lim\limits_{\xi \to -\infty} (v,u)(t,\xi) = (v_-, u_-)$, we have

(4.14) $$\lim_{\xi \to +\infty} (v,u)(t,\xi) = (v_+, u_+) \quad .$$

Because since for any $(v,u) \in S$ we have v_{ξ}, u_{ξ} and $A_{\delta} v$, $A_{\delta} u \longrightarrow 0$ as $\xi \to \pm\infty$ we can compute

$$\frac{d}{dt}(v_+ - v_-) = \int_{-\infty}^{+\infty} v_{\xi t}\, d\xi = \int_{-\infty}^{+\infty} v_{t\xi}\, d\xi =$$

$$= \Big[\frac{\partial}{\partial\xi}(w\, A_{\delta}\, v\, A_{\delta}\, u + v\, A_{\delta}\, (w\, A_{\delta}\, u)) - v_{\xi}\, C_{\delta}\, (w\, A_{\delta}\, u) - v\, \frac{\partial}{\partial\xi} C_{\delta}\, (w\, A_{\delta}\, u)\Big]_{-\infty}^{+\infty} = 0,$$

where $\dfrac{w\, A_{\delta}\, u}{\delta} \in L^{\sigma}_{\rho}$ is used in the last two terms.

Thus for the solution $(v,u)(t,.)$ in S, $(v_{\pm}, u_{\pm})$ are constant in t, and so it is sufficient to seek the solution $(v,u)(t,.)$ in $S_1 \subset S$, where

(4.15) $$S_1 = \bigcup_{\rho > 0} \big\{ (v,u)(\xi) \ ; \ v,u \in X_{\rho},\ (v,u)(\pm\infty) = (v_{\pm}, u_{\pm}) \big\}.$$

Furthermore we assume a little more than $v > 0$, i.e.

(4.16) $$| v(t,\xi) - v_- |_{\sigma,\rho} \leq \min\left\{\frac{v_-}{4}, \frac{v_-}{4C^2}\right\} \equiv R_0$$

for the solution a priori, where C is the constant in (4.9). In this case, we have the estimate for $w = 1/\{v^2 + (A_\delta v)^2\}$.

Lemma 4.2. Let $v \in X_\rho$ and $|v - v_-|_{\sigma,\rho} \leq R_0$. Then we have for a constant $C_0 = C(R_0)$ independent of $\delta \in (0,\delta_0)$, where w is defined in (4.2) and $w_- = w(v_-)$,

(4.17) $$|w - w_-|_{\sigma,\rho} \leq \frac{20}{3 v_-^3} | v - v_-|_{\sigma,\rho} ,$$
$$|w_\xi|_{L^\sigma_{,\rho}} \leq C_0 |v_\xi|_{L^\sigma_{,\rho}} .$$

Also let $v_{i,\xi} \in L^\sigma_\rho$, $i = 1,2$ and $|v_i - v_-|_{\sigma,\rho} \leq R_0$. Put $w_i = w(v_i)$, $i = 1,2$. Then we have

(4.18) $$|w_2 - w_1|_{\sigma,\rho} \leq C_0 |v_2 - v_1|_{\sigma,\rho} ,$$
$$|w_{2,\xi} - w_{1,\xi}|_{L^\sigma_{,\rho}} \leq C_0 |v_{2,\xi} - v_{1,\xi}|_{L^\sigma_{,\rho}} .$$

for a constant $C_0 = C(R_0)$ independent of $\delta \in (0, \delta_0]$.

Proof - By (4.7) (4.9) and by our assumption we have

$$|v^2 + (A_\delta v)^2|_{\sigma,\rho} \geq v_-^2 - |(2v_- + (v-v_-))(v-v_-) + (A_\delta (v-v_-))^2|_{\sigma\,\rho}$$

$$\geq v_-^2 - \{(2v_- + v_-/4) + C^2 v_-/4 C^2\}|v-v_-|_{\sigma,\rho} = v_-^2 - \frac{5}{2} v_- |v-v_-|_{\sigma,\rho}$$

Therefore w is bounded for $|v-v_-|_{\sigma,\rho} < v_-/4$ and we have

$$|w-w_-|_{\sigma,\rho} \leq \frac{1}{v_-^2} \frac{5 | v-v_-|_{\sigma,\rho}}{2 v_-} \left(\frac{1}{1 - 5/8}\right) = \frac{20}{3 v_-^3} | v-v_-|_{\sigma,\rho} .$$

$$|w_2-w_1|_{\sigma,\rho} = |w_1 w_2 (v_1^2 - v_2^2 + (A_\delta v_1)^2 - (A_\delta v_2)^2)|_{\sigma,\rho}$$

$$= |w_1 w_2 ((v_1 + v_2)(v_1-v_2) + (A_\delta v_1 + A_\delta v_2)(A_\delta (v_1 - v_2))|_{\sigma,\rho} \leq C_0 |v_1 - v_2|_{\sigma,\rho} .$$

The other inequalities are proved analogously.

qed.

Lemma 4.3. Let $w,v,u \in X_\rho$ for some $\rho > 0$. (cf. (4.5)). Then there exists a constant C independent of $\delta \in (0, \delta_0]$ such that the following estimates hold for any $0 < \rho' < \rho$:

(4.19) $\quad \|(w\, A_\delta\, v\, A_\delta\, u)_\xi\|_{\rho'} \leq \frac{C}{\rho-\rho'} \|w\|_\rho\, \|v\|_\rho\, \|u\|_\rho$,

(4.20) $\quad \|(v\, A_\delta\, (w\, A_\delta\, u))_\xi\|_{\rho'} \leq \frac{C}{\rho-\rho'} \|v\|_\rho\, \|w\|_\rho\, \|u\|_\rho$,

(4.21) $\quad \|(v\, C_\delta\, (w\, A_\delta\, u))_\xi\|_{\rho'} \leq \frac{C}{\rho-\rho'} \|v\|_\rho\, \|w\|_\rho\, \|u\|_\rho$

Proof - By (4.7), (4.8) and by (4.9) in lemma 4.1

$$\|w\, A_\delta v\, A_\delta\, u)_\xi\|_{\rho'} \leq |(w\, A_\delta\, v\, A_\delta\, u)_\xi|_{\sigma,\rho'} + |(w\, A_\delta\, v\, A_\delta\, u)_{\xi\xi}|_{L^\sigma,\rho'}$$

$$\leq \frac{1}{\rho-\rho'}\{|w\, A_\delta\, v\, A_\delta\, u|_{\sigma,\rho} + |(w\, A_\delta\, v\, A_\delta\, u)_\xi|_{L^\sigma,\rho}\}$$

$$\leq \frac{C}{\rho-\rho'}\{|w|_{\sigma,\rho}\,|v|_{\sigma,\rho}\,|u|_{\sigma,\rho} + |w_\xi|_{L^\sigma,\rho}\,|v|_{\sigma,\rho}\,|u|_{\sigma,\rho}$$

$$+ |w|_{\sigma,\rho}\,|v_\xi|_{L^\sigma,\rho}\,|u|_{\sigma,\rho} + |w|_{\sigma,\rho}\,|v|_{\sigma,\rho}\,|u_\xi|_{L^\sigma,\rho}\} \leq \frac{C}{\rho-\rho'}\|w\|_\rho\,\|v\|_\rho\,\|u\|_\rho \quad .$$

The inequality (4.20) is proved analogously by using (4.9).

The last inequality (4.21) is proved by (4.10) and (4.11) :

$$\|(v\, C_\delta\, (w\, A_\delta\, u))_\xi\|_{\rho'} \leq \frac{1}{\rho-\rho'}\{|v\, C_\delta\, (w\, A_\delta\, u)|_{\sigma,\rho} + |(v\, C_\delta\, (w\, A_\delta\, u))_\xi|_{L^\sigma,\rho}\}$$

$$\leq \frac{C}{\rho-\rho'}\{(|v|_{\sigma,\rho} + |v_\xi|_{L^\sigma,\rho})|C_\delta\, (w\, A_\delta\, u)|_{\sigma,\rho} + |v|_{\sigma,\rho}\left|\frac{\partial}{\partial\xi}\, C\, (w\, A_\delta\, u)\right|_{L^\sigma,\rho}$$

$$\leq \frac{C}{\rho-\rho'}\|v\|_\rho \left\|\delta\, C_\delta\, \left(\frac{w\, A_\delta\, u}{\delta}\right)\right\|_\rho \leq \frac{C}{\rho-\rho'}\|v\|_\rho \left|\frac{w\, A_\delta\, u}{\delta}\right|_{L^\sigma,\rho}$$

$$\leq \frac{C}{\rho-\rho'}\|v\|_\rho\,|w|_{\sigma,\rho}\,|u_\xi|_{L^\sigma,\rho} \leq \frac{C}{\rho-\rho'}\|v\|_\rho\,\|w\|_\rho\,\|u\|_\rho \quad .$$

qed.

Now we can apply the abstract Cauchy-Kowalewski theorem 2.1 to the Cauchy problem (4.1) (4.2) (4.3). Set $U(t) = (v_1(t,\xi),\, u_1(t,\xi)) = (v(t,\xi) - v_0(\xi)$ $u(t,\xi) - u_0(\xi))$, where (v_0, u_0) are the initial data given in (4.12). Since the solution $(v(t),\, u(t))$ is sought in S_1, $(v_1, u_1)(t, \pm\infty) = 0$. Define for

$U(t,\pm\infty) = 0$

(4.22) $$\| U(t) \|_\rho = \max (\| v_1(t) \|_\rho , \| u_1(t) \|_\rho)$$

The equation (4.1) with (4.3) can be rewritten in the variable U :

(4.23) $$\frac{dU}{dt} = F(v_0,u_0,U) \quad , \quad U(0) = 0 \quad ,$$

where $F(v_0,u_0,U)$ satisfies, in the scale of the Banach spaces S with the norm (4.22), all conditions (i)(ii)(iii) in § 2 by lemmas 4.2, 4.3 provided that (4.16) is satisfied. But if

(4.24) $$| v_0(.) - v_- |_{\sigma,\rho} < R_0 /2 \qquad \text{and}$$

(4.25) $$| v_1 (t,.) |_{\sigma,\rho} < R_0/2 \qquad ,$$

then (4.16) is satisfied, i.e.,

$$| v(t,.) - v_- |_{\sigma,\rho} \leq | v_0(.) - v_- |_{\sigma,\rho} + | v_1(t,.) |_{\sigma,\rho} < R_0 \qquad .$$

Therefore if the initial data v_0, $u_0 \in X_{\rho_0}$ and v_0 satisfies (4.24), then we can take $R = R_0/2$ in the theorem 2.1, which gives us a constant $a > 0$ such that for any $0 < \delta \leq \delta_0$ there exists a unique solution $U(t)$ of (4.23) which is analytic in t, $|t| < a \ (\rho_0 - \rho)$, $0 \leq \rho < \rho_0$ with the value in $X_\rho \otimes X_\rho$ and has the bound

(4.26) $$\| U(t) \|_\rho \leq R \qquad .$$

<u>Theorem 4.1</u>.- Let v_0, $u_0 \in X_{\rho_0}$ for some $\rho_0 > 0$ and suppose (4.24). Then there exists a constant $a > 0$ such that for any $0 < \delta < \delta_0$ the solution $(v(t),u(t))$ of the Cauchy problem (4.1)-(4.3) exists uniquely, which is analytic of t, $|t| < a \ (\rho_0 - \rho)$, $0 \leq \rho < \rho_0$ with the values in $X_\rho \otimes X_\rho$ and has the bound

(4.27) $$\| v(t,.) - v_- \|_\rho \ , \| u(t,.) - u_- \|_\rho \leq R_0$$

$$\text{for} \quad |t| < a(\rho_0 - \rho), \ \ 0 \leq \rho < \rho_0 \qquad .$$

Remember that the solution of (4.1) depends on δ and write it by $(v^\delta(t),u^\delta(t))$, $\delta \in (0,\delta_0)$. By theorem 4.1., the solution $(v^\delta(t),u^\delta(t))$ has the uniform estimate (4.27) in the fixed region

$$D_{a,\rho_0} = \{ |t| < a(\rho_0 - \rho), \quad 0 \leq \rho < \rho_0 \}$$

independent of $\delta \in (0, \delta_0)$. Thus by the equation (4.1) we have the uniform estimate for the time derivative, independent of δ,

$$(4.28) \qquad \| v_t^{\delta}(t) \|_{\rho'} \quad , \; \| u_t^{\delta}(t) \|_{\rho'} \leq \frac{C R_0}{\rho - \rho'}$$

$$\text{for} \quad 0 \leq \rho' < \rho, \quad (t, \rho) \in D_{a,\rho_0} \; .$$

Therefore the Ascoli-Arzela's lemma for the space X_ρ gives the existence of a convergent subsequence $(v^{\delta'}(t), u^{\delta'}(t))$ in X_ρ as $\delta' \to 0$ uniformly on any compact in D_{a,ρ_0}, i.e.,

$$(4.29) \qquad \lim_{\delta' \to 0} (v^{\delta'}(t), \; u^{\delta'}(t)) = (v^0(t), u^0(t)) \quad .$$

Also by lemma 3.1 we have uniformly in any compact of D_{a,ρ_0}

$$(4.30) \begin{cases} \frac{1}{\delta'} A_{\delta'} u^{\delta'} = \frac{1}{\delta'} A_{\delta'} u^0 + \frac{1}{\delta'} A_{\delta'} (u^{\delta'} - u^0) \to u^0_\xi \quad , \\ \delta' C_{\delta'} v^{\delta'} = \delta' C_{\delta'} v^0 + \delta' C_{\delta'} (v^{\delta'} - v^0) \to \int_{-\infty}^{\xi} v^0 \, d\xi \quad , \end{cases}$$

$$\text{in } X_\rho \quad \text{as } \delta' \to 0 .$$

Along this subsequence we can pass to the limit of the integrated form in t of the equation (4.1) :

$$(4.31) \begin{cases} v^{\delta'}(t, \xi) = v_0(\xi) + \int_0^t \frac{\partial}{\partial \xi} \{ \dots \} \, ds \\ u^{\delta'}(t, \xi) = u_0(\xi) + \int_0^t - \frac{1}{\delta'} A_{\delta'} v - \frac{\partial}{\partial \xi} \{ \dots\dots \} \, ds \quad . \end{cases}$$

By (4.29) and (4.30) the integrands converge as $\delta' \to 0$ to

$$\begin{cases} \frac{\partial}{\partial \xi} \{ - v^0 \int_{-\infty}^{\xi} w^0 u^0_\xi \, d\xi \} , \qquad w^0 = 1/(v^0) , \\ - v^0_\xi - \frac{\partial}{\partial \xi} \{ \frac{w^0}{2} (u^0)^2 + u^0 \int_{-\infty}^{\xi} w^0 u^0_\xi \, d\xi \} \quad \text{respectively} \end{cases}$$

which are by (4.28) continuous of t, $|t| < a(\rho_0 - \rho)$ in X_ρ. Therefore as the limit of $\delta' = 0$ we have after differentiation in t

$$(4.32)\quad \begin{cases} y^0 = x^0_\xi \quad , \quad v^0 = x^0_\xi \; , \quad u^0 = \varphi^0_\xi \quad , \\ v^0{}_t = -\,(v^0 \int_{-\infty}^{\xi} (u^0_\xi \, (v^0)^2 d\,\xi)_\xi \quad , \\ u^0{}_t = -\,v^0_\xi \; - \; ((u^0)^2/2(v^0)^2 + u^0 \int_{-\infty}^{\xi} w^0 \, u^0_\xi \, d\,\xi\,)_\xi \quad , \end{cases}$$

which is the same as the differentiation in ξ of (3.48). Since lemma 4.1 (4.10) and (4.11), lemma 4.2 and lemma 4.3 are valid for $\delta = 0$, the limit equation (4.32) has the unique solution in X_ρ for D_{a,ρ_0} by the same abstract theorem. Thus the whole sequence v^δ, u^δ converges to this limit as $\delta \to 0$.

Theorem 4.2. - Let $v_0, u_0 \in X_{\rho_0}$ for some $\rho_0 > 0$ and suppose (4.24) is satisfied. Then there exists a constant $a > 0$ such that the solution $v^\delta(t)$, $u^\delta(t)$ of the Cauchy problem (4.1) - (4.3) converges to $v^0(t)$, $u^0(t)$ in X_ρ for $(t, \rho) \in D_{a,\rho_0}$ as $\delta \to 0$, the limit of which is the unique solution of (3.48), i.e., the corresponding $\Gamma^0(t,x)$ and $\Phi^0(t,x)$ satisfies the non-linear shallow water equation (3.16) (3.17).

Remark - In the solution above $y^0 = x^0_\xi\,(t,\xi) = v^0(t,\xi)$ and $\varphi^0_\xi\,(t,\xi) = u^0(t,\xi)$ may have the different values $v^0(t,\pm\infty) = v_0(\pm\infty)$, $u^0\,(t,\pm\infty) = u_0(\pm\infty)$ as $\xi \to \pm\infty$, which contains the shock wave type solution locally in time though.

There are many naive questions, for example,

(i) the convergence of the full series (3.13) ?

(ii) the limit globally in time as $\delta \to 0$, or the limit in the class of less regular functions as $\delta \to 0$? cf. Nalimov (1974).

(iii) Korteweg and de Vries equation as the limit ?

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CHAPTER IV BOLTZMANN EQUATION IN THE RAREFIED GAS DYNAMICS

1 - INTRODUCTION

The nonlinear hyperbolic conservation laws are treated more or less in chapter 1 for the macroscopic description of the compressible inviscid gas motion. The compressible viscous fluid motions are considered by Nash (1962), Itaya (1971,1976) and Tani (preprint) on the existence of solutions to the initial (-boundary) value problems locally in time. The global existence of the solutions to the Cauchy problem of some model compressible viscous fluid equations are given by Kanel' (1968) and Itaya (1976). The relations between the compressible Euler equation (nonlinear hyperbolic conservation laws) and the compressible Navier-Stokes equation (the compressible viscous fluid equations) are not well considered in general. cf. the last remark (iii) in chapter 1.

In contrast to the macroscopic descriptions of the gas motion mentioned above Boltzmann (1872) and Maxwell (1867) used the distribution function in velocity as well as physical space to describe the microscopic behavior of rarefied gas. Here we consider the initial value problem to the Boltzmann equation in the rarefied gas dynamics and the macroscopic limit as the mean free path ε tends to zero at the level of the compressible Euler equation. The dimensionless Boltzmann equation can be written for the mass density distribution function $F(t,x,v)$, $t \geq 0$, $x \in \mathbb{R}^3$: the space variable, $v \in \mathbb{R}^3$: the velocity variable, in the form : cf. Grad (1958)

$$\frac{\partial F}{\partial t} + \sum v_j \frac{\partial F}{\partial x_j} = \frac{1}{\varepsilon} Q(F,F) , \tag{1.1}$$

where ε is the mean free path and

$$Q(F,G) = \frac{1}{2} \int (F'G'_* + F'_* G - FG_* - F_* G) \quad Vr\, dr\, d\phi\, d v_* , \tag{1.2}$$

where $V = |v-v_*|$, v' and v'_* are the velocities after the collision of the molecules with the velocities v, v_*, r, ϕ are the polar coordinate in the impact plane, $F_* = F(t,x,v_*)$, $F' = F(t,x,v')$, $F'_* = F(t,x,v'_*)$ and G_*, G', G'_* are defined analogously.

Define the summational invariants

(1.3) $$\{\Psi_j\}_{j=0}^{4} \equiv \{1, v_j\ (j = 1,2,3),\ v^2\} \quad ,$$

which satisfy

(1.4) $$\int \psi_j \ Q(F,G)\, dv = 0 \qquad \text{for } j = 0,1,\ldots,4 \quad .$$

The macroscopic (hydrodynamical) quantities are defined as follows : The mass density and fluid flow velocity are given by

(1.5) $$\rho(t,x) \equiv \int F(t,x,v)\, dv,$$

(1.6) $$u(t,x) \equiv \frac{1}{\rho}\int v\, F(t,x,v)\, dv.$$

Set the velocity relative to the mean $c = v - u$. Then the stress tensor and heat-flow vector are defined by

(1.7) $$P_{ij} \equiv \int c_i\, c_j\, F(t,x,v)\, dv = p_{ij} + p\ \delta_{ij} \quad ,$$

(1.8) $$q_i = \frac{1}{2}\int c_i\, c^2\, F(t,x,v)\, dv,$$

where p is the scalar pressure $= \frac{1}{3}\sum p_{kk}$.

The internal energy per unit mass is

(1.9) $$e \equiv \frac{1}{\rho}\int \frac{1}{2}\, c^2\, F(t,x,v)\, dv \quad .$$

The conservation laws for ρ, u, e can be written in the form by (1.4)

(1.10) $$\begin{cases} \dfrac{\partial\rho}{\partial t} + \sum \dfrac{\partial \rho u_j}{\partial x_j} = 0 \\[2ex] \dfrac{\partial \rho u_i}{\partial t} + \sum \dfrac{\partial}{\partial x_j}(\rho u_i\, u_j + p_{ij} + p\,\delta_{ij}) = 0 \\[2ex] \dfrac{\partial}{\partial t}\rho(e + u^2/2) + \sum \dfrac{\partial}{\partial x_j}\left\{\rho u_j\ (e + u^2/2) + \sum u_k\ (p_{kj} + p\,\delta_{kj}) + q_j\right\} = 0, \end{cases}$$

where the equation of state of gas is that of the ideal gas, i.e., (1.11) $RT \equiv p/\rho = \frac{2}{3} e$.

If the distribution function F is locally Maxwellian, i.e.,

(1.12) $$F(t,x,v) = \frac{\rho(t,x)}{(2\pi RT(t,x))^{3/2}} \exp\left(-\frac{(u(t,x)-v)^2}{2RT(t,x)}\right) ,$$

then the conservation laws (1.10) can be simplified to $p_{ij} = q_i = 0$

(1.13) $$\begin{cases} \frac{\partial \rho}{\partial t} + \sum \frac{\partial \rho u_j}{\partial x_j} = 0 \\ \frac{\partial \rho u_i}{\partial t} + \sum \frac{\partial}{\partial x_j}(\rho u_i u_j + p\,\delta_{ij}) = 0 \\ \frac{\partial}{\partial t}\rho(e + u^2/2) + \sum \frac{\partial}{\partial x_j}(\rho u_j (e + u^2/2) + p u_j) = 0 , \end{cases}$$

which may be considered as the compressible Euler equation derived from the Boltzmann equation and is the same as the system for the ideal compressible gas motion in the Eulerian coordinate. The system (1.13) of the nonlinear hyperbolic conservation laws is also the first approximation of the Chapman-Enskog procedure. The second approximation of the Chapman-Enskog expansion is the compressible Navier-Stokes equation. cf. Chapman-Cowling (1952).

Following Grad (1963 a) we consider the Boltzmann equation (1.1) for the gas molecules with the cut-off hard potentials around the absolute Maxwellian state :

(1.14) $$M(v) = (2\pi)^{-3/2} \exp(-v^2/2) .$$

The initial value problem (1.1) with the initial data

(1.15) $$F(0,x,v) = F(x,v) ,$$

whose deviation from the absolute Maxwellian $M(v)$ is assumed small, was solved for fixed ε locally in time by Grad (1965) and globally in time by Ukai (1974, 1976), Nishida-Imai (1976) and then Shizuta (preprint). The solutions decay to the absolute Maxwellian $M(v)$ as t tends to infinity. It is described in § 3 after the preliminaries on the linearized Boltzmann equation in § 2. The initial-boundary value problem in the bounded domain is solved in the large in time for the small initial data by Guiraud (1974) with the boundary condition of random reflection and by Asano-Shizuta (1977) with the specular boundary condition. The stationary shock wave solutions are obtained by Nicolaenko-Thurber (1975) and by Nicolaenko (1974).

The asymptotic problem of the Boltzmann equation (1.1) as the mean free path ε tends to zero and the relations to the hydrodynamical equations by the Chapman-Enskog expansion are considered by Grad (1965) for the "semilinear" Boltzmann equation locally in time and by McLennan (1965), Ellis-Pinsky (1975) ans Pinsky (1976) for the linear Boltzmann equation. In § 4 we solve the initial value problem of the nonlinear Boltzmann equation for any $0<\varepsilon \leqslant 1$ with the small analytic initial data locally in time. This is done by use of the abstract Cauchy-Kowalewski theorem in the form of remark 2.3. chapter 3. Then in § 5, the asymptotics of the solutions as the mean free path ε tends to zero is investigated in a finite time interval and it is shown that the Boltzmann equation with small analytic initial data can be approximated locally in time as $\varepsilon \to 0$ by its compressible Euler equation (1.13).

2 - NOTATIONS AND LINEARIZED BOLTZMANN EQUATION

$x,v \in \mathbb{R}^3$ are the space - and velocity - variables and $k \in \mathbb{R}^3$ is the variable for the Fourier transform in x. $L^p(.)$ (. = x,v or k) denotes the Lebesgue space of measurable functions whose p-th power $(1 \leqslant p < +\infty)$ is summable in $\mathbb{R}^3$ with the norm $|f|_{L^p(.)}$. $H^\ell(x)$, $\ell \geqslant 0$ denotes the Sobolev space of $L^2(x)$-functions together with the ℓ-th derivatives, $\hat{H}^\ell(k)$ is the Fourier transform of $H^\ell(x)$ with the norm

$$|f|_{H^\ell(x)} = |(1+k^2)^{\ell/2} \hat{f}(k)|_{L^2(k)} = |\hat{f}|_{\hat{H}^\ell(k)} .$$

Let H be the Lebesgue space of square summable functions in $(x,v) \in \mathbb{R}^6$ with the norm

$$\|f\| = \left(\int |f(x,v)|^2 \, dx\, dv\right)^{1/2} . \tag{2.1}$$

Let us introduce the (partial) Fourier transform in x of $f \in H$ by

$$\hat{f}(k,v) = \frac{1}{(2\pi)^{3/2}} \int e^{-ik.x} f(x,v)\, dx \tag{2.2}$$

and denote $\hat{H} = \{\hat{f} \;;\; f \in H\}$ with the norm

$$(2.3) \qquad \|\hat{f}\| \equiv \left(\int |\hat{f}(k,v)|^2 \, dk \, dv\right)^{1/2} = \|f\| \quad .$$

Définition 2.1.- Let the Hilbert space H_ℓ , $\ell \geq 0$ be a subspace of H, which consists of $H^\ell(x)$-valued L^2-functions in $v \in \mathbb{R}^3$, i.e.,

$$(2.4) \begin{cases} H_\ell = L^2(v;H^\ell(x)) & \text{with the norm} \\ \|f\|_\ell \equiv \left(\int |f(.,v)|_{H^\ell(x)}^2 \, dv\right)^{1/2} & \end{cases}$$

$$= \left(\int (1+k^2)^\ell \, |\hat{f}(k,v)|^2 \, dk \, dv\right)^{1/2} = \|\hat{f}\|_\ell < +\infty \quad .$$

Also we use the space $L^2(v;L^p(x))$, $1 \leq p \leq 2$ which consists of $L^p(x)$-valued L^2-function in $v \in \mathbb{R}^3$ with the norm

$$(2.5) \quad \|f\|_{L^{2,p}} = \left(\int f(.,v)|_{L^p(x)}^2 \, dv\right)^{1/2} < +\infty \quad ,$$

where $H = L^2(v;L^2(x)) = H_0$.

Définition 2.2.- Let $B_{m,\ell}$, $m,\ell \geq 0$ be a subspace of H_ℓ , which consists of $H^\ell(x)$-valued continuous function in v, with the property

$$(2.6) \quad (1+v^2)^{m/2} \, |f(.,v)|_{H^\ell(x)} \to 0 \qquad \text{as} \quad |v| \to +\infty \quad .$$

The norm for $f \in B_{m,\ell}$ is defined by

$$(2.7) \quad \|f\|_{m,\ell} = \sup_v \, (1+v^2)^{m/2} \, |\hat{f}(.,v)|_{H^\ell(x)}$$

$$= \sup_v \, (1+v^2)^{m/2} \, |\hat{f}(.,v)|_{\hat{H}(k)} < +\infty \quad .$$

It is easy to see that by Fubini's theorem

$$(2.8) \quad \|f\|_\ell \leq C \, \|f\|_{m,\ell} \qquad \text{for } m > 3/2 \, , \quad \ell \geq 0 \quad ,$$

and that by Sobolev's lemma $f \in B_{m,\ell}$ for $m,\ell > 3/2$ is continuous in x and v.

Definition 2.3.- $S_\ell = \bigcup_{\rho \geq 0} H_{\ell,\rho}$ for some $\ell \geq 0$ is a scale of the Hilbert spaces such that $H_{\ell,0} = H_\ell$ and

$$(2.9) \qquad H_{\ell,\rho} = \left\{ f \in H_\ell \ ; \ |||f|||_{\ell,\rho} \equiv \| e^{|k|\rho} \hat{f}(k,v) \|_\ell < +\infty \right\}$$

$S_{m,\ell} = \bigcup_{\rho \geq 0} B_{m,\ell,\rho}$ for some $m,\ell \geq 0$ is a scale of Banach spaces such that $B_{m,\ell,0} = B_{m,\ell}$ and

$$(2.10) \qquad B_{m,\ell,\rho} = \left\{ f \in B_{m,\ell} \ ; \ |||f|||_{m,\ell,\rho} \equiv \| e^{|k|\rho} \hat{f}(k,v) \|_{m,\ell} < +\infty \right.$$

with the property

$$\left. \lim_{|v| \to +\infty} (1+v^2)^{m/2} | e^{|k|\rho} \hat{f}(k,v) |_{\hat{H}^\ell(k)} = 0 \right\}$$

Lemma 2.1.- The scale of the Hilbert space S_ℓ for any $\ell \geq 0$ has the property

$$(2.11) \qquad ||| |k|^\sigma \hat{f}(k,v) |||_{\ell,\rho'} \leq \frac{C}{(\rho-\rho')^\sigma} |||f|||_{\ell,\rho}$$

for any $f \in H_{\ell,\rho}$ and any $\rho' < \rho$, $0 < \sigma \leq 1$.

In order to linearize the Boltzmann equation (1.1) around the absolute Maxwellian M(v) we set

$$(2.12) \qquad F(t,x,v) = M + M^{1/2} f(t,x,v)$$

If we substitute (2.12) into (1.1) and follow Grad (1963) (1965) for the gas molecules with the cut-off hard potential, we have the equation for f(t,x,v)

$$(2.13) \qquad \frac{\partial f}{\partial t} + \sum v_j \frac{\partial f}{\partial x_j} = \frac{1}{\varepsilon} (L f + \nu \Gamma (f,f)) .$$

Here L is a nonpositive linear operator acting on $v \in \mathbb{R}^3$,

$$(2.14) \qquad (Lf,f)_{L^2(v)} \leq 0 \qquad \text{for} \qquad f, Lf \in L^2(v)$$

and

(2.15) $\qquad Lf = 0 \quad \text{iff} \quad f = \{\Psi_j\}_{j=0}^{4} = \{M^{1/2},\ v_j M^{1/2},\ v^2 M^{1/2}\}$.

It can be decomposed as

(2.16) $\qquad L = -\nu(v) + K \quad \text{in} \quad L^2(v)$

where $\nu(v)$ is a monotone nondecreasing function in $|v|$ and

(2.17) $\qquad 0 < \nu_0 \leq \nu(v) \leq \nu_1 (1 + |v|)$,

K is a compact self-adjoint operator on $L^2(v)$, which has the smoothing properties :

(2.18) $$\begin{cases} |||Kf|||_{j,\ell,\rho} \leq K |||f|||_{j-1,\ell,\rho} & \text{for any} \quad j \geq 1 , \\ |||Kf|||_{0,\ell,\rho} \leq K |||f|||_{\ell,\rho} & \end{cases}$$

for some constant $K = K(j) < +\infty$ and any $\ell \geq 0, \rho \geq 0$.

The nonlinear operator

(2.19) $\qquad \nu\Gamma(f,g) = \frac{1}{2} \int (f' g'_* + f'_* g' - fg_* - f_* g)\, M(v_*)^{1/2}\, V\, r dr d\phi dv_*$

acts on $v \in \mathbb{R}^3$ and is bilinear in f and g.

Lemma 2.2.- Let $f(x,v), g(x,v) \in B_{m,\ell,\rho}$ for some $m > 5/2, \ell > 3/2$ and $\rho \geq 0$. Then we have

(2.20) $$|||\nu\Gamma(f,g)|||_{\ell,\rho} \leq C |||\Gamma(f,g)|||_{m,\ell,\rho}$$

$$\leq C |||f|||_{m,\ell,\rho} |||g|||_{m,\ell,\rho} \qquad \text{and then}$$

$$(\nu\Gamma(f,g), \Psi_j)_{L^2(v)} = 0 , \qquad j = 0, 1, \ldots, 4.$$

Proof - The first inequality is easily obtained by (2.8) and (2.17). The second is proved by Grad (1965) and by Handsdorff-Young's inequality. In fact for $\ell = 2$ we have

$$|||\Gamma(f,g)|||_{m,2,\rho} = \sup_v (1+v^2)^{m/2} \| (1+k^2) \; e^{|k|\rho}\hat{\Gamma}(f,g)(k,v)\|_{L^2(k)}$$

$$\leq C \left\{ \sup(1+v^2)^{m/2} \| e^{|k|\rho} \hat{f}(k,v)\|_{L^1(k)} \right\} \cdot$$

$$\cdot \left\{ \sup (1+v^2)^{m/2} \| (1+k^2) \; e^{|k|\rho} \hat{g}(k,v)\|_{L^2(k)} \right\} +$$

$$+ C \left\{ \sup (1+v^2)^{m/2} \| (1+k^2) e^{|k|\rho} \hat{f}(k,v)\|_{L^2(k)} \right\} \cdot$$

$$\cdot \left\{ \sup (1+v^2)^{m/2} \| e^{|k|\rho} \hat{g}(k,v)\|_{L^1(k)} \right\}$$

$$\leq C \left\{ \sup (1+v^2)^{m/2} \| (1+k^2) \; e^{|k|\rho} \hat{f}\|_{L^2(k)} \right\} \cdot$$

$$\cdot \left\{ \sup (1+v^2)^{m/2} \| (1+k^2) \; e^{|k|\rho} \hat{g}\|_{L^2(k)} \right\}$$

$$= C \; |||f|||_{m,2,\rho} |||g|||_{m,2,\rho} \; .$$

Now our aim in this section is to summarize some results on the linear Boltzmann equation.

$$\frac{\partial f}{\partial t} = -\sum v_j \frac{\partial f}{\partial x_j} + \frac{1}{\varepsilon} Lf. \tag{2.21}$$

Consider two operators

$$\begin{cases} \frac{1}{\varepsilon} A_\varepsilon = -\sum v_j \frac{\partial f}{\partial x_j} - \frac{1}{\varepsilon} \nu(v) \; , \\ \frac{1}{\varepsilon} B_\varepsilon = -\sum v_j \frac{\partial f}{\partial x_j} - \frac{1}{\varepsilon} Lf \end{cases} \tag{2.22}$$

with the domain $D(\frac{1}{\varepsilon} A_\varepsilon) = D(\frac{1}{\varepsilon} B_\varepsilon)$ maximal in H_ℓ, $\ell \geq 0$.

$\frac{1}{\varepsilon} A_\varepsilon$ generates a strongly continuous semigroup in H_ℓ , i.e.,

$$e^{\frac{t}{\varepsilon} A_\varepsilon} f = e^{-\frac{t}{\varepsilon}\nu(v)} f(x-\frac{t}{\varepsilon} v,v) \tag{2.23}$$

$$= \frac{1}{(2\pi)^{3/2}} \int e^{ik.x} \; e^{\frac{t}{\varepsilon} A_{\varepsilon k}} \; \hat{f}(k,v) \, dk,$$

where

(2.24) $$A_{\varepsilon k} = - i\, \varepsilon\, k.v - \nu(v) .$$

Since $B_\varepsilon = A_\varepsilon + K$ and K is a bounded perturbation, the linear Boltzmann operator $\frac{1}{\varepsilon} B_\varepsilon$ generates also a strongly continuous semigroup

$\left\{ e^{\frac{t}{\varepsilon} B_\varepsilon} \right\}_{t \geq 0}$ in H_ℓ for any $\varepsilon \in (0,1]$.

Then we have

<u>Theorem 2.1</u>.- The linear Boltzmann semigroup is represented by

(2.25) $$e^{\frac{t}{\varepsilon} B_\varepsilon} f = \frac{1}{(2\pi)^{3/2}} \int e^{ik.x}\, e^{\frac{t}{\varepsilon} B_{\varepsilon k}}\, \hat{f}(k,v)\, dk$$

for $\hat{f}(k,v) \in \hat{H}_\ell$,

where for each $k \in \mathbb{R}^3$

(2.26) $$B_{\varepsilon k} = - i \varepsilon\, k.v - \nu(v) + K$$

is a unbounded linear operator in $L^2(v)$ with the definition domain $D(B_{\varepsilon k}) = \{ f \in L^2(v),\ B_{\varepsilon k} f \in L^2(v) \}$ and generates a strongly continuous semigroup such that for $f \in L^2(v)$

(2.27) $$| e^{\frac{t}{\varepsilon} B_{\varepsilon k}} f |_{L^2(v)} \leq |f|_{L^2(v)} .$$

Furthermore there exist $\delta, \beta_1, \beta_2 > 0$ such that the following (i) (ii) are valid for any $f \in D(B_{\varepsilon k})$.

(i) for any k, $|\varepsilon k| < \delta$

(2.28) $$e^{\frac{t}{\varepsilon} B_{\varepsilon k}} f = \sum_{j=1}^{5} e^{\frac{t}{\varepsilon}\alpha_j (\varepsilon k)} (e_j(-\varepsilon k), f)_{L^2(v)}\, e_j(\varepsilon k)$$
$$+ e^{\frac{t}{\varepsilon} A_{\varepsilon k}} f + e^{-\frac{t}{\varepsilon}\beta_1} Z_1(\varepsilon k, t/\varepsilon)\, f,$$

where α_j, e_j are the eignevalues and the eigenfunctions of $B_{\varepsilon k}$ such that

$$(2.29)\begin{cases} \alpha_j(\varepsilon k) = \sum_{n=1}^{3} a_{j,n}\,(i\,\varepsilon|k|\,)^n + O(|\varepsilon\, k|^4\,) \\ e_j\,(\varepsilon\, k) = \sum_{n=0}^{3} e_{j,n}\,(k/|k|\,)\,(i\varepsilon|k|\,)^n + O(|\varepsilon\, k^4), \end{cases}$$

$a_{j,n}$ are constants, $a_{j,2} > 0$ and

$$(e_j(-\varepsilon\, k),\ e_n(\varepsilon k))_{L^2(v)} = \delta_{j,n}\ , \qquad j,n = 1,\dots,5 \quad .$$

(ii) for any $k, |\varepsilon k| > \delta$

$$(2.30) \quad e^{\frac{t}{\varepsilon}B_{\varepsilon k}}\, f = e^{\frac{t}{\varepsilon}A_{\varepsilon k}}\, f + e^{-\frac{t}{\varepsilon}\beta_2}\, Z_2\,(\varepsilon k, t/\varepsilon\,)f,$$

where

$$Z_j\,(\varepsilon\, k, t/\varepsilon\,)f = \lim_{\gamma\to\infty} \frac{1}{2\pi}\int_{-i\gamma}^{i\gamma} e^{i\frac{t}{\varepsilon}\gamma}\, Z(-\beta_j + i\,\gamma,\ \varepsilon k)f\ \ d\,\gamma\,,$$

(2.31)

$$Z(\lambda, \varepsilon\, k) = (\lambda - A_{\varepsilon\, k})^{-1}\,(I-K(\lambda - A_{\varepsilon k})^{-1})^{-1}\, K\,(\lambda - A_{\varepsilon\, k})^{-1}$$

and

$$\|\, Z_j\,(\varepsilon\, k),\ t/\varepsilon\,)f\|_{L^2(v)} \leq C\ \|\, f\|_{L^2(v)} \qquad ,$$

where C is independent of ε , k , $t \geq 0$.

<u>Proof</u> - cf. Ellis-Pinsky (1975), Ukai (1976) and Nishida-Imai (1976).

3 - THE INITIAL VALUE PROBLEM OF THE BOLTZMANN EQUATION

First we obtain the decay of solutions to the initial value problem of the linear Boltzmann equation

(3.1) $$\frac{\partial f(t)}{\partial t} = - \sum v_j \frac{\partial f}{\partial x_j} + Lf \equiv Bf \quad ,$$

where and in this paragraph ε is assumed to be 1. Let the initial data

(3.2) $$f(0) = f(x,v) \in H_\ell \qquad \text{for some} \quad \ell \geq 0 \quad .$$

This Cauchy problem is solved by the linear Boltzmann semigroup in § 2, i.e.,

(3.3) $$f(t) = e^{tB} f \quad , \qquad \text{in} \quad t \geq 0 \quad ,$$

which is strongly continuous of $t \geq 0$ in H_ℓ . By theorem 2.1 and by Planchrel theorem we have

(3.4) $$\| f(t) \|_\ell \leq \| f \|_\ell \qquad \text{in} \quad t \geq 0 \; .$$

Theorem 3.1.

(i) Let the initials f belong to H_ℓ for some $\ell \geq 0$. Then the solution $f(t)$ of (3.3) decays to zero :

(3.5) $$\| f(t) \|_\ell \to 0 \qquad \text{as} \quad t \to + \infty$$

(ii) Let $f \in H_\ell \cap L^2 (v;L^1(x))$ for some $\ell \geq 0$ and

(3.6) $$\int \psi_j(v) \, f(x,v) \, dv = 0 \qquad \text{for a.a.} x \in \mathbb{R}^3, \qquad j = 0,1,\ldots,4.$$

Then the decay estimate has the order as follows :

(3.7) $$\| f(t) \|_\ell \leq \frac{C_1 \left(\| f \|_\ell + \| f \|_{L^{2,1}} \right)}{(1 + t)^{5/4}}$$

Remark - If $f \in H_\ell \cap L^2(v;L^p(x))$ for some $\ell \geq 0$ $2 \geq p \geq 1$, then the decay estimate is better than (i), i.e.,

$$\|\partial_x^\alpha f(t)\| \leq \frac{C\,(\|f\|_{|\alpha|} + \|f\|_{L^{2,p}})}{(1+t)^{\beta+|\alpha|/2}}$$

for $|\alpha| \leq \ell$,

where $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \dots (\partial/\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\beta = \frac{3}{2}(\frac{1}{p} - \frac{1}{2})$. And also these estimates can be carried to the solution of nonlinear Boltzmann equation, but here we restrict ourselves to the basic general case (i), (3.5). See Ukai (1976) and Nishida-Imai (1976).

Proof.- cf. Arseniev (1965), Scharf (1969) for the special case and Ukai (1976), Nishida-Imai (1976) in general. By Fubini theorem and by Planchrel theorem we can compute for $f(t)$

$$\|f(t)\|_\ell^2 = \iint (1+k^2)^\ell \,|e^{tB_k}\,\hat{f}(k,v)|^2\, dv\, dk$$

$$= \int_{|k|<\delta} \Big(\int |\quad|^2 dv\Big) dk + \int_{|k|>\delta} \Big(\int |\quad|^2 dv\Big) dk \equiv I + I_1,$$

where δ is defined in Theorem 2.1. By theorem 2.1 (ii) we get the estimate with $\beta_0 = \min(\beta_2, \nu(0))$

$$I_1 \leq C^2 e^{-2\beta_0 t} \int_{|k|>\delta} (1+k^2)^\ell\, |\hat{f}(k,.)|^2_{L^2(v)}\, dk$$

$$\leq C^2 e^{-2\beta_0 t} \|f\|_\ell^2 ,$$

which means the exponential decay.

By theorem 2.1 (i) for I we have

$$I = I_2 + I_3 ,$$

where the integrand of I_2 is the first term in the right hand side of (2.28) and that of I_3 is the second and third ones in that of (2.28) respectively.

Then theorem 2.1 (2.28), (2.31) with $\beta_0 = \min(\beta_1, \nu(0))$ gives

$$I_3 \leq C^2 e^{-2\beta_0 t} \int_{|k|<\delta} (1+k^2)^\ell\, |\hat{f}(k,.)|^2_{L^2(v)}\, dk$$

$$\leq C^2 e^{-2\beta_0 t} \|f\|_\ell^2 .$$

If we set $\alpha_0 = \min_{j=0,1,\ldots,4} \alpha_{j,2} > 0$ in (2.29)

we can calculate for I_2 as follows :

$$I_2 = \int_{|k|<\delta} (1+k^2)^{\ell} \Big| \Sigma e^{t\alpha_j(k)} (e_j(-k), \hat{f})_{L^2(v)} e_j(k) \Big|_{L^2(v)} dk$$

$$\leq C^2 \int_{|k|<\delta} (1+k^2)^{\ell} e^{-t\alpha_0 k^2} \, | \hat{f}(k,.)|^2_{L^2(v)} dk$$

$$\to 0 \quad \text{as} \quad t \to +\infty \quad ,$$

where the decay to zero is assured by Lebesgue theorem. The proof of (ii) is given in the same way, if we note that for $j = 0,1,\ldots,4$

$$(e_j(0), \hat{f}(k,.))_{L^2(v)} = \frac{1}{(2\pi)^{3/2}} \int e^{-ik.x} \Big(\int \psi_j \, f(x,v) \, dv\Big) \, dx = 0$$

and that

$$I_2 \leq C^2 \int_{|k|<\delta} (1+k^2)^{\ell} \Big| \Sigma e^{t\alpha_j(k)} (e_j(-k) - e_j(0), \hat{f})_{L^2(v)} e_j(k) \Big|^2_{L^2(v)} dk$$

$$\leq C^2 \int_{|k|<\delta} k^2 e^{-t\alpha_0 k^2} \, | \hat{f}(k,.)|^2_{L^2(v)} dk$$

$$\leq C^2 \sup_{|k|<\delta} | \hat{f}(k,.)|^2_{L^2(v)} \cdot \int_{|k|<\delta} k^2 e^{-t\alpha_0 k^2} dk$$

$$\leq C^2 \| f \|^2_{L^2(v;L^1(x))} / (1+t)^{5/2} \quad .$$

qed.

Before we treat the nonlinear Boltzmann equation we improve the decay estimates in theorem 3.1 into those in the space of $B_{m,\ell}$ $(m \geq 3, \ell \geq 2)$.

Definition 3.1.- $\mathring{\mathcal{E}}([0,\infty) ; X)$ denotes the space of functions $f(t)$ which is continuous of $t \in [0,\infty)$ with the values in the Banach space X and which decays to zero in X as $t \to \infty$. The norm of $f(t) \in \mathring{\mathcal{E}}([0,\infty);X)$, where $X = H_\ell$ or $B_{m,\ell}$, $(\ell, m \geq 0)$, is defined by

$$(3.8)\quad \begin{cases} ||| f(.) |||_{\ell} = \max_{0 \leq t < \infty} \| f(t) \|_{\ell} \\ ||| f(.) |||_{m,\ell} = \max_{0 \leq t < \infty} \| f(t) \|_{m,\ell} \end{cases}$$

Theorem 3.2.- The linear Boltzmann operator B generates the strongly continuous semigroup e^{tB} also in $B_{m,\ell}$ ($m \geqslant 2$, $\ell \geqslant 0$). Let $f \in B_{m,\ell}$. Then we have a constant $C_1 < +\infty$ such that for $f(t) = e^{tB} f$

$$(3.9)\quad \begin{cases} \| f(t) \|_{m,\ell} \leqslant C_1 \| f \|_{m,\ell} \quad \text{and} \\ \| f(t) \|_{m,\ell} \longrightarrow 0 \quad \text{as} \quad t \to \infty \quad . \end{cases}$$

Moreover define

$$(3.10)\qquad h(t) = \int_0^t e^{(t-s)B}\, \nu\Gamma(g(s), g(s))\, ds,$$

where $g(t) \in \overset{\circ}{\mathcal{C}}([0,\infty); B_{m,\ell})$ for some $m \geqslant 3$, $\ell \geqslant 2$. Then $h(t) \in \overset{\circ}{\mathcal{C}}([0,\infty); B_{m,\ell})$ and we have a constant $C_2 < \infty$ such that

$$(3.10)\qquad ||| h(.) |||_{m,\ell} \leqslant C_2\, ||| g(.) |||^2_{m,\ell}$$

Proof - Following Grad (1965) we use the representation

$$(3.11)\quad \begin{aligned} f(t) &= e^{tB} f + \int_0^t e^{(t-s)B}\, \nu\Gamma(g(s),g(s))\, ds \\ &= e^{tA} f + \int_0^t e^{(t-s)A}\, \nu\Gamma(g(s),g(s))\, ds + \int_0^t e^{(t-s)A}\, Kf(s)\, ds \quad . \end{aligned}$$

e^{tB} is a strongly continuous semigroup in $B_{m,\ell}$ ($m \geqslant 2$, $\ell \geqslant 0$) because of the definition of the space $B_{m,\ell}$ with (2.6) and by (3.11) with $g \equiv 0$ and (2.18). The decay (3.9) follows from (3.11) with $g \equiv 0$ and from (2.18) :

$$(3.12)\quad \begin{aligned} \| f(t) \|_{0,\ell} &\leqslant e^{-\nu_0 t} \| f \|_{0,\ell} + \Big\| \int_0^t e^{(t-s)A}\, Kf(s) ds \Big\|_{0,\ell} \\ &\leqslant e^{-\nu_0 t} \| f \|_{0,\ell} + C \Big(\int_0^{t/2} + \int_{t/2}^t \Big) e^{-\nu_0 (t-s)} \| f(s) \|_{\ell}\, ds \\ &\longrightarrow 0 \quad \text{as} \quad t \to \infty , \end{aligned}$$

where (3.5) is used. Successively for $j = 1,2,\dots,m$

$$(3.13)\qquad \|f(t)\|_{j,\ell} \leq e^{-\nu_0 t}\|f\|_{j,\ell} + C\left(\int_0^{t/2} + \int_{t/2}^{t}\right) e^{-\nu_0(t-s)} \|f(s)\|_{j-1,\ell}\, ds$$

$$\to 0 \quad \text{as} \quad t \to \infty .$$

For the latter half of the theorem we note that if $g(s)$ $B_{m,\ell}$ for some $m > 5/2$ $\ell > 3/2$, then by lemma 2.2.

$$\nu\Gamma(g(s),g(s) \in H_{m-1,\ell} \cap L^2(v;L^1(x)) \subset H_\ell \cap L^2(v;L^1(x)$$

and

$$(\nu\Gamma(g(s),g(s)),\ \psi_j)_{L^2(v)} = 0 \qquad , \qquad j = 0,1,\dots,4.$$

Thus the rapid decay estimate (ii) in theorem 3.1 applies to this case and we have

$$\|h(t)\|_\ell \leq \int_0^t \frac{C(\|\nu\Gamma(g(s),g(s))\|_\ell + \|\nu\Gamma(g(s),g(s))\|_{L^{2,1}}}{(1+t-s)^{5/4}}\, ds$$

$$\leq C\int_0^t \frac{\|g(s)\|_{m,\ell}^2}{(1+t-s)^{5/4}}\, ds \leq C\left(\int_0^{t/2} + \int_{t/2}^{t}\right) ds$$

$$= \frac{C(\max_{0\leq s\leq t/2}\|g(s)\|_{m,\ell})^2}{(1+t/2)^{1/4}} + C(\max_{t/2\leq s\leq t}\|g(s)\|_{m,\ell})^2$$

$$\leq C\,(|||g(.)|||_{m,\ell})^2 \quad \text{and tends to zero as} \quad t \to +\infty .$$

To get the decay of $h(t)$ in $B_{m,\ell}$ we use (3.11) with $f = 0$ in the same way as (3.12) (3.13).

$$\|h(t)\|_{j,\ell} \leq \sup_v \int_0^t e^{-(t-s)\nu(v)} \nu(v)\|\Gamma(g(s),g(s))\|_{j,\ell}\, ds$$
$$+ \int_0^t e^{-(t-s)\nu_0}\, C\|h(s)\|_{j-1,\ell}\, ds$$

$$\leq C\Big\{ \max_{0\leq s\leq t/2}\|g(s)\|_{m,\ell}^2 \cdot e^{-\nu_0 t/2} + \max_{t/2\leq s\leq t}\|g(s)\|_{m,\ell}^2$$

$$+ (\max_{0\leq s\leq t/2}\|h(s)\|_{j-1,\ell})\, e^{-\nu_0 t/2} + \max_{t/2\leq s\leq t}\|h(s)\|_{j-1,\ell} \Big\}$$

for $j = m, m-1, \ldots, 2, 1$ and also

$$\|h(t)\|_{0,\ell} \leq C\left\{\left(\max_{0\leq s\leq t/2}\|g(s)\|_{m,\ell}\right)^2 e^{-\nu_0 t/2} + \left(\max_{t/2\leq s\leq t}\|g(s)\|_{m,\ell}\right)^2\right.$$

$$\begin{cases} \leq C_2 |||g(\cdot)|||^2_{m,\ell} \\ \longrightarrow 0 \qquad \text{as} \quad t \to \infty \end{cases}.$$

qed.

Now we consider the nonlinear Boltzmann equation

(3.14) $$\frac{\partial f(t)}{\partial t} = B\, f(t) + \nu\Gamma(f(t),f(t)) \quad \text{in} \quad t \geq 0 ,$$

in the space $B_{m,\ell}$ $(m \geq 3, \ell \geq 2)$ with the initial condition

(3.15) $$f(0) = f(x,v) \in B_{m,\ell} \quad (m \geq 3, \ell \geq 2) .$$

The solution is constructed by the successive approximation $(n = 1,2,\ldots)$

(3.16) $$f^{(n)}(t) = e^{tB} f + \int_0^t e^{(t-s)B}\, \nu\Gamma(f^{(n-1)}(s), f^{(n-1)}(s))\, ds$$

in the Banach space $\mathring{\mathcal{C}}([0,+\infty); B_{m,\ell})$, $f^{(0)}(t) \equiv 0$. Let the initial data $f(0)$ have $E = \|f(0)\|_{m,\ell} < \infty$ for some $m \geq 3$, $\ell \geq 2$. Then by theorem 3.2 and by (3.16) we have for the same m, ℓ

$$\begin{cases} f^{(n)}(t) \in \mathring{\mathcal{C}}([0,\infty); B_{m,\ell}) \quad \text{and} \\ |||f^{(n)}(\cdot)|||_{m,\ell} \leq C_1 E + C_2 \left(|||f^{(n-1)}(\cdot)|||_{m,\ell}\right)^2 , \\ |||f^{(n+1)}(\cdot) - f^{(n)}(\cdot)|||_{m,\ell} \leq C_2 \left(|||f^{(n)}(\cdot)|||_{m,\ell} + |||f^{(n-1)}(\cdot)|||_{m,\ell}\right) \end{cases}$$

$$|||f^{(n)}(\cdot) - f^{(n-1)}(\cdot)|||_{m,\ell} .$$

for $n = 1,2,\ldots$.

Therefore if we suppose $0 \leq E < 1/4\, C_1 C_2$ and set $a = 1 - \sqrt{1 - 4 C_1 C_2 E} < 1$, we get

$$\begin{cases} |||f^{(n)}(.)|||_{m,\ell} \leq a/2\, C_2 \text{ and} \\ |||f^{(n+1)}(.) - f^{(n)}(.)|||_{m,\ell} \leq a|||f^{(n)}(.) - f^{n-1)}(.)|||_{m,\ell} \end{cases} .$$

Then $f^{(n)}(t)$ converges in $\overset{\circ}{\mathcal{E}}([0,\infty) ; B_{m,\ell})$ to $f(t)$, which is a unique solution of (3.14) (3.15) and decays to zero in $B_{m,\ell}$ as $t \to \infty$.

Theorem 3.3.- Let the initial data $f \in B_{m,\ell}$ for some $m \geq 3$, $\ell \geq 2$. Then there exists a constant $E_0 > 0$ such that if $E = \|f\|_{m,\ell} < E_0$, the solution $f(t)$ of Boltzmann equation (3.14) (3.15) exists in the space $B_{m,\ell}$ uniquely in the large in time and decays to zero as $t \to \infty$.

Remark 3.1.-

(i) Theorem 3.3. means that the solution to the initial value problem for Boltzmann equation (1.1) converges to the absolute Maxwellian distribution as $t \to \infty$, provided that the initial deviation from it is small in the norm of $B_{m,\ell}$ ($m \geq 3$, $\ell \geq 2$).

(ii) If $m \geq 3$ and $\ell \geq 3$, the solution is smooth and satisfies Boltzmann equation in the classical sense.

(iii) The uniqueness of the solution is just proved in a small (in the norm of $B_{m,\ell}$ ($m \geq 3$, $\ell \geq 2$)) neighbourhood of the absolute Maxwellian distribution cf. Shizuta (preprint).

4 - THE FLUID DYNAMICAL LIMIT OF BOLTZMANN EQUATION AT THE LEVEL OF COMPRESSIBLE EULER EQUATION

Let us consider the initial value problem for Boltzmann equation with $\varepsilon \in (0,1]$

$$(4.1) \quad \frac{\partial F_\varepsilon(t)}{\partial t} = - \Sigma v_j \frac{\partial F_\varepsilon(t)}{\partial x_j} + \frac{1}{\varepsilon} Q(F_\varepsilon(t), F_\varepsilon(t)) \quad \text{in} \quad t \geq 0 \quad ,$$

(4.2) $$F_{\varepsilon}(0) = F(x,v) \geq 0 \quad .$$

First we note the non-negativity of the solution $F_{\varepsilon}(t,x,v)$ for fixed $\varepsilon \in (0,1]$.

Theorem 4.1.- Let $F(x,v) = M(v) + M(v)^{1/2} f(x,v) \geq 0$ and $f(x,v) \in B_{m,\ell}$ for some $m \geq 3, \ell \geq 2$. Then there exist two constants $E_0 > 0$ and $t_0 > 0$ such that if $\| f \|_{m,\ell} < E_0$, then there exists a unique non-negative solution to (4.1) (4.2) in $0 \leq t \leq \varepsilon t_0$.

The solution is given by the iteration which preserves the non-negativity.

(4.3) $$\frac{\partial F_{n+1}(t)}{\partial t} + v.\ \frac{\partial F_{n+1}(t)}{\partial x} = \frac{1}{\varepsilon}\int (F'_{n,*}\ F'_n - F_{n,*}\ F_{n+1})\ d\omega$$

where $d\omega = V\ r\ dr\ d\phi\ d\ v_*$,

(4.4) $F_{n+1}(0) = F(x,v)$, $n = 0,1,2,\ldots$ and

(4.5) $F_0(t) = F(x,v) \geq 0$.

The proof of the convergence of the iteration uses a modified argument of Grad (1965). By the uniqueness of solutions near to the absolute Maxwellian for problem (4.1) (4.2) the solution as the limit of $n \to \infty$ coincides to the solution given by Grad (1965). See Nishida (preprint).

We seek the solution of (4.1) (4.2) in $0 \leq t < t_0$, where t_0 is independent of $\varepsilon \in (0,1]$, again around the absolute Maxwellian distribution, i.e., of the integral equation

(4.6) $$f_{\varepsilon}(t) = e^{\frac{t}{\varepsilon} B_{\varepsilon}} f(0) + \int_0^t e^{\frac{t-s}{\varepsilon} B_{\varepsilon}} \frac{1}{\varepsilon} \nu \Gamma(f_{\varepsilon}(s), f_{\varepsilon}(s))\ ds$$

for $\varepsilon \in (0,1]$.

Let $f(0) \in B_{m,\ell,\rho_0}$ for some $m \geq 3$, $\ell \geq 2$, $\rho_0 > 0$. The solution of (4.6) is sought in the Banach space $\mathbf{B}$, which is defined by

Definition 4.1.-

$\mathbf{B} = \{ f(t)$; continuous function of t with the values in $B_{m,\ell,\rho}$, which has the norm

$$(4.7)\qquad N_a[f] = \sup_{\substack{0\le \rho<\rho_0 \\ 0\le t<a(\rho_0-\rho)}} |||f(t)|||_{m,\ell,\rho}(1-t/a(\rho_0-\rho)) < \infty$$

for a suitable small $a > 0\}$.

Theorem 4.2.- Let the initial data have the norm

$$(4.8)\qquad E = |||f(0)|||_{m,\ell,\rho_0} < +\infty \qquad \text{for some } m \geq 3, \ell \geq 2, \rho_0 > 0 .$$

Then there exists $E_1 > 0$, $a > 0$ and $C_1 < \infty$ such that for any $f(0)$ with $E < E_1$ and for any $\varepsilon \in (0,1]$ the equation (4.6) has the unique solution $f_\varepsilon(t)$, which is continuous of t, $0 \leq t < a(\rho_0-\rho)$ with the values in $B_{m,\ell,\rho}$, $0 < \rho < \rho_0$ and has the uniform bounds

$$(4.9)\qquad |||f_\varepsilon(t)|||_{m,\ell,\rho_0} \leq C_1 E \quad \text{in} \quad 0 \leq t < a(\rho_0-\rho),\ 0 \leq \rho < \rho_0 ,$$

where C_1 is independent of $\varepsilon \in (0,1]$.

The proof of theorem 4.2 is based on the following proposition.

Proposition 4.1.- The solution of linear Boltzmann equation has a uniform estimate :

$$(4.10)\qquad |||e^{\frac{t}{\varepsilon}B_\varepsilon} f(0)|||_{m,\ell,\rho_0} \leq C|||f(0)|||_{m,\ell,\rho_0}$$

for $m \geq 3, \ell \geq 2, \rho_0 > 0$,

where C is independent of $\varepsilon \in (0,1]$. Furthermore let us consider the function for any $f(t), g(t) \in \mathbf{B}$, $m \geq 3, \ell \geq 2$,

$$(4.11)\qquad h(t) = \int_0^t e^{\frac{t-s}{\varepsilon}B_\varepsilon} \frac{1}{\varepsilon}\nu\Gamma(f(s), g(s))\, ds.$$

Then it has a uniform estimate

$$(4.12)\qquad N_b[h] \leq C R N_b[g] \leq C R B_a[g] \quad \text{for any } b < a ,$$

where $N_b[h]$ is defined by (4.7) with b replacing a and

$$(4.13) \qquad R = \sup_{\substack{0\leq s<b(\rho_0-\rho) \\ 0\leq\rho<\rho_0}} |||f(s)|||_{m,\ell,\rho} .$$

Proof of proposition 4.1.-

By theorem 2.1 (2.27) and Planchrel theorem we have

$$|||e^{\frac{t}{\varepsilon}B_\varepsilon} f(0)|||^2_{\ell,\rho_0} = \int |e^{\frac{t}{\varepsilon}B_{\varepsilon k}} e^{|k|\rho_0} (1+k^2)^{\ell/2} \hat{f}(0,k,v)|^2_{L^2(v)} dk$$

$$\leq |||f(0)|||^2_{\ell,\rho_0} \quad \text{for any } \ell\geq 0, \rho_0>0 .$$

It is improved to the estimate in the norm of B_{m,ℓ,ρ_0}, $m\geq 3$, if we remember the representation

$$(4.14) \quad e^{\frac{t}{\varepsilon}B_\varepsilon} f(0) = e^{\frac{t}{\varepsilon}A_\varepsilon} f(0) + \int_0^t e^{\frac{t-s}{\varepsilon}A_\varepsilon} \frac{K}{\varepsilon} (e^{\frac{s}{\varepsilon}A_\varepsilon} f(0)) \, ds$$

and the same argument used in the proof of (3.9).

The latter half of the proposition is proved as follows : since $(e_j(0), \nu\Gamma(f,g))_{L^2(v)} = 0$, $j = 1,2,\ldots,5$, we have by theorem 2.1

$$h(t) = \int_0^t \Big[\frac{1}{(2\pi)^{3/2}} \int_{|\varepsilon k|<\delta} \Big\{ \sum_{j=1}^5 e^{\frac{t-s}{\varepsilon}\alpha_j(\varepsilon k)} ik\, (e'_j(-\theta\varepsilon k), (\nu\Gamma)^\wedge)\, e_j(\varepsilon k)$$

$$+ e^{\frac{t-s}{\varepsilon}A_{\varepsilon k}} \frac{1}{\varepsilon} (\nu\Gamma)^\wedge + e^{-\frac{t-s}{\varepsilon}\beta_1} \frac{1}{\varepsilon} Z_1(\varepsilon k, t/\varepsilon)(\nu\Gamma)^\wedge \Big\} dk$$

$$+ \int_{|\varepsilon k|>\delta} \Big\{ e^{\frac{t-s}{\varepsilon}A_{\varepsilon k}} \frac{1}{\varepsilon} (\nu\Gamma)^\wedge + e^{-\frac{t-s}{\varepsilon}\beta_2} \frac{1}{\varepsilon} Z_2(\varepsilon k, t/\varepsilon)(\nu\Gamma)^\wedge \Big\} dk \Big] ds .$$

The norm in $H_{\ell,\rho}$ has the estimate by the same theorem

$$|||h(t)|||_{\ell,\rho} \leq C\int_0^t \Big[\Big(\int (1+k^2)^\ell e^{2|k|\rho} k^2 |(\nu\Gamma)^\wedge(s)|^2_{L^2(v)} dk \Big)^{1/2}$$

$$+ \frac{e^{-\frac{t-s}{\varepsilon}\beta_0}}{\varepsilon} |||\nu\Gamma(s)|||_{\ell,\rho} \Big] ds$$

$$\leq C\left\{\int_0^t \frac{|||f(s)|||_{m,\ell,\rho(s)}\,|||g(s)|||_{m,\ell,\rho(s)}}{\rho(s)-\rho}\,ds+\int_0^t \frac{e^{-\frac{t-s}{\varepsilon}\beta_0}}{\varepsilon}|||f(s)|||_{m,\ell,\rho}\,|||g(s)|||_{m,\ell,\rho}\,ds\right.$$

for some choice of $\rho(s)$, $\rho<\rho(s)<\rho_0 - s/a$, where we used lemmas 2.1 and 2.2.

It can be estimated by (4.13) in $0\leq t<b(\rho_0-\rho)$, $0\leq\rho<\rho_0$ for any $b<a$

$$|||h(t)|||_{\ell,\rho} \leq C\,R\,\left(\int_0^t \frac{|||g(s)|||_{m,\ell,\rho(s)}}{\rho(s)-\rho}\,ds+\int_0^t \frac{e^{-\frac{t-s}{\varepsilon}\beta_0}}{\varepsilon}|||g(s)|||_{m,\ell,\rho}\,ds\right)$$

$$\leq C\,R\,N_b[g]\left(\int_0^t \frac{ds}{(\rho(s)-\rho)(1-s/b(\rho_0-\rho(s)))}+\int_0^t \frac{e^{-\frac{t-s}{\varepsilon}\beta_0}}{\varepsilon}\,\frac{ds}{1-s/b\,(\rho_0-\rho)}\right)$$

with $\rho<\rho(s)<\rho_0 - s/b$.

Therefore if we choose $\rho(s) = (\rho_0 - s/b + \rho)/2$, we have

$$\sup_{\substack{0\leq\rho<\rho_0\\ 0\leq t<b(\rho_0-\rho)}} |||h(t)|||_{\ell,\rho}\,(1-t/b(\rho_0-\rho))\leq C(4b+1/\beta_0)\,R\,N_b[g] \tag{4.15}$$

In order to obtain the estimate for $N_b[h]$ from (4.15) we use the equivalent representation

$$h(t)=\int_0^t e^{\frac{t-s}{\varepsilon}A_\varepsilon}\frac{1}{\varepsilon}\nu\Gamma(f(s),g(s))\,ds+\int_0^t e^{\frac{t-s}{\varepsilon}A_\varepsilon}\frac{K}{\varepsilon}h(s)\,ds \tag{4.16}$$

and the same argument as that for (3.10). Thus we arrive at

$$N_b[h]\leq C\,R\,N_b[g]\leq C\,R\,N_a[g] \quad .$$

qed of proposition 4.1.

Now we introduce the same approximation as (3.16) to solve (4.6), i.e.,

$$f_0(t) = e^{\frac{t}{\varepsilon} B_\varepsilon} f(0)$$

$$g_0(t) = \int_0^t e^{\frac{t-s}{\varepsilon} B_\varepsilon} \frac{1}{\varepsilon} \nu\Gamma(f_0(s), f_0(s))\, ds,$$

$$f_1(t) = g_0(t) + f_0(t),$$

$$g_n(t) = \int_0^t e^{\frac{t-s}{\varepsilon} B_\varepsilon} \frac{1}{\varepsilon}\{\nu\Gamma(f_n(s), g_{n-1}(s)) + \nu\Gamma(g_{n-1}(s), f_{n-1}(s))\}\, ds,$$

$$f_{n+1}(t) = g_n(t) + f_n(t)$$

$$= f_0(t) + \int_0^t e^{\frac{t-s}{\varepsilon} B_\varepsilon} \frac{1}{\varepsilon}\nu\Gamma(f_n(s), f_n(s))\, ds,$$

$$n = 1,2,\ldots \quad .$$

It is easy from proposition 4.1 to see that

$$(4.18) \quad |||f_0(t)|||_{m,\ell,\rho} \leq C\, |||f(0)|||_{m,\ell,\rho} \leq C\, |||f(0)|||_{m,\ell,\rho_0} \equiv R_0$$

and

$$(4.19) \quad \mu_0 \equiv \sup_{\substack{0 \leq \rho < \rho_0 \\ 0 \leq t < a_0(\rho_0 - \rho)}} |||g_0(t)|||_{m,\ell,\rho} \leq C R_0^2$$

for any $a_0 > 0$.

Then it follows from (4.17) and (4.19) that

$$(4.20) \quad |||f_1(t)|||_{m,\ell,\rho} \leq R_0 + \mu_0$$

in $0 \leq \rho < \rho_0$, $0 \leq t < a_0(\rho_0 - \rho)$.

Define $a_1 = a_0 > 0$ and

$$(4.21) \quad a_{n+1} = a_n\,(1 - 1/(n+1)^2) \qquad \text{for } n = 1,2,\ldots$$

and

$$(4.22) \quad N_n[g] = N_{a_n}[g] \qquad \text{for } n = 0,1,2,\ldots \quad .$$

By use of proposition 4.1 and by the same argument as that for remark 2.3 chapter 3, we have for $k=1,2,\ldots$

$$(4.23) \qquad \mu_{k+1} \equiv N_{k+1} \left[g_{k+1} \right] \leq C\, R\, \mu_k \leq \frac{\mu_0}{(k+2)^4}$$

provided

$$(4.24) \qquad R < \left(\tfrac{2}{3}\right)^4 / C \qquad , \text{ and also}$$

$$(4.25) \qquad |||f_{k+1}(t)|||_{m,\ell,\rho} \leq R_0 + C\, R_0^2 \,\Sigma\, (j+1)^{-2} < R$$

$$\text{in } \; 0 \leq t < a_{k+2}(\rho_0 - \rho),\; 0 \leq \rho < \rho_0 \;,$$

provided that R_0 is small. Thus if we choose R_0 small, (4.24) and (4.25) are valid. Therefore there exists

$$\lim_{k \to \infty} f_{k+1}(t) = f(t) \qquad ,$$

the limit of which is the solution of (4.6) for $\varepsilon \in (0,1]$ and has the uniform bounds by (4.25)

$$(4.26) \qquad |||f(t)|||_{m,\ell,\rho} \leq R \qquad \text{in} \quad 0 \leq t < a(\rho_0 - \rho) \;,$$

where R and $a = \lim_{n \to \infty} a_n$ are independent of $\varepsilon \in (0,1]$.

qed of theorem 4.2

In order to take the limit of $f_\varepsilon(t)$ as $\varepsilon \to 0$ we need more than the uniform bounds (4.9). The uniform continuity in t is given by the following.

Theorem 4.3.- Let the initial data $f(0) \in B_{m,\ell,\rho_0}$ for some $m \geq 3, \ell \geq 2$, $\rho_0 > 0$ and let

$$E = ||f(0)||_{m,\ell,\rho_0} < E_1 \qquad ,$$

where E_1 is defined in theorem 4.2. Then there exist constants $0 < E_2 \leq E_1$ and $C_2 < \infty$ such that if $E < E_2$, then the solution $f_\varepsilon(t)$ of (4.6) has the uniform Hölder-continuity in t :

(4.7) $$|||f_\varepsilon(t) - f_\varepsilon(s)|||_{m-\sigma,\ell-\sigma,\rho} \leq C_2 E \left\{ \frac{t-s}{s(1-t/a(\rho_0-\rho))} \right\}^\sigma$$

for $0 < s < t$ and for a fixed $\sigma \in (0,1/2)$,

where C_2 is independent of $\varepsilon \in (0,1]$.

The proof needs the Hölder-continuity of the solution for the linear Boltzmann equation and of the function h(t) (4.11). See Nishida (preprint). We only remark that the Hölder-coefficient has the singularity of $1/t^\sigma$ as $t \to 0$, which corresponds to the initial layer of the rarefied gas motion described by Boltzmann equation.

It follows from theorems 4.2 and 4.3 that by Ascoli-Arzela lemma we can choose a convergent subsequence as $\varepsilon \to 0$ such that

(4.28) $$f_\varepsilon(t) \to f_0(t) \quad \text{in } B_{m-\sigma,\ell-\sigma,\rho}, \quad 0 \leq t < a(\rho_0-\rho), \ 0 \leq \rho < \rho_0 .$$

The limit function has the bound

(4.29) $$|||f_0(t)|||_{m,\ell,\rho} \leq C_1 E \quad \text{in } 0 \leq t < a(\rho_0-\rho), \quad 0 \leq \rho < \rho_0 .$$

and the Hölder-continuity of $\sigma \in (0,1/2)$.

(4.30) $$|||f_0(t) - f_0(s)|||_{m-\sigma,\ell-\sigma,\rho} \leq C_2 E \left\{ \frac{t-s}{s(1-t/a(\rho_0-\rho))} \right\}^\sigma .$$

Now we turn to the original mass density distribution function

(4.31) $$F_\varepsilon(t,x,v) = M(v) + M(v)^{1/2} f_\varepsilon(t,x,v) \quad ,$$

which satisfies Boltzmann equation (4.1) (4.2). Taking the limit of the equation (4.1) in the integrated form in t along the subsequence (4.28) as $\varepsilon \to 0$, we have by the uniform bound (4.29)

(4.32) $$Q(F_0(t,x,v), F_0(t,x,v)) = 0 \quad \text{in } 0 < t < a\rho_0 \quad ,$$

where $$F_0(t,x,v) = M(v) + M(v)^{1/2} f_0(t,x,v) \quad .$$

If we assume that $F(0,x,v) = F(x,v) \geq 0$ and $\rho(0,x) = \int F(x,v)\,dv > 0$ in $x \in \mathbb{R}^3$, the solution has the same properties by theorem (4.1) and by the mass conservation laws (1.10) :

(4.33) $\qquad F_0\ (t,x,v) \geqslant 0$

(4.34) $\qquad \rho\,(t,x) = \int F_0\ (t,x,v)\ dv > 0 \qquad .$

It follows from (4.32) (4.33) (4.34) that $F_0(t,x,v) > 0$ and then $F_0(t,x,v)$ is locally Maxwellian. Thus we can obtain the conservation laws (1.13) for $F_0\ (t,x,v)$ from (1.10) for $F_\varepsilon\ (t,x,v)$ as the limit of $\varepsilon \to 0$ along the subsequence of (4.28). The uniqueness of the solution to the initial value problem (1.13) guarantees the convergence of full sequence F_ε to F_0 as $\varepsilon \to 0$.

Theorem - Let the initial data $F(x,v) = M(v) + M(v)^{1/2}\ f(x,v) \geqslant 0$ with $\rho(0,x) = \int F(x,v)\ dv > 0$ in $x \in \mathbb{R}^3$, and let $f(x,v) \in B_{m,\ell,\rho_0}$ for some $m \geqslant 3$, $\ell \geqslant 2$, $\rho_0 > 0$ and set $|||f|||_{m,\ell,\rho_0} = E$. If $E < E_2$, where E_2 is defined in Theorem 4.3., then the solution $F_\varepsilon\ (t,x,v)$ of Boltzmann equation (4.1) (4.2) exists uniquely in $B_{m,\ell,\rho}$, $0 \leqslant t < a(\rho_0 - \rho)$, $0 \leqslant \rho < \rho_0$ for any $\varepsilon \in (0,1]$ and is non-negative there, where a is defined in Theorem 4.2. Furthermore there exists

$$\lim_{\varepsilon \to 0} F_\varepsilon\ (t,x,v) = F_0\ (t,x,v) \quad \text{in} \quad B_{m,\ell,\rho}\ , \quad 0 < t < a(\rho_0 - \rho), \quad 0 < \rho < \rho_0 \quad ,$$

where $F_0(t,x,v)$ is locally Maxwellian distribution. Therefore its fluid dynamical quantities satisfy the conservation laws (1.13).

At last we note that the system (1.13) with (1.11) is hyperbolic and has two genuinely nonlinear characteristic fields (cf. Chapter 1), and so it developes in general shock waves in finite time even for the analytic initial data.

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