FOURIER SERIES AND WAVELETS

Jean-Pierre Kahane
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PART I. - FOURIER SERIES

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PART II. - WAVELETS

Preface

Like the others in this collection, the present book has different aspects : history, classical mathematics, contemporary mathematics. The period of time extends from 1807, when Fourier wrote his first memoir on the Analytic Theory of Heat, to 1994 and the last developments on wavelets. The work is divided into two parts. The first, written by Jean-Pierre Kahane, deals with Fourier series in the classical sense, decomposition of a function into harmonic components. The second, by Pierre-Gilles Lemarié-Rieusset, expounds the modern theory of wavelets, the most recent tool in pure and applied harmonic analysis. There is an interplay between these two topics. Some common features appear in their history, their linkage with physics and numerical computation, their role and impact in mathematics. Since the first part is more classical, emphasis was put on the historical aspect ; how problems appear and move in the course of time. The history being shorter in the second part, the purely mathematical exposition - including original contributions - plays a central role.

From the Fourier point of view mathematical analysis originates from the study of Nature and expresses natural laws in the most general and powerful way. At first, Fourier series are a general method, including a good numerical algorithm, to describe and to compute the functions which occur in the heat diffusion and equilibrium. Then they become an interesting object by themselves and the germ of new theories, developed by the followers of Fourier. In succession we see Dirichlet and the convergence problem, Riemann and real analysis, Cantor and set theory, Lebesgue and functional analysis, probabilistic methods, algebraic structures. Classical Fourier series are still a seminal branch of modern mathematics, as well as a tool of constant use by physicists and engineers. The fast Fourier transform extended this use enormously in the past thirty years.

Interaction with physics and construction of efficient algorithms for numerical computation, which appear in Fourier series from the very beginning, are also at the heart of wavelet theory. Here the initiators were engineers and physicists, and mathematicians came later. But in no time wavelets became a unifying language and method outside and inside mathematics. Now they play a decisive role in the new network which expands between mathematical analysis, theoretical physics, signal analysis, image analysis, telecommunications, fast methods of computation, thanks to which new applications were found for purely mathematical theories.

The book is meant to give an idea of these movements as well as solid information on Fourier series and the state of the art about wavelets. On these matters the authors have personal experience and personal views. This is clear in the choice of the original papers by Fourier, Dirichlet, Riemann, Cantor, reproduced in the first part of the book, as in the choice and treatment of purely mathematical questions, both in the first and second parts.

The authors are grateful to a number of colleagues and collaborators for their help in scientific, linguistic or bibliographic matters, among others Fan Ai-hua, Olivier Gebuhrer, Monique Hakim, Geoffrey Howson, Lee Lorch, Yves Meyer, Hélène Nocton, Hervé Queffélec, Jean-Bernard Robert, Jan Stegeman, Guido Weiss.

Josette Dumas had to convert our handwritten manuscripts into a real book. If the reader appreciates the presentation of our work the merit belongs to her.

The figures of Part II haven been drawn with help of MICRONDE, a software developed at Orsay by Y. and M. Misiti, G. Oppenheim and J. M. Poggi as a preliminary version of a MATLAB wavelet toolbox.

FOURIER SERIES

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Introduction

WHAT ARE FOURIER SERIES ABOUT ?

The subject matter of Fourier series consists essentially of two formulas :

$$f(x) = \sum c_n e^{inx} \tag{1}$$

$$c_n = \int f(x) e^{-inx} \frac{dx}{2\pi}. \tag{2}$$

The first involves a series and the second an integral.

One can look at these formulas in two ways. First, we can start with a convergent trigonometric series and define $f(x)$ as its sum. What kind of functions do we obtain through formula (1) ? Being given such a function, are the coefficients well defined ? Can they be computed by formula (2) ? This circle of ideas was initiated by Riemann and continued by Cantor, Lebesgue, Denjoy. Strikingly enough, the Riemann integral, the Lebesgue integral, the Denjoy integral were either introduced in relation to or immediately applied to this specific question. It is also the core of the Cantor theory of uniqueness of trigonometric series, the source of the Cantor theory of transfinite induction. In the twentieth century this was continued mainly by the Polish and Russian schools. Trigonometric series, as a subject in pure mathematics, provide an interesting and constant interplay among different theories : functions of a real variable, functions of a complex variable, theory of sets, theory of numbers.

Secondly, we can start with a function $f(x)$, apply formula (2) and look at the series in (1). Does it actually converge to $f(x)$? This is the point of view of Fourier. Fourier says that (2) can be applied to an arbitrary function, and that it is possible to prove that the series in (1) converges to that function. He was wrong : first, the integrable functions should be defined ; then, the convergence of Fourier series is a delicate matter, where difficult theorems coexist with strange counter-examples. The names of Dirichlet, du Bois-Reymond, Kolmogorov, Carleson, are associated with the main landmarks in this respect. However the intuition of Fourier was essentially right. Given a periodic phenomenon f, with period 2π (it can be a function, a measure, a distribution in the sense of Schwartz, maybe something else), it is simply the duty of mathematicians to define a notion of integral such that (2) has a meaning ; the c_n are called the Fourier coefficients. Then, the series in (1) is defined in a formal way ; it is called the Fourier series of f. Again, it is the duty of mathematicians to say how to obtain f from its Fourier series : summability methods and convergence in functions spaces fit this purpose exactly. Actually Fourier series were one of the sources of functional analysis.

Fourier had predecessors. Daniel Bernoulli had the idea of expressing the solution of the problem of vibrating strings with the help of trigonometric series - that is, to express the motion of the string as the superposition of motions corresponding to pure harmonics. Euler applied formula (2) in particular cases. But, as Riemann observes, Fourier was the first to consider (1) and (2) as a whole : you analyse f through the Fourier formulas (2), you synthesize f through its Fourier series in (1). Analysis and synthesis are two complementary aspects of what is now called harmonic analysis.

Moreover, the motivation of Fourier was not the theory of vibrating strings - where harmonics introduce themselves in a rather natural way. It was the theory of heat. Fourier series is only a part of a large project : to understand and predict heat diffusion. Fourier had to build a mathematical model for the propagation of heat. It is the so called heat equation, or Fourier equation. He had to show how to use it in a number of particular cases. Fourier series are a practical tool for the computation of the temperature at a given point, in a given body, with given boundary conditions. This will be explained by Fourier himself when we quote him in this book.

Therefore, the heritage of Fourier is not only the collection of problems, results, theories, concepts coming from formulas (1) and (2). It is more their *raison d'être*. Fourier addressed an important problem of nature. He was able to construct a good mathematical model. Then he wanted a general and powerful method to solve a type of equation : it was formulas (1) and (2). The tool being discovered, he tried to point out its actual extension as well as its precision in order to perform numerical computations. Mathematical modeling and algorithmic mathematics are part of the heritage of Fourier.

In particular, other connections with physics and numerical methods will be mentioned in this book. The historical part goes from Fourier to fast Fourier transform. However the unity of the book - expressed in its title - is the correspondence between the historical part and a self-contained mathematical exposition of the quite recent theory of wavelets. Wavelets are linked with physics and engineering; they are now the most general and powerful method for a scaled harmonic analysis needed in many ways, inside and outside mathematics. Of course, analysis and synthesis will appear in different forms, but the trace of formulas (1) and (2) will be quite visible.

It is time to say that formulas (1) and (2) were never written by Fourier. Complex exponentials were not used in Fourier series until well into the twentieth century. Therefore, in the historical part, (1) and (2) may appear in the form

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{3}$$

$$\begin{cases} a_n = \dfrac{1}{\pi}\displaystyle\int f(x) \cos nx \, dx \\ b_n = \dfrac{1}{\pi}\displaystyle\int f(x) \sin nx \, dx \end{cases} \tag{4}$$

or any equivalent form.

Chapter 1

WHO WAS FOURIER ?

Fourier is a pure product of the French Revolution. His life is a very interesting cross-section of French history in the years 1770-1830.

Joseph Fourier was born on the 21st of March, 1768, in Auxerre, on the Yonne, in a rather poor family. His grand-parents were peasants, his father a tailor, with three children from a first marriage and thirteen (among them, Joseph) in a second. Both his father and mother died when he was very young. At the age of ten years he was an orphan, but already noticed as a bright boy. He was taught French and Latin by the organist of the cathedral, then sent to the Military College of Auxerre.

The military colleges had just been created in twelve small towns in order to replace the Military School in Paris, founded in 1760. Among them, Beaumont and Brienne are known now because Laplace and Bonaparte studied there. In Auxerre the college was run by Benedictine monks and the principles of education were quite progressive : to mix children of different classes, to practise living languages, German in particular, as well as Latin, to be familiar with the use of geographical maps. Corporal punishment was forbidden but there was rather hard physical training. Joseph Fourier was happy and successful. He read Latin and occasionally wrote sermons (Arago says that some of them were delivered by high Church officials in Paris who did not bother writing sermons themselves). He discovered mathematics in the books of Bezout and Clairaut, worked day and night, graduated from college at fourteen, and was appointed as a teacher in the same college at sixteen and a half. It was a bright record, but not yet a stable position.

For the college graduates there were two possible paths : Army or Church. Fourier chose to serve in the artillery, the most scientific arm. He was backed by Legendre, who already knew the interest and first works of Fourier on the localization of roots of an algebraic equation. But Fourier was poor and of poor origin. The minister of war answered Legendre : *"Fourier n'étant pas noble ne pourrait entrer dans l'artillerie, quand même il serait un second Newton"* (were he even a second Newton, Fourier could not enter artillery since he is not a noble). Fourier had to choose the Church.

He entered the Benedictine abbey of Saint-Benoit at Fleury, near Orleans, as a novice and mathematics teacher. From 1787 to 1789 he stayed there in complete isolation. Complete isolation is what his letters express. However the circumstances of his departure from Fleury suggest that he had some connection with the external world. He was supposed to take his vows on November 5, 1789. On November 2 the National Assembly ordered a suspension of religious vows. Fourier learned of this and, though already known as *"abbé Fourier"*, decided to give up the vows *"par respect pour les décrets de l'Assemblée Nationale"* (in consideration of the decrees of the National Assembly), and to leave Fleury.

He settled back in the military school at Auxerre as a humanities teacher. Already on December 9, 1789, he had presented a memoir on algebraic equations at the Academy of Sciences and the work was refereed by Monge, Legendre and Cousin. He was not involved in public affairs until 1793. This is the year which begins with the execution of Louis XVI and goes on to the European coalition against the French Republic, the uprisings in Vendée and many other regions, the mass levy of 300,000 men, and the revolutionary committees. In this most troubled time Fourier got involved and proved active and efficient by taking part in the revolutionary committee in Auxerre, recruiting volunteers in Burgundy, organizing food and military supplies in Orléans, moderating excesses in many places. In a letter written some two years later he explains his position :

> *"à mesure que les idées naturelles d'égalité se développèrent on a pu concevoir l'espérance sublime d'établir parmi nous un gouvernement libre exempt de rois et de prêtres, et d'affranchir de ce double joug la terre d'Europe depuis si longtemps usurpée. Je me passionnai aisément pour cette cause, qui est selon moi la plus grande et la plus belle qu'aucune nation ait jamais entreprise".*
>
> (as the natural ideas of equality were developing it became possible to conceive the sublime hope to establish among us a free government and to deliver the European soil from the kings and priests who have usurped it for so long. I became passionately fond of this cause, which I consider as the greatest and the best ever attempted by any nation).

Fourier's mission to Orléans, a very troubled city, proved very successful as far as food and military supplies were concerned. However - maybe therefore - Fourier, who was on the *"sans-culottes"* side, was denounced to the *Convention*, arrested, liberated, arrested again, and once more liberated after the fall of Robespierre in July 1794. He returned to Auxerre and chose a newly created position : *"instituteur salarié par la nation"* ("State paid teacher" is not a good translation, since *nation* here evokes people as well as structure).

Then a new life began. In October 1794 the *Ecole normale* was created. The pupils were selected on a local basis, taking into account both involvement in teaching and devotion to revolutionary ideals. Fourier, *"professeur de physique et d'éloquence au collège national établi à Auxerre"*, was chosen and sent to Paris in December. There were 1,500 students and a handful of teachers : Lagrange, Laplace, Monge, Haüy, Berthollet, the greatest scientists of the time. The students were not well prepared - Fourier was an exception -, the weather was cold, the amphitheatre (still existing, in the Museum of Natural History) was small, it was difficult to hear and to be heard, but the lectures were superb, and followed by debates between teachers and pupils. Traces exist. Not only were some lessons published, those of Laplace in particular, but stenographic records were taken - they waited until 1992 before being published (Dhombres 1992). For example, we see a discussion between Citizen Fourier and Citizen Monge on the definition of planes - Fourier, addressing Monge as *vous*, insisting that such a definition is necessary, as well as the definition of spheres given in Monge's course, and proposing to define planes using distances ; Monge, addressing Fourier as *tu*, acknowledging Fourier's point of view but keeping his, taking the plane as a primary notion. Obviously Fourier took advantage of the opportunities presented. However, for most students the level was too high. Winter cold, poor conditions, discrepancy between students and teachers resulted in the collapse of the school. The *Ecole normale* was recreated later on a more selective basis.

In the meantime the *Convention* founded the *Ecole polytechnique* as a science and military school. Fourier, highly appreciated by students and colleagues, taught there from 1795 to 1798 on several topics : differential calculus, integral calculus, statics, dynamics, hydrostatics, probabilities.

The first paper which he published, *"Mémoire sur la statique..."*, was printed in *Journal de l'Ecole Polytechnique* in 1798. His courses at the *Ecole Polytechnique* contained also original results, such as his theory of localisation of real roots of an algebraic equation.

The scene changed again when Napoléon Bonaparte, who was about the same age as Fourier, led the French expedition to Egypt. Bonaparte had been elected a member of the *Institut de France*, probably because he brought back from Italy some geometrical constructions where ruler and compass are replaced by compass alone. He was proud of this membership and made use of it. In Egypt, he signed his orders : *"le membre de l'Institut, commandant l'armée d'Orient"*. He founded a copy of the French Institute in Cairo. Gaspard Monge was elected as President of this *Institut d'Egypte* and Fourier as *Secrétaire perpétuel*. Bonaparte left Egypt one year later to become First Consul in Paris, then emperor. Monge also went back to Paris. Fourier had increasing duties. He organized the activities of *Institut d'Egypte*, edited the proceedings in the journal *La Décade égyptienne*, wrote research papers on a great variety of subjects, from oases to the theory of equations, presented obituaries of the French generals Kléber and Desaix, wrote articles for *Le Courrier de l'Egypte*, directed a scientific expedition in Upper Egypt investigating the monuments and inscriptions, and conducted diplomatic negotiations, first with Mourad-Bey through his beautiful and celebrated wife Sitty Néfiçah, then with the English forces when the French had to withdraw from Egypt. The information he gathered was collected later in a monumental work called *Description de l'Egypte*, published in 1809, for which Fourier wrote an extensive *Préface historique*. Fourier's contributions qualified him as an important Egyptologist.

After his return to France, at the end of 1801, Napoléon gave him an important position as *Préfet de l'Isère*. It could have been the end of his scientific activities, but it was not. Not only did he write thousands of pages on Egypt, but also his main mathematical work, on the theory of heat, when he was prefect in Grenoble. Moreover, he was a good politician and administrator, supervising schools, mines, roads, health and agriculture. One of his achievements was to effect the long-wanted draining of the marshes in Bourgoin : political power was not sufficient, scientific competence and diplomatic talent were needed. He was clever enough to have good collaborators, to make friends in Grenoble (among them the Champollion family, whom he introduced to archeology), and to save time for his own scientific work.

Before Fourier, the theory of heat was not a very clear matter. The French Academy had proposed a competition in 1736 on the theme : *étude de la nature et de la propagation du feu*, meaning the nature and propagation of heat. But all candidates, including Euler and Voltaire, misunderstood the question and treated how fires develop. At the end of the 18th century there were debates on the nature of *"calorique"*, the term for heat in learned French. Is it a substance like a chemical element ? The approach of Fourier is quite new : whatever may be the nature of heat, here is the way it should evolve. He does not speak of *calorique* any more : he writes about *la propagation de la chaleur.* This is an important topic from many points of view : temperature in houses, and other such practical matters, and also the most fundamental questions on the origin of heat on the Earth and the evolution of the Solar system.

Fourier sent a first *Mémoire sur la propagation de la chaleur dans les corps solides* to the *Institut de France* in December, 1807. The astronomer Delambre was *secrétaire perpétuel.* Delambre asked Lagrange, Laplace, Lacroix and Monge to read the paper. Apparently Lagrange strongly opposed what Fourier wrote on trigonometric series. The monograph was not published. Only a short and biased review by Denis Poisson, a new mathematical star at the time, appeared in the *Bulletin de la Société Philomathique.*

Then the subject was proposed for a competition. Fourier wrote an extended manuscript and sent it just in time, at the end of September, 1811 with a beautiful quotation attributed to Plato : *et ignem regunt numeri* (meaning that also heat is governed by numbers). Again Lagrange, Laplace and Lacroix were examiners. Fourier won the Prize, but clearly not the complete approval of the Jury :

> *"cette pièce renferme les véritables équations différentielles de la transmission de la chaleur, soit à l'intérieur des corps, soit à leur surface : et la nouveauté du sujet, jointe à son importance, a déterminé la Classe à couronner cet Ouvrage, en observant cependant que la manière dont l'Auteur parvient à ses équations n'est pas exempte de difficultés, et que son analyse, pour les intégrer, laisse encore quelque chose à désirer, soit relativement à la généralité, soit même du côté de la rigueur."*

In brief, the Fourier heat equation was crowned, but Fourier series were considered as a disgrace. Again, the *mémoire* was not published.

Fourier - then Baron Fourier - went on working on the topic, preparing a printed version of his manuscript. Only in 1822 was the *Théorie analytique de la chaleur*, containing the whole mathematical theory, including Fourier series and Fourier integrals, published.

In the meantime many things had changed in France and for Fourier. Napoléon abdicated in 1814, came back from exile in 1815, crossed Grenoble, conquered France and stayed in power for hundred days, was defeated at Waterloo and sent to the remote island of *Sainte-Hélène*, while King Louis XVIII seized, left, and seized the crown again, with the help of the European Allied forces and to the great joy of the émigrés. Fourier kept his office during the first Restoration. He opposed Napoléon, was dismissed one day as *Préfet de l'Isère*, and installed the following day as *Préfet du Rhône* in Lyons, a much more important place than Grenoble. He resigned just before the second Restoration.

Then he settled in Paris. With the help of a former student he obtained a very quiet position as director of the Bureau of Statistics in Paris, which he kept for life. In May 1816 he was elected to the *Institut de France* but Louis XVIII refused to approve the election - Monge already had been dismissed. However, after the recreation of the *Académies* in place of the classes of *Institut de France*, Fourier was elected again, in 1817, and accepted by the King. Five years later, he became *secrétaire perpétuel*.

Around 1820 Fourier had good friends in Paris but also some influential competitors and enemies. Among the great mathematicians of the preceding generation, Lagrange, Monge, Laplace, only Laplace remained active and - contrary to Lagrange - he had a great esteem for Fourier. Fourier was admired by younger people : Navier, Sturm, Liouville, Dirichlet. Cauchy was a hard competitor and certainly not a friend. Poisson, a declining star, was not a friend either. The main action of Fourier against his competitors was to have his works of 1807 and 1811 published, to publish first of all the extended version *(Théorie analytique de la chaleur)* in 1822, and also to publish or prepare for publication his earlier works on algebraic equations and inequalities.

His health declined. Together with Cauchy, to whom he sent the papers of Abel and Galois which he had received as *secrétaire perpétuel*, he can be considered as responsible for their lack of recognition.

After his death in 1830 his reputation increased on the world scene and somehow decreased in France. His obituary was read by Arago, a physicist, at the Academy of Sciences ; another, more detailed, by Victor Cousin, at the *Académie Française*. Neither one pays tribute to his role in mathematics. In his novel *"Les Misérables"*, published in 1862, Victor Hugo evokes the year 1817 with a series of short statements; here is one about Joseph Fourier and the phalansterian Charles Fourier : *"il y avait à l'académie des sciences un Fourier célèbre que la postérité a oublié et dans je ne sais quel grenier un Fourier obscur dont l'avenir se souviendra"* (there was at the Academy of Sciences a celebrated Fourier whose name is forgotten now, and in some unknown attic an obscure Fourier who will be remembered in times to come). Actually the collected works of Fourier were never published. When Darboux gathered the material for an edition of Fourier's works, he left out the whole of what Fourier called *"Analyse indéterminée"*, including what we call linear programming now. Darboux explains that Fourier gave an "exaggerated importance" to these things. Until quite recently the French did not consider Fourier as one of their great men. As an example, the first editions of *Encyclopaedia Universalis* (1972) ignored Fourier.

In the meantime, the Fourier equation of heat, Fourier series and integrals, and Fourier analysis became common words among mathematicians, physicists and engineers all around the world. In recent times there has been a new interest in the man and his work. For those who want to know more about the man, we recommend the obituaries written by Arago and by Cousin, and the books by I. Grattan-Guiness and J. R. Ravetz (1972), John Herivel (1975), Jean Dhombres and Jean-Bernard Robert (1995) - the preliminary papers of J.-B. Robert were the main source for this chapter. The next chapter will give an idea of his work on trigonometric series, together with his views on mathematical analysis in general.

Chapter 2

THE BEGINNING OF FOURIER SERIES

1. The Analytical Theory of Heat. Introduction.

The Analytical Theory of Heat (1822) contains the substance of the previous works of Fourier (1807, 1811), then unpublished. There Fourier series appear as a general tool for solving a collection of problems coming from the natural world. Fourier gives a series of examples before stating that an *entirely arbitrary function* can be decomposed into a trigonometric series, and that *it is easy to show that these series converge.*

In view of the importance of this book we shall reproduce the whole of its introduction, *discours préliminaire.* It contains no mathematics, but the main motivations and the general approach of Fourier. Fourier mentions Archimedes, Galileo, Newton and his *natural philosophy :* there is a small number of fundamental laws, which govern all natural phenomena. However, Newtonian mechanics does not apply to the effects of heat. Fourier had to use the best instruments, to observe the significant facts and to discover three main physical factors : capacity, internal conductivity, external conductivity. He emphasizes the interest of such an investigation in practical matters and for the understanding of climates, winds, oceanic streams, temperatures below the surface of the earth, temperature in space, the origin and evolution of heat in the solar system. Then he explains the main features of his theory. The principles are deduced from a very small number of fundamental facts, without paying attention to the possible causes. The differential equation of the propagation of heat (the Fourier heat equation) reduces the physical questions to pure Analysis. After being established, it had to be solved, that is, to go from a general expression to the specific solution constrained by the data. This difficult investigation needed a special analysis, based on new theorems. The resulting method led eventually to numerical applications. Fourier adds that the same theorems apply to questions of General Analysis and Dynamics for which a solution had been wanted for a long time (this means a final solution for the historical controversy on vibrating strings, that we shall consider in a moment).

Then Fourier develops his ideas on nature, mathematical discoveries, mathematical analysis. "The thorough study of nature is the most fertile ground for mathematical discoveries". "Analytic equations... apply to all general phenomena. No language can be as universal and simple, as free from error and abstruseness, as able to express the constant relations of natural beings." "Considered from this point of view, Mathematical Analysis has the same extension as nature itself... Its main feature is clarity ; it is not made to express confused notions. It unites the most diverse phenomena and discovers their most secret relations... It looks like a faculty of the human reason that is meant to supply the brevity of life and the imperfection of our senses."

The end of the *Discours préliminaire* explains the different steps of the work (1807, 1811, 1816, 1821) and announces the publication of the memoir of 1811 in the Collection of the *Académie des Sciences* - a way for Fourier to secure priority before his challengers.

After the *Discours préliminaire* the book consists of more than 500 pages organized in nine chapters, divided into sections and subsections, numbered from 1 to 433.

2. Chapters I, II, III.

The first chapter is again of an introductory nature. It describes a series of bodies on which heat phenomena can be observed : metallic annuli, spheres, cubes, infinite prisms. It expounds the physical aspects of heat propagation and explains the purpose of a mathematical treatment : to obtain solutions in the form of easily computable series or integrals. Explicit formulas for stationary temperatures and for flux of heat through surfaces are given in particular cases.

The second contains the differential equations. Before going to the general case Fourier investigates the cases of annuli, spheres, cylinders and cubes. Only then does he give the general formula for the propagation of heat inside a homogeneous body :

$$(A) \qquad \frac{\partial \nu}{\partial t} = \frac{K}{CD}\left(\frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 \nu}{\partial z^2}\right)$$

where K is the internal conductibility and CD the caloric capacity by volume unit, and the general condition at the boundary, involving the flux of heat and the external conductibility.

In the third chapter, trigonometric series are introduced as a way to solve the equilibrium problem for an infinite rectangular body. Here is an overview of the content. The body is described as $0 \leq x < \infty$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $-\infty < z < \infty$. For the equilibrium problem (A) is reduced to

$$(a) \qquad \frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} = 0, \qquad \nu = \nu(x, y).$$

Fourier assumes that ν, the temperature, is maintained at value 0 on the half-planar parts of the boundary and value 1 on the strip part of the boundary, that is

$$\nu(x, -\frac{\pi}{2}) = \nu(x, \frac{\pi}{2}) = 0 \qquad (x \geq 0)$$
$$\nu(0, y) = 1 \qquad (-\frac{\pi}{2} < y < \frac{\pi}{2}).$$

In modern words, this is a typical Dirichlet problem, and the solution can be expressed in closed terms, namely

$$\nu = \frac{2}{\pi} \text{arc tang} \frac{2\cos y}{e^x - e^{-x}}.$$

Fourier provides this formula after a long detour, using a method much more important than the result. He looks for special solutions of (a) of the type $\nu = F(x)f(y)$. Taking (a) and the first boundary condition into account he obtains $e^{-mx}\cos my$ with $m = 1, 3, 5, 7, \cdots$. Then a candidate is

$$(b) \qquad v = ae^{-x}\cos y + be^{-3x}\cos 3y + ce^{-5x}\cos 5y + de^{-7x}\cos 7y + \cdots.$$

It provides a solution if the constant 1 can be expressed as

$$(c) \qquad 1 = a\cos y + b\cos 3y + c\cos 5y + d\cos 7y + \cdots$$

for $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Formula ($c$) is the first occurrence of Fourier series in the book (n. 169).

Before going further Fourier explains how equilibrium is realized, from a physical point of view, when $\nu = e^{-x}\cos y$. Clearly the boundary condition $\nu(0, y) = 1$ is not so simple as $\nu(0, y) = \cos y$!

Then a rather bizarre series of computations begins. Fourier truncates (c), differentiates a number of times, solves the resulting system, goes to limits, and using Wallis' formula for infinite products, concludes that $a = \frac{4}{\pi}$, $b = -\frac{4}{3\pi}$, $c = \frac{4}{5\pi}$, $d = -\frac{4}{7\pi}$, etc... Therefore (c) becomes

$$(d) \qquad \frac{\pi}{4} = \cos y - \frac{1}{3}\cos 3y + \frac{1}{5}\cos 5y - \frac{1}{7}\cos 7y + \cdots (-\frac{\pi}{2} < y < \frac{\pi}{2}).$$

Fourier is not puzzled by the convergence problem. He says first that convergence is easy to prove - and gives a precise definition of convergence on this occasion (n. 177) - before giving a full proof a few pages later (n. 179). What is important for him is the behaviour of the series which converges to $\frac{\pi}{4}$ on $(0, \frac{\pi}{2})$, $-\frac{\pi}{4}$ on $(\frac{\pi}{2}, \frac{3\pi}{2})$, and so on. "The limit of the series is positive and negative alternately. By the way, the convergence is not rapid enough in order to provide an easy approximation, but it suffices for the truth of the equation" (n. 177).

The proof of (d) consists in expressing the partial sums as integrals and going to the limit. It is precisely what Dirichlet did later in a general context. However the main interest of Fourier is to use (b) in order to obtain the temperature *inside* the body. He writes

$$\frac{\pi}{4}\nu = e^{-x}\cos y - \frac{1}{3}e^{-3x}\cos 3y + \frac{1}{5}e^{-5x}\cos 5y - \frac{1}{7}e^{-7x}\cos 7y + \cdots$$

and says that this series, due to the exponential factor, is *extremely convergent* (n. 191). Then he derives the closed formula given above.

On the way Fourier describes the graph of a partial sum of the series in (d) and he states that "it tends to coincide" (*"il tend à se confondre"*) with the graph of the limit, completed by vertical lines. Fourier does not describe these vertical lines, but clearly assumes that they simply join the horizontal parts. In other words, he did not notice the so-called Gibbs phenomenon. It is his only misleading statement about formula (d) (n. 178).

Afterwards (n. 207) he forgets about the theory of heat and concentrates on the *development of arbitrary functions into trigonometric series* for fifty pages. He begins in the same fantastic way as in expressing the constant 1 as a cosine series. Denoting the arbitrary function by φ, he feels free to develop φ into a sine or cosine series, to differentiate, to solve linear equations, to go to limits, and to discuss a number of examples before giving what we now call Fourier formulas in the form

$$\int \varphi(x)\sin nx\,dx \quad , \quad \int \varphi(x)\cos nx\,dx$$

(n. 219). He applies these formulas to functions defined on $(0,\pi)$, expanding cosine into a series of sines, sine into a series of cosines, characteristic functions of intervals and trapezoidal functions in both ways. He observes that his analysis applies to the problem of vibrating strings. He justifies the point of view of Daniel Bernoulli, that every function can be expressed as a trigonometric series, by saying that the best proof consists in exhibiting the coefficients - a very disputable statement.

Then he summarizes the main properties of Fourier series (trigonometric series whose coefficients are obtained through Fourier formulas) related to the theory of heat (n. 235) :

1. all series converge (!)

2. if f and φ have cosine coefficients a, b, c, $d, \cdots$ α, β, γ, $\delta, \cdots$ there is a simple integral formula (what we call convolution now) for the sum :

$$a\alpha + b\beta\cos x + c\gamma\cos 2x + d\delta\cos 3x + \cdots$$

3. Fourier series can be written

$$\begin{aligned}\pi F(x) &= \frac{1}{2}\int F(\alpha)d\alpha + \cos x\int F(\alpha)\cos\alpha d\alpha + \cdots + \sin x\int F(\alpha)\sin\alpha d\alpha + \cdots \\ &= \int F(\alpha)d\alpha\left(\frac{1}{2} + \cos(x-\alpha) + \cos 2(x-\alpha) + \cdots\right).\end{aligned}$$

4. Fourier integrals can be written in similar ways.

These properties are expressed again, and generalized, at the very end of the book (n. 416 to 419). "We must remark that our demonstration applies to an entirely arbitrary function" (n. 417). "There is no function or part of a function that could not be expressed as a trigonometric series. The value of the second member is periodic and the interval of period is X, that is, the value of the second member does not change when $x + X$ replaces x. The succession of values is renewed in each interval X. The trigonometric series in the second member converges..." (n. 418). The formulas which follow (n. 419) are astonishing ; when $a < x < b$, then

$$f(x) = \frac{1}{2\pi}\int_a^b f(\alpha)d\alpha\int_{-\infty}^{\infty}\cos p(x-\alpha)dp$$

$$\frac{d^{2i}f(x)}{dx^{2i}} = \pm\frac{1}{2\pi}\int_a^b f(x)d\alpha \int_{-\infty}^{\infty} p^{2i}\cos p(x-\alpha)dp$$

$$\frac{d^{2i+1}f(x)}{dx^{2i+1}} = \mp\frac{1}{2\pi}\int_a^b f(\alpha)d\alpha \int_{-\infty}^{\infty} p^{2i+1}\sin p(x-\alpha)dp.$$

Of course, these statements and formulas could be considered as nonsense. However, a good part of mathematical analysis of the nineteenth and twentieth century consisted in giving a sense to these pieces of nonsense. In particular, the last formulas are one of the goals and achievements of the *Theory of Distributions* of Laurent Schwartz.

We shall summarize chapters IV to IX more briefly.

3. Chapters IV to IX.

Chapter IV considers the motion of heat in an annulus. The heat equation is reduced to

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad (x \in (0, 2\pi)) \ ;$$

the elementary solutions are

$$a\, e^{-kn^2 t}\sin nx$$

and the general solution is expressed as

$$\frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\sum_i e^{-i^2 kt}\cos i(\alpha - x)d\alpha \qquad (p.\ 301).$$

Chapter V considers the same problem in a solid sphere. The equation is reduced in the same way but now the elementary solutions do not correspond to integral values of n. The relevant values of n are solutions of an equation

$$\mathrm{tg}\, nr = Anr \quad (A, r\ :\ \text{constants}).$$

The general solution is an anharmonic series. This chapter has a particular importance in the theory of heat inside the earth.

Chapter VI deals with a solid circular cylinder, and here appears an anticipation of Bessel functions. The Bessel functions

$$J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - z\sin\theta)d\theta$$

were introduced by Bessel in 1826. Fourier already considered $J_0(z)$, proved that its zeros are real, and studied expansions of the form

$$f(x) = \sum_1^{\infty} a_m J_0(j_m x)$$

where the j_m denote the positive zeros of $J_0(z)$ (n. 308 and 314 to 320). It is the first example of the so-called series of Fourier-Bessel (see Watson's treatise on Bessel functions).

Chapter VII deals with a rectangular prism (= cylinder). In the equilibrium problem the elementary solutions are of the form

$$ae^{-mx}\cos ny \cos pz \qquad (m = \sqrt{n^2 + p^2})$$

and the solution is a double Fourier series with exponentially decreasing coefficients.

The diffusion of heat in the cube (Chapter VIII) is a superposition of elementary solutions of the type

$$ae^{-kt(n^2+p^2+q^2)}\cos nx \cos py \cos qz,$$

again, a multiple Fourier series.

The last chapter is by far the longest. It is a general study of the diffusion of heat, using all the mathematical tools that Fourier introduced previously. Moreover, Fourier integrals play a central role. We have already quoted a few statements made by Fourier at the end of this chapter.

This overview barely gives an idea of the richness of the book. The title is important : first of all it is a theory of propagation of heat. The mathematical aspects are of different kinds : a model for diffusion, the right way to consider a partial differential equation together with initial or boundary data, ordinary Fourier series as a tool for a class of problems, their use when they are weighted in order to become "extremely convergent", Fourier series in several variables, anharmonic Fourier series, series of Bessel functions, Fourier integrals, convolutions, use of the formalism of distributions, and also, for specific examples, good convergence proofs through a method that Dirichlet was able to use in a rather general case a few years later.

Fourier considers mathematical analysis very much in the tradition of the eighteenth century : a general method for discovering natural laws and finding numerical values. On the other hand the problems that he left played a crucial role in a new conception of mathematical analysis as a fundamental part of pure mathematics, as it developed during the nineteenth century. An interesting counterpart is the point of view of Jacobi, that we shall see at the end of the chapter on Dirichlet.

4. Back to the introduction.

Joseph Fourier : Théorie analytique de la chaleur, chez Firmin Didot, père et fils, 1822. Discours préliminaire, pp. i-xxii.

Les causes primordiales ne nous sont point connues ; mais elles sont assujetties à des lois simples et constantes, que l'on peut découvrir par l'observation, et dont l'étude est l'objet de la philosophie naturelle.

La chaleur pénètre, comme la gravité, toutes les substances de l'univers, ses rayons occupent toutes les parties de l'espace. Le but de notre ouvrage est d'exposer les lois mathématiques que suit cet élément. Cette théorie formera désormais une des branches les plus importantes de la physique générale.

Les connaissances que les plus anciens peuples avaient pu acquérir dans la mécanique rationnelle ne nous sont point parvenues, et l'histoire de cette science, si l'on excepte les premiers théorèmes sur l'harmonie, ne remonte point au-delà des découvertes d'Archimède. Ce grand géomètre expliqua les principes mathématiques de l'équilibre des solides et des fluides. Il s'écoula environ dix-huit siècles avant que Galilée, premier inventeur des théories dynamiques, découvrit les lois du mouvement des corps graves. Newton embrassa dans cette science nouvelle tout le système de l'univers. Les successeurs de ces philosophes ont donné à ces théories une étendue et une perfection admirables ; ils nous ont appris que les phénomènes les plus divers sont soumis à un petit nombre de lois fondamentales, qui se reproduisent dans tous les actes de la nature. On a reconnu que les mêmes principes règlent tous les mouvements des astres, leur forme, les inégalités de leurs cours, l'équilibre et les oscillations des mers, les vibrations harmoniques de l'air et des corps sonores, la transmission de la lumière, les actions capillaires, les ondulations des liquides, enfin les effets les plus composés de toutes les forces naturelles, et l'on a confirmé cette pensée de Newton : *Quod tam paucis tam multa præstet geometria gloriatur.*

Mais quelle que soit l'étendue des théories mécaniques, elles ne s'appliquent point aux effets de la chaleur. Ils composent un ordre spécial de phénomènes qui ne peuvent s'expliquer par les principes du mouvement et de l'équilibre. On possède depuis longtemps des instruments ingénieux, propres à mesurer plusieurs de ces effets; on a recueilli des observations précieuses ; mais on ne connaît ainsi que des résultats partiels, et non la démonstration mathématique des lois qui les comprennent tous.

J'ai déduit ces lois d'une longue étude et de la comparaison attentive des faits connus jusqu'à ce jour ; je les ai tous observés de nouveau dans le cours de plusieurs années, avec les instruments les plus précis dont on ait encore fait usage.

Pour fonder cette théorie, il était d'abord nécessaire de distinguer et de définir avec précision les propriétés élémentaires qui déterminent l'action de la chaleur. J'ai reconnu ensuite que tous les phénomènes qui dépendent de cette action, se résolvent en un très petit nombre de faits généraux et simples ; et par là toute question physique de ce genre est ramenée à une recherche d'analyse mathématique. J'en ai conclu que pour déterminer en nombre les mouvements les plus variés de la chaleur, il suffit de soumettre chaque substance à trois observations fondamentales. En effet, les différents corps ne possèdent point au même degré la faculté de *contenir* la chaleur, de la *recevoir*, ou de la *transmettre* à travers leur superficie, et de la *conduire* dans l'intérieur de la masse. Ce sont trois qualités spécifiques que notre théorie distingue clairement, et qu'elle apprend à mesurer.

Il est facile de juger combien ces recherches intéressent les sciences physiques et l'économie civile, et quelle peut être leur influence sur les progrès des arts qui exigent l'emploi et la distribution du feu. Elles ont aussi une relation nécessaire avec le système du monde, et l'on connaît ces rapports, si l'on considère les grands phénomènes qui s'accomplissent près de la surface du globe terrestre.

En effet, le rayon du soleil dans lequel cette planète est incessamment plongée, pénètre l'air, la terre et les eaux ; ses éléments se divisent, changent de directions dans tous les sens, et pénétrant dans la masse du globe, ils en élèveraient de plus en plus la température moyenne, si cette chaleur ajoutée n'était pas exactement compensée par celle qui s'échappe en rayons de tous les points de la superficie, et se répand dans les cieux.

Les divers climats, inégalement exposés à l'action de la chaleur solaire, ont acquis après un temps immense des températures propres à leur situation. Cet effet est modifié par plusieurs causes accessoires, telles que l'élévation et la figure du sol, le voisinage et l'étendue des continents et des mers, l'état de la surface, la direction des vents.

L'intermittence des jours et des nuits, les alternatives des saisons occasionnent, dans la terre solide, des variations périodiques qui se renouvellent chaque jour ou chaque année ; mais ces changements sont d'autant moins sensibles, que le point où on les mesure est plus distant de la surface. On ne peut remarquer aucune variation diurne à la profondeur d'environ trois mètres ; et les variations annuelles cessent d'être appréciables à une profondeur beaucoup moindre que 60 mètres. La température des lieux profonds est donc sensiblement fixe, dans un lieu donné ; mais elle n'est pas la même pour tous les points d'un même parallèle, en général, elle s'élève lorsqu'on s'approche de l'équateur.

La chaleur que le soleil a communiquée au globe terrestre, et qui a produit la diversité des climats, est assujettie maintenant à un mouvement devenu uniforme. Elle s'avance dans l'intérieur de la masse qu'elle pénètre toute entière, et en même temps elle s'éloigne du plan de l'équateur, et va se perdre dans l'espace à travers les contrées polaires.

Dans les hautes régions de l'atmosphère, l'air très rare et diaphane ne retient qu'une faible partie de la chaleur des rayons solaires, c'est la cause principale du froid excessif des lieux élevés. Les couches inférieures, plus denses et plus échauffées par la terre et les eaux, se dilatent, et s'élèvent; elles se refroidissent par l'effet même de la dilatation. Les grands mouvements de l'air, comme les vents alizés qui soufflent entre les tropiques, ne sont point déterminés par les forces attractives de la lune et du soleil. L'action de ces astres ne produit sur un fluide aussi rare, à une aussi grande distance, que des oscillations très peu sensibles. Ce sont les changements des températures qui déplacent périodiquement toutes les parties de l'atmosphère.

Les eaux de l'Océan sont différemment exposées par leur surface aux rayons du soleil; et le fond du bassin qui les renferme est échauffé très inégalement, depuis les pôles jusqu'à l'équateur. Ces deux causes, toujours présentes, et combinées avec la gravité et la force centrifuge, entretiennent des mouvements immenses dans l'intérieur des mers. Elles en déplacent et en mêlent toutes les parties, et produisent ces courants réguliers et généraux que les navigateurs ont observés.

La chaleur rayonnante qui s'échappe de la superficie de tous les corps, et traverse les milieux élastiques, ou les espaces vides d'air, a des lois spéciales, et elle concourt aux phénomènes les plus variés. On connaissait déjà l'explication physique de plusieurs de ces faits; la théorie mathématique que j'ai formée en donne la mesure exacte. Elle consiste en quelque sorte dans une seconde catoptrique qui a ses théorèmes propres, et sert à déterminer par le calcul tous les effets de la chaleur directe ou réfléchie.

Cette énumération des objets principaux de la théorie, fait assez connaître la nature des questions que je me suis proposées. Quelles sont ces qualités élémentaires que dans chaque substance il est nécessaire d'observer, et quelles expériences sont les plus propres à les déterminer exactement ? Si des lois constantes règlent la distribution de la chaleur dans la matière solide, quelle est l'expression mathématique de ces lois ? et par quelle analyse peut-on déduire de cette expression la solution complète des questions principales ?

Pourquoi les températures terrestres cessent-elles d'être variables à une profondeur si petite par rapport au rayon du globe ? Chaque inégalité du mouvement de cette planète devant occasionner au-dessous de la surface une oscillation de la chaleur solaire, quelle relation y a-t-il entre la durée de la période et la profondeur où les températures deviennent constantes ?

Quel temps a dû s'écouler pour que les climats pussent acquérir les températures diverses qu'ils conservent aujourd'hui; et quelles causes peuvent faire varier maintenant leur chaleur moyenne ? Pourquoi les seuls changements annuels de la distance du soleil à la terre, ne causent-ils pas à la surface de cette planète des changements très considérables dans les températures ?

A quel caractère pourrait-on reconnaître que le globe terrestre n'a pas entièrement perdu sa chaleur d'origine; et quelles sont les lois exactes de la déperdition?

Si cette chaleur fondamentale n'est point totalement dissipée, comme l'indiquent plusieurs observations, elle peut être immense à de grandes profondeurs, et toutefois elle n'a plus aujourd'hui aucune influence sensible sur la température moyenne des climats. Les effets que l'on y observe sont dus à l'action des rayons solaires. Mais indépendamment de ces deux sources de chaleur, l'une fondamentale et primitive, propre au globe terrestre, l'autre due à la présence du soleil, n'y a-t-il point une cause plus universelle, qui détermine *la température du ciel*, dans la partie de l'espace qu'occupe maintenant le système solaire ? Puisque les faits observés rendent cette cause nécessaire, quelles sont dans cette question entièrement nouvelle les conséquences d'une théorie exacte ? comment pourra-t-on déterminer cette valeur constante de *la température de l'espace*, et en déduire celle qui convient à chaque planète ?

Il faut ajouter à ces questions celles qui dépendent des propriétés de la chaleur rayonnante. On connaît très distinctement la cause physique de la réflexion du froid, c'est-à-dire de la réflexion d'une moindre chaleur; mais quelle est l'expression mathématique de cet effet ?

De quels principes généraux dépendent les températeurs atmosphériques, soit que le thermomètre qui les mesure reçoive immédiatement les rayons du soleil, sur une surface métallique ou dépolie, soit que cet instrument demeure exposé, durant la nuit, sous un ciel exempt de nuages, au contact de l'air, au rayonnement des corps terrestres, et à celui des parties de l'atmosphère les plus éloignées et les plus froides.

L'intensité des rayons qui s'échappent d'un point de la superficie des corps échauffés variant avec leur inclinaison suivant une loi que les expériences ont indiquée, n'y a-t-il pas un rapport mathématique nécessaire entre cette loi et le fait général de l'équilibre de la chaleur; et quelle est la cause physique de cette inégale intensité ?

Enfin, lorsque la chaleur pénètre les masses fluides, et y détermine des mouvements intérieurs, par les changements continuels de température et de densité de chaque molécule, peut-on encore exprimer, par des équations différentielles, les lois d'un effet aussi composé; et quel changement en résulte-t-il dans les équations générales de l'hydrodynamique ?

Telles sont les questions principales que j'ai résolues, et qui n'avaient point encore été soumises au calcul. Si l'on considère de plus les rapports multipliés de cette théorie mathématique avec les usages civils et les arts techniques, on reconnaîtra toute l'étendue de ses applications. Il est manifeste qu'elle comprend une série entière de phénomènes distincts, et qu'on ne pourrait en omettre l'étude, sans retrancher une partie notable de la science de la nature.

Les principes de cette théorie sont déduits, comme ceux de la mécanique rationnelle, d'un très petit nombre de faits primordiaux, dont les géomètres ne considèrent point la cause, mais qu'ils admettent comme résultant des observations communes et confirmés par toutes les expériences.

Les équations différentielles de la propagation de la chaleur expriment les conditions les plus générales, et ramènent les questions physiques à des problèmes d'analyse pure, ce qui est proprement l'objet de la théorie. Elles ne sont pas moins rigoureusement démontrées que les équations générales de l'équilibre et du mouvement. C'est pour rendre cette comparaison plus sensible, que nous avons toujours préféré des démonstrations analogues à celles des théorèmes qui servent de fondement à la statique et à la dynamique. Ces équations subsistent encore, mais elles reçoivent une forme différente, si elles expriment la distribution de la chaleur lumineuse dans les corps diaphanes, ou les mouvements que les changements de température et de densité occasionnent dans l'intérieur des fluides. Les coefficients qu'elles renferment sont sujets à des variations dont la mesure exacte n'est pas encore connue; mais dans toutes les questions naturelles qu'il nous importe le plus de considérer, les limites des températures sont assez peu différentes, pour que l'on puisse omettre ces variations des coefficients.

Les équations du mouvement de la chaleur, comme celles qui expriment les vibrations des corps sonores, ou les dernières oscillations des liquides, appartiennent à une des branches de la science du calcul les plus récemment découvertes, et qu'il importait beaucoup de perfectionner. Après avoir établi ces équations différentielles, il fallait en obtenir les intégrales; ce qui consiste à passer d'une expression commune, à une solution propre assujettie à toutes les conditions données. Cette recherche difficile exigeait une analyse spéciale, fondée sur des théorèmes nouveaux dont nous ne pourrions ici faire connaître l'objet. La méthode qui en dérive ne laisse rien de vague et d'indéterminé dans les solutions; elle les conduit jusqu'aux dernières applications numériques, condition nécessaire de toute recherche, et sans laquelle on n'arriverait qu'à des transformations inutiles.

Ces mêmes théorèmes qui nous ont fait connaître les intégrales des équations du mouvement de la chaleur, s'appliquent immédiatement à des questions d'analyse générale et de dynamique, dont on désirait depuis longtemps la solution.

L'étude approfondie de la nature est la source la plus féconde des découvertes mathématiques. Non seulement cette étude, en offrant aux recherches un but déterminé, a l'avantage d'exclure les questions vagues et les calculs sans issue; elle est encore un moyen assuré de former l'analyse elle-même, et d'en découvrir les éléments qu'il nous importe le plus de connaître, et que cette science doit toujours conserver : ces éléments fondamentaux sont ceux qui se reproduisent dans tous les effets naturels.

On voit, par exemple, qu'une même expression, dont les géomètres avaient considéré les propriétés abstraites, et qui sous ce rapport appartient à l'analyse générale, représente aussi le mouvement de la lumière dans l'atmosphère, qu'elle détermine les lois de la diffusion de la chaleur dans la matière solide, et qu'elle entre dans toutes les questions principales de la théorie des probabilités.

Les équations analytiques, ignorées des anciens géomètres, que Descartes a introduites le premier dans l'étude des courbes et des surfaces, ne sont pas restreintes aux propriétés des figures, et à celles qui sont l'objet de la mécanique rationnelle; elles s'étendent à tous les phénomènes généraux. Il ne peut y avoir de langage plus universel et plus simple, plus exempt d'erreurs et d'obscurités, c'est-à-dire plus digne d'exprimer les rapports invariables des êtres naturels.

Considérée sous ce point de vue, l'analyse mathématique est aussi étendue que la nature elle-même; elle définit tous les rapports sensibles, mesure les temps, les espaces, les forces, les températures; cette science difficile se forme avec lenteur, mais elle conserve tous les principes qu'elle a une fois acquis; elle s'accroît et s'affermit sans cesse au milieu de tant de variations et d'erreurs de l'esprit humain.

Son attribut principal est la clarté; elle n'a point de signes pour exprimer les notions confuses. Elle rapproche les phénomènes les plus divers, et découvre les analogies secrètes qui les unissent. Si la matière nous échappe comme celle de l'air et de la lumière par son extrême ténuité, si les corps sont placés loin de nous, dans l'immensité de l'espace, si l'homme veut connaître le spectacle des cieux pour des époques successives que sépare un grand nombre de siècles, si les actions de la gravité de la chaleur s'exercent dans l'intérieur du globe solide à des profondeurs qui seront toujours inaccessibles, l'analyse mathématique peut encore saisir les lois de ces phénomènes. Elle nous les rend présents et mesurables, et semble être une faculté de la raison humaine destinée à suppléer à la brièveté de la vie et à l'imperfection des sens; et ce qui est plus remarquable encore, elle suit la même marche dans l'étude de tous les phénomènes; elle les interprète par le même langage, comme pour attester l'unité et la simplicité du plan de l'univers, et rendre encore plus manifeste cet ordre immuable qui préside à toutes les causes naturelles.

Les questions de la théorie de la chaleur offrent autant d'exemples de ces dispositions simples et constantes qui naissent des lois générales de la nature; et si l'ordre qui s'établit dans ces phénomènes pouvait être saisi par nos sens, ils nous causeraient une impression comparable à celles des résonances harmoniques.

Les formes des corps sont variées à l'infini; la distribution de la chaleur qui les pénètre peut être arbitraire et confuse; mais toutes les inégalités s'effacent rapidement et disparaissent à mesure que le temps s'écoule. La marche du phénomène devenue plus régulière et plus simple, demeure enfin assujettie à une loi déterminée qui est la même pour tous les cas, et qui ne porte plus aucune empreinte sensible de la disposition initiale.

Toutes les observations confirment ces conséquences. L'analyse dont elles dérivent sépare et exprime clairement, 1°) les conditions générales, c'est-à-dire celles qui résultent des propriétés naturelles de la chaleur; 2°) l'effet accidentel, mais subsistant, de la figure ou de l'état des surfaces; 3°) l'effet non durable de la distribution primitive.

Nous avons démontré dans cet ouvrage tous les principes de la théorie de la chaleur, et résolu toutes les questions fondamentales. On aurait pu les exposer sous une forme plus concise, omettre les questions simples, et présenter d'abord les conséquences plus générales; mais on a voulu montrer l'origine même de la théorie et ses progrès successifs. Lorsque cette connaissance est acquise, et que les principes sont entièrement fixés, il est préférable d'employer immédiatement les méthodes analytiques les plus étendues, comme nous l'avons fait dans les recherches ultérieures. C'est aussi la marche que nous suivrons désormais dans les mémoires qui seront joints à cet ouvrage, et qui en forment en quelque sorte le complément, et par là nous aurons concilié, autant qu'il peut dépendre de nous, le développement nécessaire des principes avec la précision qui convient aux applications de l'analyse.

Ces mémoires auront pour objet la théorie de la chaleur rayonnante, la question des températures terrestres, celle de la température des habitations, la comparaison des résultats théoriques avec ceux que nous avons observés dans diverses expériences, enfin la démonstration des équations différentielles du mouvement de la chaleur dans les fluides.

L'ouvrage que nous publions aujourd'hui a été écrit depuis longtemps; diverses circonstances en ont retardé et souvent interrompu l'impression. Dans cet intervalle, la science s'est enrichie d'observations importantes; les principes de notre analyse, que l'on n'avait pas saisis d'abord, ont été mieux connus; on a discuté et confirmé les résultats que nous en avions déduits. Nous avons appliqué nous-mêmes ces principes à des questions nouvelles, et changé la forme de quelques démonstrations. Les retards de la publication auront contribué à rendre l'ouvrage plus clair et plus complet.

Nos premières recherches analytiques sur la communication de la chaleur, ont eu pour objet la distribution entre des masses disjointes; on les a conservées dans la section II du chapitre III. Les questions relatives aux corps continus, qui forment la théorie proprement dite, ont été résolues plusieurs années après; cette théorie a été exposée pour la première fois dans un ouvrage manuscrit remis à l'Institut de France à la fin de l'année 1807, et dont il a été publié un extrait dans le bulletin des Sciences (Société philomatique, année 1808, page 112). Nous avons joint à ce mémoire, et remis successivement des notes assez étendues, concernant la convergence des séries, la diffusion de la chaleur dans un prisme infini, son émission dans les espaces vides d'air, les constructions propres à rendre sensibles les théorèmes principaux, et l'analyse du mouvement périodique à la surface du globe terrestre. Notre second mémoire, sur la propagation de la chaleur, a été déposé aux archives de l'Institut, le 28 septembre 1811. Il est formé du précédent et des notes déjà remises ; on y a omis des constructions géométriques, et des détails d'analyse qui n'avaient pas un rapport nécessaire avec la question physique, et l'on a ajouté l'équation générale qui exprime l'état de la surface. Ce second ouvrage a été livré à l'impression dans le cours de 1821, pour être inséré dans la collection de l'Académie des Sciences. Il est imprimé sans aucun changement ni addition; le texte est littéralement conforme au manuscrit déposé, qui fait partie des archives de l'Institut.

On pourra trouver dans ce mémoire, et dans les écrits qui l'ont précédé un premier exposé des applications que ne contient pas notre ouvrage actuel; elles seront traitées dans les mémoires subséquens, avec plus d'étendue, et, s'il nous est possible, avec plus de clarté. Les résultats de notre travail concernant ces mêmes questions, sont aussi indiqués dans divers articles déjà rendus publics. L'extrait inséré dans les Annales de chimie et de physique fait connaître l'ensemble de nos recherches, (tom. III; pag. 350, ann. 1816). Nous avons publié dans ces annales deux notes séparées, concernant la chaleur rayonnante, (tom. IV, pag. 128, ann. 1817 et tom. VI, pag. 259, ann. 1817).

Divers autres articles du même recueil présentent les résultats les plus constants de la théorie et des observations; l'utilité et l'étendue des connaissances thermologiques ne pouvaient être mieux appréciées que par les célèbres rédacteurs de ces annales.

On trouvera dans le bulletin des Sciences, (Soc. philomat., ann. 1818, pag. 1 et ann. 1820, pag. 60) l'extrait d'un mémoire sur la température constante ou variable des habitations, et l'exposé des principales conséquences de notre analyse des températures terrestres.

M. Alexandre de Humboldt, dont les recherches embrassent toutes les grandes questions de la philosophie naturelle, a considéré sous un point de vue nouveau et très important, les observations des températures propres aux divers climats. (Mémoire sur les lignes isothermes, *Société d'Arcueil*, tom. III, pag. 462); (Mémoire sur la limite inférieure des neiges perpétuelles, *Annales de Chimie et de Physique*, tom. V, pag. 102, ann. 1817).

Quant aux équations différentielles du mouvement de la chaleur dans les liquides, il en a été fait mention dans l'histoire annuelle de l'Académie des Sciences. Cet extrait de notre mémoire en montre clairement l'objet et le principe. (*Analyse des travaux de l'Académie des Sciences*, par M. De Lambre, année 1820).

L'examen des forces répulsives que la chaleur produit, et qui déterminent les propriétés statiques des gaz, n'appartient pas au sujet analytique que nous avons considéré. Cette question liée à la théorie de la chaleur rayonnante vient d'être traitée par l'illustre auteur de la *Mécanique céleste* à qui toutes les branches principales de l'analyse mathématique doivent des découvertes importantes. (Connaissance des temps, pour les années 1824 et 1825).

Les théories nouvelles, expliquées dans notre ouvrage sont réunies pour toujours aux sciences mathématiques, et reposent comme elles sur des fondements invariables; elles conserveront tous les éléments qu'elles possèdent aujourd'hui, et elles acquerront continuellement plus d'étendue. On perfectionnera les instruments et l'on multipliera les expériences. L'analyse que nous avons formée sera déduite de méthodes plus générales, c'est-à-dire plus simples et plus fécondes, communes à plusieurs classes de phénomènes. On déterminera pour les substances solides ou liquides, pour les vapeurs et pour les gaz permanents, toutes les qualités spécifiques relatives à la chaleur, et les variations des coefficients qui les expriment. On observera, dans les divers lieux du globe, les températures du sol à diverses profondeurs, l'intensité de la chaleur solaire, et ses effets, ou constants ou variables, dans l'atmosphère, dans l'Océan et les lacs; et l'on connaîtra cette température constante du Ciel, qui est propre aux régions planétaires. La théorie elle-même dirigera toutes ces mesures, et en assignera la précision. Elle ne peut faire désormais aucun progrès considérable qui ne soit fondé sur ces expériences; car l'analyse mathématique peut déduire des phénomènes généraux et simples l'expression des lois de la nature; mais l'application spéciale de ces lois à des effets très composés exige une longue suite d'observations exactes.

Chapter 3

PREDECESSORS AND CHALLENGERS

1. The prehistory of harmonic analysis.

To decompose complex motions into a sum of simple motions is the core of the Ptolemean astronomy. Actually, before Kepler and Newton, the best way to describe the motion of planets was to combine circular motions. This can be considered as the historical source of harmonic analysis and synthesis.

The very term of harmony appears in the context of cosmogony already in Plato: the harmony of spheres. However, it is borrowed from music. In Plato's world, mathematics consists of five parts : numbers, plane geometry, solid geometry, astronomy and music. Astronomy and music are like sisters (*Republic*, VII, 530(d)). Harmonic motion is grasped through the eyes and the ears respectively. Pure sounds echo perfect circles.

After printing made the use of mathematical tables possible, the trigonometric functions became an important tool in astronomy and navigation. Trigonometric and logarithmic tables gave the status of elementary functions to sines, cosines and logarithms. Funny names were given. The graph of $\sin x$ over $(0,\pi)$ was named a trochoid.

2. Vibrating strings, D. Bernoulli, Euler, d'Alembert.

Then music came back. In 1715 Brook Taylor published his book *De methodo incrementorum* where the famous Taylor formula appeared. A section was devoted to vibrating strings and Taylor recognized the role of trochoids in producing pure sounds. Vibrations, propagation of sounds and the mathematical theory of music became important scientific topics.

Leonard Euler (1707-1783) was interested in these topics throughout his life. His first published paper is entitled *Dissertatio physica de sono* (1727) and he wrote thirty three articles in the field between 1727 and 1782. His main mathematical papers on the propagation of sound are dated 1748 and 1759 and they take advantage of contributions by d'Alembert and Lagrange respectively. We shall come back to the famous controversy involving D. Bernoulli and the three of them in a moment.

Jean Le Rond d'Alembert (1717-1783) gave the equation of vibrating strings and what he considered as a partial solution in 1747. Let us express it in modern terms. Here are the equations :

$$(A) \qquad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

the boundary conditions :

$$(B) \qquad y(0,t) = y(\ell,t) = 0 \qquad (t \geq 0),$$

(the string is fixed at points $x = 0$ and $x = \ell$) and the initial conditions :

$$(C) \qquad y(x,0) = \varphi(x), \qquad \frac{\partial}{\partial t} y(x,0) = \psi(x)$$

(the initial shape and velocity are given). The general solution of (A) is

$$y = \Psi(ct + x) - \Phi(ct - x)$$

and the general solution of (A) and (B) is

$$(D) \qquad y = \Psi(ct + x) - \Psi(ct - x),$$

Ψ being a 2ℓ-periodic function ($\Psi(x + 2\ell) = \Psi(x)$). In order to have a physical interpretation, (D) should be restricted to the domain $0 \leq x \leq \ell$, $t \geq 0$. Moreover, (A) has a meaning when Ψ is twice differentiable. This excludes the most natural initial conditions, such as a piecewise linear function φ. This was the point of view of d'Alembert : for him, (D) could not be the general solution of the physical problem.

Euler took a different view. For him (D) was *the* general solution, regardless of regularity properties of Ψ (the term in use was continuity ; a piecewise linear function was called a discontinuous function). The main reason was heuristic. Given arbitrary initial conditions, there is a way to match φ, ψ and Ψ through equations (C) and (D). This being possible, Ψ has to be the solution of the problem. There is no restriction on φ or ψ : they can be "discontinuous" functions. D'Alembert strongly opposed this view of Euler.

Then Daniel Bernoulli (1700-1782) came on the stage. The *Memoirs of the Academy of Sciences* in Berlin, dated 1753, published two papers of Daniel Bernoulli on vibrating strings and trigonometric series : *"Réflexions et éclaircissements sur les nouvelles vibrations des cordes"*, and *"Sur le mélange de plusieurs espèces de vibrations simples isochrones, qui peuvent exister dans un même système de corps"*; the publication occurred in 1755. D. Bernoulli observed that a string fixed at points 0 and ℓ can produce a fundamental sound (this corresponds to the Taylor solution), and harmonics corresponding to integral multiples of the fundamental frequency. A convenient mixture of the fundamental sound and its harmonics should give any sound that the string is able to produce. In other words, we have a collection of particular solutions of (A) and (B), namely

$$\sin n\frac{\pi x}{\ell} \cos n\frac{\pi ct}{\ell} \quad , \quad \sin \frac{\pi x}{\ell} \sin n\frac{\pi ct}{\ell}$$

$(n = 1, 2, 3, \cdots)$. The general solution should be

$$y = \sum_{n=1}^{\infty} \sin n\frac{\pi x}{\ell} \left(a_n \cos n\frac{\pi ct}{\ell} + b_n \sin n\frac{\pi ct}{\ell} \right).$$

Euler disagreed. The initial conditions would read

$$y(x,0) = \sum_{n=1}^{\infty} a_n \sin n\frac{\pi x}{\ell} = \varphi(x)$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n cn \sin n\frac{\pi x}{\ell} = \psi(x).$$

Both series involve "continuous" functions, but φ and ψ can be "discontinuous". There seemed to be a contradiction. This controversy became famous immediately in scientific circles.

3. Lagrange.

A very accurate description of the controversy can be found in the memoir of Joseph Louis de Lagrange *"Recherches sur la nature et la propagation du son"*, published in 1759 in Turin. Lagrange (1736-1813) was a young man then, but already considered as a master (*"M. de la Grange, très savant Géomètre à Turin"*, said Euler the same year). He introduced a new method. As a first step he assumed that the string carries a finite number of equidistant masses and he analysed the resultant motion. A long and careful study led him to an important formula which seems to justify Bernoulli and to anticipate Fourier ; here is the exact reproduction (p. 100) :

$$\begin{aligned} y = \frac{2}{a}\int dx\, Y \Big(& \sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\cos\frac{\varpi Ht}{2T} + \sin\frac{2\varpi X}{2a}\sin\frac{2\varpi X}{2a}\cos\frac{2\varpi Ht}{2T} \\ & \sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\cos\frac{3\varpi Ht}{2T} + \cdots \Big) \\ + \frac{4T}{\varpi Ha}\int dx\, V \Big(& \sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\sin\frac{\varpi Ht}{2T} + \frac{1}{2}\sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\sin\frac{2\varpi Ht}{2T} \\ & \frac{1}{3}\sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\sin\frac{3\varpi Ht}{2T} + \cdots \Big). \end{aligned}$$

Now let us explain the notation : $\varpi = 2\pi$, $a = \ell$ (length of the string), $\frac{Ha}{T} = c$, $Y = \varphi$ (initial positions), $V = \psi$ (initial data), X is the generic position of a mass, therefore $Y = Y(X)$ and $V = V(X)$, and moreover $\int dx \cdots$ means the normalized summation of the X's. These are exactly *Fourier formulas on a cyclic group*, applied to a system of vibrating point masses localized on the group. However this is nothing for Lagrange but a step to recover Euler's statements. He lets $\frac{x}{a} + \frac{Ht}{T}$ and $\frac{x}{a} - \frac{Ht}{T}$ appear, he shows how to extend data and results on a discrete subgroup of the line, he goes to the limit, and recovers the construction that Euler had given (how to obtain $y(x,t)$ when $\varphi(x)$ and $\psi(x)$ are given functions on $(0,\ell)$).

Here is Lagrange's conclusion of this study (p. 107).

> *"Voilà donc la théorie de ce grand Géomètre mise hors de toute atteinte et établie sur des principes directs et lumineux, qui ne tiennent en aucune façon à la loi de continuité que demande M. d'Alembert ; voilà encore comment il peut se faire que la même formule qui a servi pour appuyer et démontrer la théorie de M. Bernoulli sur le mélange des vibrations isochrones, lorsque le nombre des corps mobiles était fini, nous en dévoile l'insuffisance dans le cas où le nombre de ces corps devient infini. En effet le changement que subit la formule, en passant*

d'un cas dans l'autre, est tel que les mouvements simples qui composaient les mouvements absolus de tout le système s'anéantissent pour la plupart, et que ceux qui restent se défigurent et s'altèrent de façon qu'ils deviennent absolument méconnaissables. Il est vraiment fâcheux qu'une théorie aussi ingénieuse, et qui aurait pu sans doute jeter de grandes lumières sur des matières également obscures et importantes, se trouve démentie dans le cas principal, qui est celui auquel se rapportent tous les petits mouvements réciproques qui ont lieu dans la nature."

(That is the way to save and establish the theory of this great Geometer on direct and clear principles, free of the continuity requirements of Mr. d'Alembert. This is also a way by which the same formula used to support and demonstrate the theory of Mr. Bernoulli on the mixing of isochronous vibrations when the number of bodies is finite proves that this theory is inadequate in the case of infinitely many bodies. For, changing from the finite to the infinite case, the simple motions which compose all motions of the system would disappear in general, and those that remain are distorted in such a way that it is impossible to recognize them. It is really a pity that such an ingenious theory, which could be hoped to apply in other obscure and important matters, is just disproved in the principal case to which are referred all small reciprocal motions occurring in nature.)

The reaction of Euler was enthusiastic. Not only had Lagrange justified his views, but he justified also the use of "discontinuous" functions in Analysis :

"M. de la Grange ayant justifié pleinement ma solution, et cela de manière incontestable, je ne doute pas qu'on ne reconnaisse bientôt la nécessité des fonctions discontinues dans l'analyse, surtout quand on verra que c'est l'unique moyen d'expliquer la propagation du son." (Mémoires de l'Académie des sciences de Berlin (1759), published in 1766, pp. 187-188).

Though special trigonometric series appeared in many others places in the eighteenth century, and Fourier formulas were used by Euler in particular instances, we shall stop there. The discussion on vibrating strings played a crucial role in extending methods and views in Analysis. This can be seen through the main memoirs, but scientific communication was also exchange of letters, so that some of the ideas of Euler and Bernoulli were expressed already in their correspondence in the years 1741-43 (see *Daniel Bernoulli's Werke*, Band II) and certainly known by d'Alembert. Priorities may be difficult to establish.

As far as Fourier is concerned, the influence of the theory of vibrating strings is obvious. First, the equation was the paradigm of the heat equation. Then, the treatment of the equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ by Fourier is inspired directly by Bernoulli's treatment of the equation $\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$. Finally, Fourier was happy to effect the introduction of "discontinuous" functions (including discontinuous in the modern sense) in a more efficient way than any of his predecessors.

The reluctance of Lagrange has clear reasons also. Lagrange knew and practised Fourier analysis on cyclic groups. He had worked a lot in order to go from there (purely algebraic computations) to the Euler solution for continuously vibrating strings. That was the aim and the achievement. There was no simple way to go from Fourier analysis on cyclic groups to Fourier analysis on the circle. Therefore this could not be a "general" nor a "rigorous" method.

Lagrange had no reason to object to the use which Fourier made of the series

$$\frac{1}{2} + \cos(x-\alpha) + \cos 2(x-\alpha) + \cos 3(x-\alpha) + \cdots$$

In his memoir of 1759 he proves in different ways that this series equals 0 except at $x = \alpha$. One proof (p. 111) leads to the formula

$$\frac{\cos mx - \cos(m+1)x}{2(1-\cos x)} = 0$$

in the case where m is infinite (dans le cas où m est un nombre infini). Bernoulli came back to this and other similar series in 1772. Euler played with apparently divergent series in a very clever way during his whole life. Only in the course of the nineteenth century did these formulas look absurd, in view of the formal definition of convergent series. They can be justified, of course, as soon as series and summability processes are considered as two faces of the same theory.

The point of view of Fourier on the above series is more precise and more modern. He expresses very clearly that it is essentially the Dirac measure at α (α being the variable) : integrated against $f(\alpha)\frac{d\alpha}{\pi}$, it gives $f(x)$.

4. Euler and Fourier formulas. Clairaut.

Given a linear combination of $\cos nx$ and $\sin nx$, the coefficients can be computed through formulas

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad , \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

These formulas were known and used in the more general situation of a trigonometric series before Fourier. Then, assuming

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty}(a_n\cos nx + b_n\sin nx),$$

a formal integration gives a_0, a_n, b_n. According to Lebesgue *(Leçons sur les séries trigonométriques*, p. 23), the formula for a_0 was already used by d'Alembert in 1754, and the formula for a_n by Clairaut in 1757. Euler stated and proved them in a natural manner (trigonometric identities and formal integration) in a memoir of 1777. At the time of Lebesgue's book (1906), formulas for a_n and b_n were called formulas of Euler and Fourier.

Actually, Clairaut anticipated Lagrange. Alexis Clairaut (1713-1765) was an astronomer. His contribution to trigonometric sums appeared in a memoir on the solar orbit. Assuming.

$$
\begin{aligned}
H &= A + B\cos\frac{c}{n} + C\cos\frac{2c}{n} + D\cos\frac{3c}{n} + \cdots \\
I &= A + B\cos\frac{2c}{n} + C\cos\frac{4c}{n} + D\cos\frac{6c}{n} + \cdots \\
K &= \cdots \qquad L = \cdots \qquad M = \cdots \qquad \cdots
\end{aligned}
$$

he derived

$$
\begin{aligned}
A &= \frac{1}{n}(H + I + K + L + M + \cdots) \\
S &= \frac{1}{2n}(H\cos\frac{pc}{n} + I\cos\frac{2pc}{n} + \cdots),
\end{aligned}
$$

that is, discrete Fourier formulas for even functions (pp. 546-547, *Mémoire sur l'orbite apparente du soleil)*. When did these formulas appear ? This is not clear. The memoir is dated July 9, 1757, and it is contained in the yearly issue, 1754, of *Histoire de l'Académie royale des sciences*. Moreover the first words (p. 521) mention a first version of the work in the form of a lecture given in 1747.

The only possible conclusion is that discrete Fourier formulas were in the air around 1750.

5. Poisson, Cauchy.

The main challenger of Fourier during his lifetime was Siméon Denis Poisson (1781-1840). Poisson mistreated Fourier, as we have already said, and was mistreated himself by his successors. There is no book on Poisson, and his works were never collected. However his name appears frequently in probability (the Poisson distribution) and in analysis (the Abel-Poisson summability process, the Poisson summation formula on Fourier integrals, the Poisson kernel).

In 1823, immediately after the publication of *The Analytical Theory of Heat* by Fourier, Poisson published a long article on the distribution of heat in solid bodies (*Journal de l'Ecole Polytechnique*, 19ième cahier, tome XII, pp. 1-144, 145-162), an expansion of a lecture which he gave in 1815. Many of his contributions to Fourier analysis, including complex Fourier transforms (p. 126) and the use of the Poisson kernel (p. 155), can be found there. We shall not discuss these contributions. Let us just point out that the Abel-Poisson summability process has a very strong relation with Fourier series. In the sense of Abel-Poisson, that is

$$
\sum_{\text{Abel-Poisson}} u_n = \lim_{r\uparrow 1} \sum u_n r^n \qquad (n \geq 0)
$$

all Fourier series of continuous functions converge uniformly to the corresponding functions, and the main theorems in Fourier series can be derived from there. However this was not recognized before the end of the nineteenth century, and did not play a role in the history of Fourier series. The problem of ordinary convergence came first.

Another challenger of Fourier was Augustin-Louis Cauchy (1789-1857). Cauchy worked also on the propagation of heat, and the contributions of Cauchy to the theory of series are well known. From a historical point of view his paper of 1827, *Mémoire sur le développement des fonctions en séries périodiques*, deserves attention. He made a number of mistakes in it. First, he did not distinguish ordinary convergence and Abel-Poisson summability. Then, he used analytic continuation in a case when it does not work. Finally he made the classical mistake : if two series have equivalent terms and one converges, necessarily the other converges too. These mistakes had a very positive influence within mathematics. As we shall see in the next chapters, they stimulated Dirichlet and Riemann.

Let us add a comment on Cauchy's memoir : Fourier is not mentioned.

6. For further information.

These few pages may give an incentive for additional reading.

The history of the controversy on vibrating strings is given in the learned study of C. Truesdell published in the collected works of Euler. Basic and entertaining information can be found in memoirs 1 and 25 of d'Alembert's *Opuscules mathématiques*, but it is not so easy to find these books in libraries.

The most important study on the history of trigonometric series is the long article of H. Burkhardt in *Encyclopädie der mathematischen Wissenschaften* (1914). The historical introduction of Carlslaw's book is a good guide. The book of Paplauskas, in Russian, is quite extensive. Riemann's thesis and Lebesgue's book which we shall encounter in the following chapters, are very interesting and accurate from the historical point of view.

For the period 1900-1925 that we shall consider later, the best report is due to Plancherel in l'*Enseignement mathématique* (1924).

Chapter 4

DIRICHLET AND THE CONVERGENCE PROBLEM

1. Dirichlet.

The article of Dirichlet on Fourier series is a turning point in the theory and also in the way mathematical analysis is approached and written. Its intention is simply to give a correct statement and a correct proof on the convergence of Fourier series. The result is a paradigm of what is correctness in analysis. In view of its interest, we reproduce it in its entirety at the end of this chapter, followed by a celebrated quotation from Jacobi. After saying a few words on Dirichlet himself, we shall summarize his article and indicate strong and weak points. Then we shall give a historical survey of the problem up to recent times.

Peter Gustav Lejeune-Dirichlet (1805-1859) obtained his Abitur from a Jesuit college in Cologne at the early age of sixteen. He wanted to study mathematics and went to Paris in 1822. He worked as tutor of the children of General Fay, who died in 1825. During this short period he met a number of French mathematicians and was impressed by Fourier particularly.

He returned to Germany in 1826, obtained a Privatdozent "habilitation" at the University of Breslau, was appointed extraordinary professor there, and then moved to Berlin. In 1831 he became a member of the Berlin Academy of Sciences, and earned his living in the military academy with a very heavy teaching load (thirteen lectures a week and other duties). When Gauss died in 1855, Dirichlet became his successor and settled in Göttingen. He died soon afterwards.

Dirichlet was a close friend of Karl Gustav Jacobi (1804-1851) and certainly shared some of his views on mathematics. His main interest and achievements were in the theory of numbers. He can be considered as the founder of the analytic theory of numbers through the use of Dirichlet series. His work in analysis, apart from Fourier series, deals with expansions into spherical harmonics and potential theory. Dirichlet's box principle and the Dirichlet problem in potential theory are well known.

Let us concentrate now on the article that he wrote in Berlin in 1829 : *Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données.*

2. Comments on the article.

The first sentence of Dirichlet's article is a statement "à *la Fourier*" : the series of sines and cosines by which an arbitrary function can be represented in a given interval have a remarkable property among others, namely, to be convergent. Then Dirichlet pays tribute to the pioneer role of Fourier but observes that no demonstration was ever given of the above statement. He explains why the approach of Cauchy is incorrect. Then he develops his argument in a succession of steps.

He considers first a decreasing and continuous function (in the modern sense) between 0 and h, $0 < h < \frac{\pi}{2}$, say $f(\beta)$, and the integral

$$\int_0^h \frac{\sin i\beta}{\sin \beta} f(\beta) d\beta. \tag{1}$$

He introduces subdivision points $\frac{\nu\pi}{i}$, $i = 1, 2, \cdots, r$, such that $\frac{r\pi}{i} \leq h < \frac{(r+1)\pi}{i}$, and considers the integrals between two consecutive points. These integrals have alternating signs and decreasing absolute values. Dirichlet writes the ν-th integral in the form $\pm\rho_\nu K_\nu$, where

$$f\left(\frac{(\nu-1)\pi}{i}\right) \leq \rho_\nu \leq f\left(\frac{\nu\pi}{i}\right)$$

$$K_\nu = \left| \int_{(\nu-1)\frac{\pi}{i}}^{\frac{\nu\pi}{i}} \frac{\sin i\beta}{\sin \beta} d\beta \right|.$$

When i tends to infinity K_ν tends to

$$k_\nu = \left| \int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin \gamma}{\gamma} d\gamma \right|,$$

and the series $k_1 - k_2 + k_3 - k_4 + k_5 \cdots$ converges to $\frac{\pi}{2}$. Now comes a difficulty : when i increases, each term has a limit, but r, the number of terms, increases too. Dirichlet cuts the sum $\sum \pm\rho_\nu K_\nu$ into two parts, corresponding to $\nu \leq m$ and $\nu > m$, m being fixed, and studies carefully both sums as i increases to infinity. The conclusion is that (1) tends to $\frac{\pi}{2} f(0)$. Such an argument is standard now. It was a conceptual and technical *tour de force* at the time.

The result is easily extended to increasing instead of decreasing functions, then to integrals of the form

$$\int_g^h f(\beta) \frac{\sin i\beta}{\sin \beta} d\beta,$$

where $0 \leq g < h$ and f is monotonic and continuous on $[g, h]$: the limit is $\frac{\pi}{2}$ when $g = 0$ and 0 when $g > 0$. This is the key lemma.

Now Dirichlet considers a function φ defined on $[-\pi, \pi]$ and its Fourier series. He writes the partial sums as

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(\alpha) \frac{\sin(n + \frac{1}{2})(\alpha - x)}{2 \sin \frac{1}{2}(\alpha - x)} d\alpha.$$

The purpose is to cut this integral into pieces to which the lemma applies. The method works whenever φ has a finite number of discontinuity points (*"solutions de continuité"*) and a finite number of maxima and minima on $[-\pi, \pi]$. The limit of the partial sums as $m \to \infty$ is then

$$\frac{1}{2}(\varphi(x+\epsilon)+\varphi(x-\epsilon)).$$

This is the Dirichlet theorem, the first theorem on the convergence of Fourier series, and a beautiful piece of analysis.

The last page is more disputable, but it proved extremely influential. Dirichlet wishes to get rid of the conditions that he stated, and to allow an infinity of discontinuities or extrema. *"Il faut seulement⋯"* and then he states the following condition : that each interval $[a, b]$ on $[-\pi, \pi]$ contains subintervals $[r, s]$ on which φ is continuous. If this condition is not satisfied, says he, the integral has no meaning. As an example, if $\varphi(x)$ equals one constant on the rationals and another constant on the irrationals, the integrals are meaningless. This restriction - together with the fact that $\varphi(x)$ cannot be infinite - is the only one to impose. But the thing needs a few details to become perfectly clear, and Dirichlet announces a subsequent note thereabout - a note which never appeared !

The first comment is that the assertion is wrong - but it was a challenge for half a century. Secondly, the restriction given by Dirichlet depends on the notion of integral; here he refers to the Cauchy integral, the restriction would be different for the Riemann integral and for the Lebesgue integral - again, the successive enlargements of the notion of integral arose as an implicit challenge to this integrability restriction made by Dirichlet. Thirdly, the example that he gives of an everywhere discontinuous function is very new and far reaching ; the fact that it is Lebesgue integrable does not diminish its value, as the first instance of a function which cannot be expressed by ordinary formulas or depicted in a simple graphic way. Finally there is a striking contrast between the rigour and accuracy of the demonstration of the Dirichlet convergence theorem and the ambiguous language that he uses to express the simplest logical notions : does *"il faut seulement⋯"* means *it is necessary* or *it is sufficient* or both ? On the whole, the mathematical style of Dirichlet is superb and incredibly modern. But clearly the language had still to be fixed in order to fit logical concepts. Maybe it is something that we can remember, together with the mistakes of Cauchy, when we teach undergraduates : ambiguities and mistakes are part of mathematical life, and the effort to repair them is a constant challenge.

3. The convergence problem since then.

Does the Fourier series of an arbitrary function converge ?

According to the observation of Dirichlet, a restriction is necessary, namely that the integrals involved in Fourier formulas exist.

Anyway, continuous functions are locally integrable. What is the situation for continuous functions on the circle (that is, 2π-periodic on the line) ?

First, their Fourier series can diverge at a point. The counter-example to convergence was given by Paul du Bois-Reymond in 1873, in the form of a continuous function which is piecewise monotonic away from the point 0, but oscillates infinitely near 0. Simpler examples were given later by Leopold Fejér and by Henri Lebesgue. In all cases, the properties of the Dirichlet kernel

$$D_n(x) = \frac{1}{2} + \cos x + \cdots + \cos nx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

are used. Fejér used the fact that the integrals of the $D_n(x)$ on intervals I are uniformly bounded with respect to n and I ($\mid I \mid < 2\pi$), that is, the trigonometric polynomials

$$F_n(x) = \sin x + \frac{1}{2}\sin 2x + \cdots + \frac{1}{n}\sin nx$$

are bounded uniformly. Therefore,

$$\sum_1^\infty \frac{1}{n^2}\sin(3^{n^2}x)F_{2^{n^2}}(x)$$

is a continuous function, and an inspection of Fourier coefficients shows that the series diverges for $x = 0$. Lebesgue simply used the functions $\operatorname{sgn} D_n(x)$ and the fact that

$$\int_{-\pi}^{\pi} D_n(x)\operatorname{sgn} D_n(x)dx = \int_{-\pi}^{\pi} \mid D_n(x) \mid dx \to \infty \qquad (n \to \infty).$$

A convenient linear combination of continuous approximants of $\operatorname{sgn} D_n(x)$ gives the example. This may serve as an example for the principle of condensation of singularities, later established by Banach and Steinhaus. Actually, according to the Banach-Steinhaus theorem, the fact that the linear forms

$$f \to \int f D_n$$

defined on $C(\mathbf{T})$ have unbounded norms implies that they are unbounded for some f : this is the easiest way to see the thing now.

The Fourier series of a continuous function can diverge on an arbitrary set of Lebesgue measure zero. This was proved much later (Kahane-Katznelson 1966), as a way to establish the following dichotomy result : either the Fourier series of all continuous functions converge almost everywhere, or there exists a continuous function whose Fourier series diverges everywhere. In 1965, the question was still open.

The question was settled by a famous theorem of Lennart Carleson in 1966 : for all functions belonging to $L^2(\mathbf{T})$ (that is, square-integrable, in particular continuous functions), the Fourier series converges almost everywhere to the function. The same holds for $L^p(\mathbf{T})$ ($p > 1$) instead of $L^2(\mathbf{T})$ (Richard Hunt 1967).

Now what is the situation for integrable functions ?

If we consider improper Riemann integrals, the coefficients need not tend to zero (example : $\frac{1}{\operatorname{tg}\frac{x}{2}}$), therefore the Fourier series can diverge everywhere.

If we consider Lebesgue integrals and $f \in L^1(\mathbf{T})$, the coefficients tend to zero: it is the so-called Riemann-Lebesgue theorem. Here the example is far from being obvious, but there exists actually a Lebesgue integrable function whose Fourier series diverges everywhere (Kolmogorov 1926). A good reference for the divergence constructions is the book of Y. Katznelson, *An introduction to harmonic analysis.*

Let us mention an open problem for the partial sums $S_n(x)$ of a Fourier-Lebesgue series : for what sequences λ_n is it true that $S_n(x) = o(\lambda_n)$ holds almost everywhere? What is known is that $\lambda_n = \log n$ works and $\lambda_n = \log\log n$ does not.

Moreover it was announced recently that $S_n(x)$ enjoys a bounded mean oscillation property on $\mathbb{N}$, that is

$$\sup_{n,m} \frac{1}{m} \sum_{k=1}^{m} \left| S_{n+k} - \frac{1}{m} \sum_{j=0}^{m-1} S_{n+j} \right| < \infty,$$

for almost every x (Rodin 1992).

For Fourier series of functions in $L^p(\mathbf{T})$ $(p > 1)$ the problem of convergence almost everywhere is settled by the theorem of Carleson-Hunt. What about the corresponding problem in several dimensions, for functions in $L^p(\mathbf{T}^d)$, $d \geq 2$?

The first difficulty is that convergence of a multiple series is not defined in a unique way. For example, we can consider the sum of all terms whose indices are in a cube $[-R, R]^d$ and look for a limit as $R \to \infty$; this is the cubic process of summation. Or we can consider the same inside a sphere centered at 0 with radius R, and we have the spherical process of summation. The analogue of the Carleson-Hunt theorem holds for the cubic process (Charles Fefferman, Per Sjölin) but the situation is quite different for the spherical process : there is a function belonging to all $L^p(\mathbf{T}^d)$ $(p < 2)$ whose Fourier series diverges almost everywhere (Charles Fefferman 1972). The situation is not settled when $p \geq 2$.

A strikingly simple example is the indicator function of a small ball in $\mathbf{T}^d$ and its spherical partial sums considered at the centre of the ball. When $d = 2$ they converge to 1 ; when $d \geq 3$ they diverge. Then necessary and sufficient smoothness conditions for radial functions can be found in order to guarantee that the spherical partial sums at 0 converge (Pinsky 1993).

The case of Fourier series in several variables shows that it is very natural to enlarge the convergence problem, by considering summability processes other than the usual convergence of partial sums. We already mentioned the Abel-Poisson process. We shall see the Cesàro-Fejér process later ; in both cases and in many others there is no pathological behaviour. Following these processes the Fourier series of a continuous function converges uniformly, the Fourier series of a function in $L^p(\mathbf{T})$ converges in $L^p(\mathbf{T})$, and so on. A general framework is given in the first chapter of the book of Katznelson mentioned above.

There is a huge literature on orthogonal series and summability processes all along the twentieth century. We shall discuss part of it in the chapters on Lebesgue and Fejér. The main recent achievements can be found in the books of Olevskiĭ (1975) and Kašin and Saakian (1984). The questions of convergence and summability are linked with the behaviour of the L^1-norms of a sequence of kernels (D_n in the classical case) called the Lebesgue constants (after Fejér); this is both an observation and a program, as Lee Lorch pointed out in 1959. Sharp estimates of Lebesgue constants appear throughout the twentieth century and it is a living subject until now (see results and references in Lorch 1944 and Liflyand 1995).

Now we return to the nineteenth century.

4. Dirichlet and Jordan

The Dirichlet convergence theorem can be considered now as a particular case of the Dirichlet-Jordan theorem, that reads as follows : given a 2π-periodic function f with bounded variation, the Fourier series of f converges to $\frac{1}{2}(f(x+0)+f(x-0))$ at every point x.

The statement is due to Jordan (1881) and can be proved along exactly the same lines as the Dirichlet theorem. It is a nice theorem on Fourier series. However its main value is outside : its influence on Analysis is much more important than its interest for Fourier series. Let us try to explain.

Camille Jordan (1838-1922) was mainly interested in groups, algebra, and geometry related to algebra. In his *Collected research works* there is almost no paper in Analysis. The main exception is a three page note in *Comptes-Rendus*, where he introduced functions of bounded variation (called *fonctions à oscillation bornée* then) and stated his theorem. Functions of bounded variation did not appear again in the research papers of Jordan.

On the other hand, Camille Jordan was in charge of the *Cours d'Analyse de l'Ecole Polytechnique*. *"Cours d'Analyse"* should be understood in a wide sense, as a complete and up to date exposition of mathematical analysis and related matters. There were several editions of the book, and the turning point is in the second edition (1893), where Jordan develops such fundamental notions as simple curves, rectifiable curves, additive plane measures (*"étendue"*). The influence of this book was very important in promoting a new spirit in mathematical analysis, more attention to basic concepts and rigorous treatment. Rectifiable curves are just another way to consider continuous functions with bounded variation, and it is a far reaching notion. The fact that bounded variation appeared in connection with Fourier series is just a sign of the constant interplay between problems in Fourier series and fundamental notions in analysis. Other and more significant indications will be seen in the following chapters.

One more remark on Dirichlet and Jordan. Jordan proves that each function of bounded variation on an interval is integrable, and gives an example of a function of bounded variation discontinuous on a dense subset of the interval of definition. He concludes that Dirichlet was wrong in stating his necessary condition for integrability (that is, given $a < b$, there exist $a < r < s < b$ such that the function is continuous between r and s), and he says that he is surprised by such a mistake from such a mathematician. First, it would have been a fruitful mistake. Secondly, it was not a mistake at all. Jordan is not using the same notion of integral as Dirichlet - that is all.

5. Dirichlet's original paper.

G. Lejeune Dirichlet : Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données. *Crelle, Journal für die reine und angewandte Mathematik* 4 (1829), 157-169.

Les séries de sinus et de cosinus, au moyen desquelles on peut représenter une fonction arbitraire dans un intervalle donné, jouissent entre autres propriétés remarquables de celle d'être convergentes. Cette propriété n'avait pas échappé au géomètre illustre qui a ouvert une nouvelle carrière aux applications de l'analyse, en y introduisant la manière d'exprimer les fonctions arbitraires dont il est question; elle se trouve énoncée dans le Mémoire qui contient ses premières recherches sur la chaleur. Mais personne, que je sache, n'en a donné jusqu'à présent une démonstration générale. Je ne connais sur cet objet qu'un travail dû à M. Cauchy et qui fait partie des Mémoires de l'Académie des sciences de Paris pour l'année 1823. L'auteur de ce travail avoue lui-même que sa démonstration se trouve en défaut pour certaines fonctions pour lesquelles la convergence est pourtant incontestable. Un examen attentif du Mémoire cité m'a porté à croire que la démonstration qui y est exposée n'est pas même suffisante pour les cas auxquels l'auteur la croit applicable. Je vais, avant d'entrer en matière, énoncer en peu de mots les objections auxquelles la démonstration de M. Cauchy me paraît sujette. La marche que ce géomètre célèbre suit dans cette recherche, exige que l'on considère les valeurs que la fonction $\varphi(x)$ qu'il s'agit de développer, obtient, lorsqu'on y remplace la variable x par une quantité de la forme $u + v\sqrt{-1}$. La considération de ces valeurs semble étrangère à la question et l'on ne voit d'ailleurs pas bien ce que l'on doit entendre par le résultat d'une pareille substitution lorsque la fonction dans laquelle elle a lieu, ne peut pas être exprimée par une formule analytique. Je présente cette objection avec d'autant plus de confiance, que l'auteur me semble partager mon opinion sur ce point. Il insiste en effet dans plusieurs de ses ouvrages sur la nécessité de définir d'une manière précise le sens que l'on attache à une pareille substitution même lorsqu'elle est faite dans une fonction d'une loi analytique régulière; on trouve surtout dans le Mémoire qu'il a inséré dans le 19$^{\text{ième}}$ cahier du Journal de l'École Polytechnique pag. 567 et suiv., des remarques sur les difficultés que font naître les quantités imaginaires placées sous des signes de fonctions arbitraires. Quoi qu'il en soit de cette première observation, la démonstration de M. Cauchy donne encore lieu à une autre objection qui paraît ne laisser aucun doute sur son insuffisance. La considération des quantités imaginaires conduit l'auteur à un résultat sur le décroissement des termes de la série, qui est loin de prouver que ces termes forment une suite convergente. Le résultat dont il s'agit peut être énoncé comme il suit, en supposant que l'intervalle considéré s'étend depuis zéro jusqu'à 2π.

"Le rapport du terme dont le rang est n, à la quantité $A\frac{\sin nx}{n}$. (A désignant une constante déterminée, dépendante des valeurs extrêmes de la fonction) diffère de l'unité prise positivement d'une quantité qui diminue indéfiniment, à mesure que n devient plus grand."

De ce résultat et de ce que la série qui a $A\frac{\sin nx}{n}$ pour terme général, est convergente, l'auteur conclut que la série trigonométrique générale l'est également. Mais cette conclusion n'est pas permise, car il est facile de s'assurer que deux séries (du moins lorsque, comme il arrive ici, les termes n'ont pas tous le même signe) peuvent être l'une convergente, l'autre divergente, quoique le rapport de deux termes de même rang diffère aussi peu que l'on veut de l'unité prise positivement lorsque les termes sont d'un rang très avancé.

On en voit un exemple très simple dans les deux séries, ayant l'un pour terme général $\frac{(-1)^n}{\sqrt{n}}$, et l'autre $\frac{(-1)^n}{\sqrt{n}}\left(1+\frac{(-1)^n}{\sqrt{n}}\right)$. La première de ces séries est convergente, la seconde au contraire est divergente, car en la soustrayant de la première on obtient la série divergente :

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\text{ etc.}$$

et cependant le rapport de deux termes correspondants, qui est $1\pm\frac{1}{\sqrt{n}}$, converge vers l'unité à mesure que n croît.

Je vais maintenant entrer en matière, en commençant par l'examen des cas les plus simples, auxquels tous les autres peuvent être ramenés. Désignons par h un nombre positif inférieur ou tout au plus égal à $\frac{\pi}{2}$ et par $f(\beta)$ une fonction de β qui reste continue entre les limites 0 et h ; j'entends par là une fonction qui a une valeur finie et déterminée pour toute valeur de β comprise entre 0 et h, et en outre telle que la différence $f(\beta+\epsilon)-f(\beta)$ diminue sans limite lorsque ϵ devient de plus en plus petit. Supposons encore que la fonction reste toujours positive entre les limites 0 et h et qu'elle décroisse constamment depuis 0 jusqu'à h, en sorte que si p et q désignent deux nombres compris entre 0 et h, $f(p)-f(q)$ ait toujours un signe opposé à celui de $p-q$. Cela posé, considérons l'intégrale :

$$\int_0^h \frac{\sin i\beta}{\sin\beta} f(\beta)\, d\beta \tag{1}$$

dans laquelle i est une quantité positive, et voyons ce que cette intégrale deviendra à mesure que i croît. Pour cela partageons-la en plusieurs autres prises la première depuis $\beta=0$ jusqu'à $\beta=\frac{\pi}{i}$, la seconde depuis $\beta=\frac{\pi}{i}$ jusqu'à $\beta=\frac{2\pi}{i}$, et ainsi de suite, l'avant-dernière ayant pour limites $(r-1)\frac{\pi}{i}$ et $\frac{r\pi}{i}$, et la dernière $\frac{r\pi}{i}$ et h, $\frac{r\pi}{i}$ désignant le plus grand multiple de $\frac{\pi}{i}$ qui soit contenu dans h. Il est facile de voir que ces intégrales nouvelles, dont le nombre est $r+1$, sont alternativement positives et négatives, la fonction placée sous le signe d'intégration étant évidemment toujours positive entre les limites de la première, négative entre les limites de la seconde et ainsi de suite. Il n'est pas moins facile de se convaincre que chacune d'elles est plus petite que la précédente, abstraction faite du signe. En effet ν désignant un entier

$< r$, les expressions :

$$\int_{(\nu-1)\frac{\pi}{i}}^{\frac{\nu\pi}{i}} \frac{\sin i\beta}{\sin\beta} f(\beta)d\beta \quad \text{et} \quad \int_{\frac{\nu\pi}{i}}^{(\nu+1)\frac{\pi}{i}} \frac{\sin i\beta}{\sin\beta} f(\beta)d\beta$$

représentent deux intégrales consécutives. Remplaçons dans la seconde β par $\frac{\pi}{i}+\beta$; elle se changera ainsi en celle-ci :

$$\int_{(\nu-1)\frac{\pi}{i}}^{\frac{\nu\pi}{i}} \frac{\sin(i\beta+\pi)}{\sin(\beta+\frac{\pi}{i})} f\left(\beta+\frac{\pi}{i}\right) d\beta$$

ou ce qui revient au même :

$$-\int_{(\nu-1)\frac{\pi}{i}}^{\frac{\nu\pi}{i}} \frac{\sin i\beta}{\sin(\beta+\frac{\pi}{i})} f\left(\beta+\frac{\pi}{i}\right) d\beta.$$

Les deux intégrales qu'il s'agit de comparer ayant ainsi les mêmes limites, on voit sans peine que la seconde a une valeur numérique inférieure à celle de la première. Il suffit pour cela de remarquer qu'il suit de la supposition faite sur la fonction $f(\beta)$:

$$f\left(\frac{\pi}{i}+\beta\right) < f(\beta),$$

et que d'un autre côté :

$$\sin\left(\frac{\pi}{i}+\beta\right) > \sin\beta,$$

les arcs β et $\frac{\pi}{i}+\beta$ étant l'un et l'autre moindres que $\frac{\pi}{2}$, car il en résulte l'inégalité:

$$\frac{f(\beta)}{\sin\beta} > \frac{f\left(\beta+\frac{\pi}{i}\right)}{\sin\left(\beta+\frac{\pi}{i}\right)},$$

qui ayant lieu pour toutes les valeurs de β intermédiaires entre les limites $(\nu-1)\frac{\pi}{i}$ et $\frac{\nu\pi}{i}$, fait voir, comme nous l'avons dit, que chaque intégrale est plus grande que celle qui la suit, abstraction faite du signe. Cette circonstance a lieu a fortiori, lorsqu'on compare l'avant-dernière à la dernière, attendu que la différence des limites $\frac{r\pi}{i}$ et h de la dernière est inférieure à $\frac{\pi}{i}$, différence commune des limites de toutes les autres.

Examinons actuellement un peu plus en détail l'intégrale du rang ν, qui est :

$$\int_{(\nu-1)\frac{\pi}{i}}^{\frac{\nu\pi}{i}} \frac{\sin i\beta}{\sin\beta} f(\beta)d\beta.$$

Comme la fonction de β qui se trouve sous la signe $\int$ est le produit des facteurs $\frac{\sin i\beta}{\sin\beta}$ et $f(\beta)$, qui sont l'un et l'autre des fonctions continues de β entre les limites de l'intégration, et comme d'un autre côté le premier de ces facteurs conserve toujours le même signe entre ces mêmes limites, on conclura, en vertu d'un théorème connu, que l'intégrale considérée est égale à l'intégrale du premier facteur multipliée par une quantité comprise entre la valeur la plus grande et la valeur la plus petite de l'autre facteur. Le second facteur décroissant depuis la première limite jusqu'à la seconde, la quantité dont il s'agit est comprise entre $f\left(\frac{(\nu-1)\pi}{i}\right)$ et $f\left(\frac{\nu\pi}{i}\right)$. En la désignant par ρ, notre intégrale sera équivalente à :

$$\rho_\nu \int_{(\nu-1)\frac{\pi}{i}}^{\frac{\nu\pi}{i}} \frac{\sin i\beta}{\sin\beta} d\beta.$$

L'intégrale que renferme encore cette expression, dépend à la fois de ν et de i. Elle est positive ou négative selon que $\nu - 1$ est pair ou impair ; nous la désignerons désormais par K_ν, abstraction faite du signe. Nous aurons bientôt besoin de connaître la limite vers laquelle elle converge, lorsque ν restant invariable, i devient de plus en plus grand. Pour découvrir cette limite, remplaçons β par $\frac{\gamma}{i}$, γ étant une nouvelle variable. Nous aurons ainsi :

$$\int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin\gamma}{i\sin(\frac{\gamma}{i})} d\gamma.$$

Sous cette forme, il est évident qu'elle converge vers la limite :

$$\int_{(\nu-1)\pi}^{\nu\pi} \frac{\sin\gamma}{\gamma} d\gamma,$$

que pour abréger nous désignerons par k_ν, abstraction faite du signe.

On sait que l'intégrale $\int_0^\infty \frac{\sin\gamma}{\gamma} d\gamma$ a une valeur finie et égale à $\frac{\pi}{2}$. Cette intégrale peut être partagée en une infinité d'autres, prises la première depuis $\gamma = 0$ jusqu'à $\gamma = \pi$, la seconde depuis $\gamma = \pi$ jusqu'à $\gamma = 2\pi$, et ainsi de suite. Ces nouvelles intégrales sont alternativement positives et négatives, chacune d'elles à une valeur numérique inférieure à celle de la précédente, et celle du rang ν est k_ν, abstraction faite du signe. La proposition qu'on vient de citer, revient donc à dire que la suite infinie :

$$(2) \qquad k_1 - k_2 + k_3 - k_4 + k_5 - \text{ etc.}$$

est convergente et a une somme égale à $\frac{\pi}{2}$.

Les termes de cette suite allant toujours en décroissant, il suit d'une proposition connue que la somme des n premiers termes est supérieure ou inférieure à $\frac{\pi}{2}$, selon que n est impair ou pair, et que cette somme qu'on peut désigner par S_n, diffère de $\frac{\pi}{2}$ d'une quantité moindre que le terme suivant k_{n+1}.

Reprenons actuellement l'intégrale (1) et cherchons à déterminer la limite vers laquelle elle converge lorsque i croît indéfiniment. En faisant ainsi croître le nombre i, les intégrales dans lesquelles nous avons décomposé l'intégrale (1), changeront sans cesse de valeur en même temps que leur nombre augmentera ; il s'agit de connaître le résultat de ce double changement lorsqu'il continue indéfiniment. Pour cela, prenons un nombre entier m (qu'il soit supposé pair pour plus de simplicité) et supposons que le nombre m reste invariable pendant que i croît. Le nombre r, qui croît sans cesse avec i, finira bientôt par surpasser le nombre invariable m, quelque grand qu'on l'ait choisi.

Cela posé, partageons en deux groupes les intégrales dont la somme est équivalente à l'intégrale (1). Le premier groupe comprendra les m premières de ces intégrales, et le second sera composé de toutes les suivantes. On aura pour la somme du premier groupe :

$$K_1\rho_1 - K_2\rho_2 + K_3\rho_3 - K_4\rho_4 + \cdots - K_m\rho_m \tag{3}$$

et le second, dont le nombre des termes croît sans cesse avec i, a pour premiers termes:

$$K_{m+1}\rho_{m+1} - K_{m+2}\rho_{m+2} + \cdots \tag{4}$$

Considérons séparément ces deux groupes. Le nombre i croissant indéfiniment, la somme (3) convergera vers une limite qu'il est facile de déterminer. En effet, les quantités $\rho_1, \rho_2, \cdots, \rho_m$ qui sont comprises la première entre $f(0)$ et $f(\frac{\pi}{i})$, la seconde entre $f(\frac{2\pi}{i})$, et la dernière entre $f\left(\frac{(m-1)\pi}{i}\right)$ et $f\left(\frac{m\pi}{i}\right)$ convergent chacune vers la limite $f(0)$, lorsque, m restant invariable, i croît sans cesse. D'un autre côté nous avons vu que les quantités :

$$K_1, K_2, \cdots, K_m$$

convergent dans les mêmes circonstances respectivement vers les limites :

$$k_1, k_2, \cdots, k_m.$$

Donc la somme (3) converge vers la limite :

$$(k_1 - k_2 + k_3 - \cdots - k_m)f(0) = S_m f(0),$$

ce qui veut dire que la différence entre la somme (3) et $S_m f(0)$ finira toujours, abstraction faite du signe, par être constamment inférieure à ω, ω désignant une quantité positive aussi petite que l'on veut.

Considérons pareillement la somme (4), dont le nombre des termes augmente sans cesse. Ses termes étant alternativement positifs et négatifs, et chacun d'eux ayant une valeur numérique inférieure à celle du terme précédent, comme nous l'avons vu plus haut, en considérant les intégrales que ces termes représentent, il suit d'un principe connu(*) que cette somme, quel que soit le nombre de ses termes, est positive comme son premier terme $K_{m+1}\rho_{m+1}$ et a une valeur inférieure à celle de ce terme. Or ce premier terme convergeant vers la limite $k_{m+1}f(0)$, il s'ensuit que la somme (4) finira toujours par être inférieure à $k_{m+1}f(0)$ augmenté d'une quantité positive ω' aussi petite que l'on veut. En combinant ce résultat avec celui que nous avons obtenu sur la somme (3), il n'y a qu'un instant, on verra que l'intégrale (1) qui est la somme des expressions (3) et (4) finira toujours par différer de $S_m f(0)$ d'une quantité moindre, abstraction faite du signe, que $\omega+\omega'+k_{m+1}f(0)$, ω et ω' étant deux nombres d'une petitesse arbitraire. D'un autre côté S_m diffère de $\frac{\pi}{2}$ d'une quantité numériquement inférieure à k_{m+1} ; donc l'intégrale finira toujours par différer de $\frac{\pi}{2}f(0)$ d'une quantité moindre que $\omega+\omega'+2k_{m+1}f(0)$, abstraction faite du signe.

Comme m peut être choisi tellement grand que k_{m+1} soit moindre que toute grandeur donnée, il s'ensuit que l'intégrale (1) finira toujours, lorsque i croît sans limite, par différer constamment de $\frac{\pi}{2}f(0)$ d'une quantité moindre, abstraction faite du signe, qu'un nombre aussi petit que l'on veut. Il est ainsi prouvé, que l'intégrale (1) converge vers la limite $\frac{\pi}{2}f(0)$ pour des valeurs croissantes de i.

Supposons maintenant que la fonction $f(\beta)$, au lieu d'être toujours décroissante depuis 0 jusqu'à h, soit constante et égale à l'unité. On pourra dans ce cas déterminer la limite vers laquelle converge l'intégale (1) par les mêmes considérations que nous venons d'employer ; c'est ce qu'on voit tout de suite, en se rappelant que la démonstration précédente est fondée sur ce que les intégrales, dans lesquelles nous avons décomposé l'intégrale (1), forment une suite décroissante. Or ce décroissement tient à deux choses, au décroissement du facteur $f(\beta)$ et à l'accroissement du diviseur $\sin\beta$. Si $f(\beta)$ devient un nombre constant, l'accroissement de $\sin\beta$ suffira toujours pour rendre chaque intégrale de la série plus petite que la précédente. On trouvera ainsi, en supposant toujours h positive et tout au plus égale à $\frac{\pi}{2}$, que l'intégrale $\int_0^h \frac{\sin i\beta}{\sin\beta}d\beta$ converge vers la limite $\frac{\pi}{2}$. Il suit de là que l'intégrale $\int_0^h c\frac{\sin i\beta}{\sin\beta}d\beta$, dans laquelle c est une constante positive ou négative, converge vers la limite $c\frac{\pi}{2}$.

(*) Le principe sur lequel nous nous appuyons peut être énoncé de cette manière.

Les lettres $A, A', A'', \cdots$ désignant les quantités positives en nombre quelconque et telles que

$$A > A' > A'' > \text{etc.},$$

la quantité :

$$A - A' + A'' - A''' + \text{etc.}$$

est positive et inférieure à A. Cela résulte immédiatement de ce que la quantité précédente peut être mise sous l'une et l'autre de ces deux formes :

$$(A - A') + (A'' - A''') + \text{etc.},$$

$$A - (A' - A'') - (A''' - A^{IV}) - \text{etc.}$$

Nous avons supposé que la fonction $f(\beta)$ était décroissante et positive entre les limites 0 et h. La première circonstance ayant toujours lieu, c'est-à-dire la fonction étant telle que $f(p)-f(q)$ ait un signe contraire à celui de $p-q$ pour des valeurs p et q comprises entre 0 et h, supposons que $f(\beta)$ ne soit pas toujours positive. On prendra alors une constante positive c assez grande pour que $c+f(\beta)$ conserve toujours un signe positif depuis $\beta=0$ jusqu'à $\beta=h$. L'intégrale $\int_0^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$ étant égale à la différence de celles-ci :

$$\int_0^h [c+f(\beta)]\frac{\sin i\beta}{\sin\beta}d\beta \quad \text{et} \quad \int_0^h c\frac{\sin i\beta}{\sin\beta}d\beta,$$

sa limite sera la différence des limites vers lesquelles convergent ces dernières. Or ces dernières rentrent dans les cas précédemment examinés ($c+f(\beta)$ étant une fonction décroissante et positive) et convergent vers les limites $[c+f(0)]\frac{\pi}{2}$ et $c\frac{\pi}{2}$, d'où il suit que la première converge vers la limite $\frac{\pi}{2}f(0)$.

Considérons actuellement une fonction $f(\beta)$ croissante depuis 0 jusqu'à h. Dans ce cas $-f(\beta)$ sera une fonction décroissante. L'intégrale

$$\int_0^h -f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$$

convergera donc vers la limite $-\frac{\pi}{2}f(0)$, et par conséquent l'intégrale

$$\int_0^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$$

vers la limite $\frac{\pi}{2}f(0)$.

En réunissant ces résultats, on aura cet énoncé :

(5) "Quelle que soit la fonction $f(\beta)$, pourvu qu'elle reste continue entre les limites 0 et h (h étant positive et tout au plus égale à $\frac{\pi}{2}$), et qu'elle croisse ou qu'elle décroisse depuis la première de ces limites jusqu'à la seconde, l'intégrale $\int_0^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$ finira par différer constamment de $\frac{\pi}{2}f(0)$ d'une quantité moindre que tout nombre assignable, lorsqu'on y fait croître i au delà de toute limite positive."

Désignons par g un nombre positif différent de zéro et inférieur à h, et supposons que la fonction reste continue et croisse ou décroisse depuis g jusqu'à h. L'intégrale $\int_0^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$ convergera alors vers une limite qu'il est facile de découvrir. On pourrait y parvenir par des considérations analogues à celles que nous avons appliquées à l'intégrale (1) ; mais il est plus simple de ramener ce nouveau cas à ceux que nous avons considérés dans ce qui précède. La fonction n'étant donnée que depuis g jusqu'à h, reste entièrement arbitraire pour les valeurs de β comprises entre 0 et g. Supposons que l'on entende par $f(\beta)$, pour les valeurs de β comprises entre 0 et g, une fonction continue et croissante ou décroissante depuis 0 jusqu'à g, selon que $f(\beta)$

croît ou décroît depuis g jusqu'à h ; supposons encore que $f(g-\epsilon)$ diffère infiniment peu de $f(g+\epsilon)$, si ϵ décroît dans limite ; ayant satisfait d'une manière quelconque à ces conditions, ce qu'on peut toujours faire d'une infinité de manières, la fonction $f(\beta)$ remplira depuis 0 jusqu'à h les conditions exprimées dans l'énoncé (5). Les intégrales :

$$\int_0^g f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta \quad \text{et} \quad \int_0^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$$

convergeront donc l'une et l'autre vers la limite $\frac{\pi}{2}f(0)$. D'où l'on conclut que l'intégrale $\int_0^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$, qui est la différence des précédentes, a zéro pour limite.

Ce nouveau résultat peut être réuni en un seul énoncé avec celui que nous avons obtenu plus haut. On aura ainsi :

(6) "La lettre h désignant une quantité positive tout au plus égale à $\frac{\pi}{2}$, et g une quantité également positive et en outre inférieure à h, l'intégrale :

$$\int_g^h f(\beta)\frac{\sin i\beta}{\sin\beta}d\beta$$

dans laquelle la fonction $f(\beta)$ est continue entre les limites de l'intégration et a une marche toujours croissante ou toujours décroissante depuis $\beta = g$ jusqu'à $\beta = h$, convergera vers une certaine limite, lorsque le nombre i devient de plus en plus grand. Cette limite est égale à zéro, le seul cas excepté où g a une valeur nulle, dans ce cas elle a la valeur $\frac{\pi}{2}f(0)$."

Il est évident que ce résultat ne serait que légèrement modifié, si la fonction $f(\beta)$ présentait une solution de continuité pour $\beta = g$ et $\beta = h$, c'est-à-dire si $f(g)$ était différent de $f(g+\epsilon)$ et $f(h)$ de $f(h-\epsilon)$, ϵ désignant une quantité infiniment petite et positive, pourvu qu'alors les valeurs $f(g)$ et $f(h)$ ne fussent pas infinies. Il faudrait seulement dans ce cas remplacer $f(0)$ par $f(\epsilon)$ dans l'énoncé précédent, ce qu'on peut faire encore même quand il n'y a pas de solution de continuité, attendu qu'alors $f(\epsilon)$ est égale à $f(0)$.

Nous sommes maintenant en état de prouver la convergence des séries périodiques qui expriment des fonctions arbitraires entre des limites données. La marche que nous allons suivre nous conduira à établir la convergence de ces séries et à déterminer en même temps leurs valeurs. Soit $\varphi(x)$ une fonction de x, ayant une valeur finie et déterminée pour chaque valeur de x comprise entre $-\pi$ et π, et supposons qu'il s'agisse de développer cette fonction en une série de sinus et de cosinus d'arcs multiples de x. La série qui résout cette question, est, comme l'on sait :

$$(7)\quad \frac{1}{2\pi}\int\varphi(\alpha)d\alpha + \frac{1}{\pi}\left\{\begin{array}{l}\cos x\int\varphi(\alpha)\cos\alpha\, d\alpha + \cos 2x\int\varphi(\alpha)\cos 2\alpha\, d\alpha + \cdots \\ \sin x\int\varphi(\alpha)\sin\alpha\, d\alpha + \sin 2x\int\varphi(\alpha)\sin 2\alpha\, d\alpha + \cdots\end{array}\right\},$$

les intégrales qui déterminent les coefficients constants, étant prises depuis $\alpha = -\pi$ jusqu'à $\alpha = \pi$, et x désignant une quantité quelconque comprise entre $-\pi$ et π (*Théorie de la Chaleur*, n° 232 et suiv.).

Considérons les $2n+1$ premiers termes de cette série (n étant un nombre entier) et voyons vers quelle limite converge la somme de ces termes, lorsque n devient de plus en plus grand. Cette somme peut être mise sous la forme suivante:

$$\frac{1}{\pi}\int_{-\pi}^{+\pi}\varphi(\alpha)d\alpha[\frac{1}{2}+\cos(\alpha-x)+\cos 2(\alpha-x)+\cdots+\cos n(\alpha-x)],$$

ou en sommant la suite de cosinus :

$$(8) \qquad \frac{1}{\pi}\int_{-\pi}^{+\pi}\varphi(\alpha)\frac{\sin(n+1/2)(\alpha-x)}{2\sin 1/2(\alpha-x)}d\alpha.$$

Tout se réduit maintenant à déterminer la limite dont cette intégrale approche sans cesse, lorsque n croît indéfiniment. Pour cela nous la partagerons en deux autres prises l'une depuis $-\pi$ jusqu'à x, l'autre depuis x jusqu'à π. Si l'on remplace dans la première α par $x-2\beta$, et dans la seconde α par $x+2\beta$, β étant une nouvelle variable, ces deux intégrales se changeront en celles-ci, abstraction faite du facteur $\frac{1}{\pi}$:

$$(9) \quad \int_0^{1/2(\pi+x)}\frac{\sin(2n+1)\beta}{\sin\beta}\varphi(x-2\beta)d\beta \quad \text{et} \quad \int_0^{1/2(\pi-x)}\frac{\sin(2n+1)\beta}{\sin\beta}\varphi(x+2\beta)d\beta.$$

Considérons la seconde de ces deux intégrales. La quantité x étant inférieure ou tout au plus égale à π, abstraction faite du signe, $1/2(\pi-x)$ ne pourra tomber hors des limites 0 et π. Si $1/2(\pi-x)=0$, ce qui a lieu lorsque $x=\pi$, l'intégrale est nulle quel que soit ν ; dans tous les autres cas elle convergera pour des valeurs croissantes de n vers une limite que nous allons déterminer. Supposons d'abord $1/2(\pi-x)$inférieure ou tout au plus égale à $\frac{\pi}{2}$, et remarquons que la fonction $\varphi(x+2\beta)$ peut présenter plusieurs solutions de continuité depuis $\beta=0$ jusqu'à $\beta=1/2(\pi-x)$, et qu'elle peut aussi avoir plusieurs maxima et minima dans ce même intervalle. Désignons par ℓ, ℓ', ℓ'', $\cdots$, $\ell^{(\nu)}$, rangées selon l'ordre de leur grandeur, les différentes valeurs de β qui présentent l'une ou l'autre de ces circonstances, et décomposons notre intégrale en plusieurs autres prises respectivement entre les limites :

$$0 \text{ et } \ell,\ \ell \text{ et } \ell',\ \ell' \text{ et } \ell'',\cdots,\ell^{(\nu)} \text{ et } 1/2(\pi-x).$$

Toutes ces intégrales se trouveront dans le cas de l'énoncé (6). Elles convergeront donc toutes vers la limite zéro à mesure que n croît, à l'exception de la première qui converge vers la limite $\frac{\pi}{2}\varphi(x+\epsilon)$, ϵ étant un nombre infiniment petit et positif. Si $1/2(\pi-x)$ était supérieure à $1/2\pi$, ce qui arrivera lorsque x a une valeur négative, on partagerait l'intégrale en deux autres, l'une prise depuis $\beta=0$ jusqu'à $\beta=1/2\pi$, l'autre depuis $\beta=1/2\pi$ jusqu'à $\beta=1/2(\pi-x)$. La première de ces nouvelles intégrales se trouvera dans le même cas que celle que nous venons de considérer, elle convergera donc vers la limite $\frac{\pi}{2}\varphi(x+\epsilon)$. Quant à la seconde, on peut la changer en celle-ci, en y remplaçant β par $\pi-\gamma$, γ étant une nouvelle variable :

$$\int_{1/2(\pi+x)}^{1/2\pi}\varphi(x+2\pi-2\gamma)\frac{\sin(2n+1)(\pi-\gamma)}{\sin(\pi-\gamma)}d\gamma,$$

ou ce qui revient au même, n étant un entier :

$$\int_{1/2(\pi+x)}^{1/2\pi} \varphi(x+2\pi-2\gamma)\frac{\sin(2n+1)\gamma}{\sin\gamma}d\gamma.$$

Elle a ainsi une forme analogue à celle de la précédente ; en la décomposant comme précédemment en plusieurs autres, on verra qu'elle converge vers la limite zéro, le seul cas excepté, où $1/2(\pi+x)$ a une valeur nulle, c'est-à-dire lorsque $x=-\pi$; dans ce cas elle approche continuellement de la limite $\varphi(\pi-\epsilon)$, ϵ ayant toujours la même signification. En résumant tout ce qui précède, on trouvera que la seconde des intégrales (9) est nulle lorsque $x=\pi$, qu'elle converge vers la limite $\frac{\pi}{2}[\varphi(\pi-\epsilon)+\varphi(-\pi+\epsilon)]$ lorsque $x=-\pi$, et que dans tous les autres cas elle approche continuellement de la limite $\frac{\pi}{2}\varphi(x+\epsilon)$.

La première des intégrales (9) est entièrement analogue à la seconde ; en y appliquant des considérations semblables, on trouvera qu'elle est nulle lorsque $x=-\pi$, qu'elle converge vers la limite $\frac{\pi}{2}[\varphi(\pi-\epsilon)+\varphi(-\pi+\epsilon)]$ lorsque $x=\pi$ et que dans tous les autres cas elle a pour limite $\frac{\pi}{2}\varphi(x-\epsilon)$. Connaissant ainsi les limites de chacune des intégrales (9), il est facile de trouver la limite dont l'intégrale (8) approche sans cesse, lorsque n devient de plus en plus grand ; il suffit pour cela de se rappeler que cette intégrale est égale à la somme des intégrales (9) divisée par π. Or, l'intégrale (8) étant équivalente à la somme des $2n+1$ premiers termes de la série (7), il est prouvé que cette série est convergente, et l'on trouve au moyen des résultats précédents qu'elle est égale à :

$$1/2[\varphi(x+\epsilon)+\varphi(x-\epsilon)]$$

pour toute valeur de x comprise entre $-\pi$ et π, et que pour chacune des valeurs extrêmes π et $-\pi$, elle est égale à :

$$1/2[\varphi(\pi-\epsilon)+\varphi(-\pi+\epsilon)].$$

L'exposé précédent embrasse tous les cas ; il se simplifie lorsque la valeur de x qu'on considère n'est pas une de celles qui présentent une solution de continuité. En effet les quantités $\varphi(x+\epsilon)$ et $\varphi(x-\epsilon)$ étant alors l'une et l'autre équivalentes à $\varphi(x)$, on voit que la série a pour valeur $\varphi(x)$.

Les considérations précédentes prouvent d'une manière rigoureuse que, si la fonction $\varphi(x)$, dont toutes les valeurs sont supposées finies et déterminées, ne présente qu'un nombre fini de solutions de continuité entre les limites $-\pi$ et π, et si en outre elle n'a qu'un nombre déterminé de maxima et de minima entre ces mêmes limites, la série (7), dont les coefficients sont des intégrales définies dépendantes de la fonction $\varphi(x)$, est convergente et a une valeur généralement exprimée par :

$$1/2[\varphi(x+\epsilon)+\varphi(x-\epsilon)],$$

où ϵ désigne un nombre infiniment petit. Il nous resterait à considérer les cas où les suppositions que nous avons faites sur le nombre des solutions de continuité et sur celui des valeurs maxima et minima cessent d'avoir lieu. Ces cas singuliers peuvent être ramenés à ceux que nous venons de considérer. Il faut seulement, pour que la série (8) présente un sens lorsque les solutions de continuité sont en nombre infini, que la fonction $\varphi(x)$ remplisse la condition suivante.

Il est nécessaire qu'alors la fonction $\varphi(x)$ soit telle que, si l'on désigne par a et b deux quantités quelconques comprises entre $-\pi$ et π, on puisse toujours placer entre a et b d'autres quantités r et s assez rapprochées pour que la fonction reste continue dans l'intervalle de r à s. On sentira facilement la nécessité de cette restriction en considérant que les différents termes de la série sont des intégrales définies et en remontant à la notion fondamentale des intégrales. On verra alors que l'intégrale d'une fonction ne signifie quelque chose qu'autant que la fonction satisfait à la condition précédemment énoncée. On aurait un exemple d'une fonction qui ne remplit pas cette condition, si l'on supposait $\varphi(x)$ égale à une constante déterminée c lorsque la variable x obtient une valeur rationnelle, et égale à une autre constante d, lorsque cette variable est irrationnelle. La fonction ainsi définie a des valeurs finies et déterminées pour toute valeur de x, et cependant on ne saurait la substituer dans la série, attendu que les différentes intégrales qui entrent dans cette série, perdraient toute signification dans ce cas. La restriction que je viens de préciser, et celle de ne pas devenir infinie, sont les seules auxquelles la fonction $\varphi(x)$ soit sujette et tous les cas qu'elles n'excluent pas peuvent être ramenés à ceux que nous avons considérés dans ce qui précède. Mais la chose, pour être faite avec toute la clarté qu'on peut désirer, exige quelques détails liés aux principes fondamentaux de l'analyse infinitésimale et qui seront exposés dans une autre note, où je m'occuperai aussi de quelques autres propriétés assez remarquables de la série (7).

Berlin, Janvier 1829.

6. A quotation of Jacobi.

C. G. J. Jacobi : lettre à Legendre, 2 juillet 1830. Gesammelte Werke, B. I, Berlin 1881, p. 454.

"M. Poisson n'aurait pas dû reproduire dans son rapport une phrase peu adroite de feu M. Fourier, où ce dernier nous fait reproche, à Abel et à moi, de ne pas nous être occupés de préférence du mouvement de la chaleur. Il est vrai que M. Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que, sous ce titre, une question de nombres vaut autant qu'une question du système du monde."

("The only purpose of science is the honour of the human spirit")

Chapter 5

RIEMANN AND REAL ANALYSIS

1. Riemann.

The life of Bernhardt Riemann was short : he was born in Breselenz (Hanover) in 1826; he died less than forty years later in Selasca (Italy), in 1866. He did not write mathematics very early. His first publication, 1851, was his inaugural dissertation (Ph. D. thesis) on functions of a complex variable, in which he created the notion of a Riemann surface. The whole of his mathematical production, including unpublished papers, was gathered after his death, in the form of a book of a few hundred pages. This book was translated into French and had a tremendous effect on mathematics. As an introduction to the French translation, Charles Hermite declared that the work of Riemann was "the most beautiful and the greatest of our time". Very likely no single book contains so many deep ideas in so many parts of mathematics.

Riemann surfaces, algebraic functions, elliptic integrals, abelian functions, theta functions, differential equations with algebraic coefficients, theoretical physics, partial differential equations, minimal surfaces, potential theory, conformal mappings, prime numbers and the zeta function, non-euclidean geometry, Riemannian metrics are some of the key words in Riemann's works. When Felix Klein analysed their common features he found that Riemann developed a profound geometric intuition about all of the above topics. And he wondered whether it was the case also in his work on trigonometric series, where Riemann established the principles of Infinitesimal Analysis. We shall discuss this point briefly later.

Here are a few chronological landmarks. The first teacher of Bernhardt Riemann was his father, a Protestant minister. There is no extraordinary report about him in high school. He enrolled at the University of Göttingen in 1846, to study theology and philology. In 1847 he moved to Berlin and studied mathematics with Jacobi, Dirichlet, Steiner. He moved back to Göttingen in 1849, met the physicist W. E. Weber, and studied physics, philosophy, education. As we have already said, his doctoral thesis, on analytic functions, was presented in 1851. Two memoirs were needed for the "habilitation" to teach at Göttingen. The first memoir was written on trigonometric series and we shall return to it in a moment. The second was on a topic chosen by the Faculty on a list of three submitted by Riemann ; Gauss chose: *"Ueber die Hypothesen welche der Geometrie zu Grunde liegen"*. Both memoirs were presented in 1854 but published only after Riemann's death. Then Riemann stayed in Göttingen as a Privatdozent, until he succeeded Dirichlet, in 1859. He married in 1862. The same year he suffered from pleuritis and had to move to Italy. He never recovered and died in 1866. Apparently he had been affected by tuberculosis long before.

The memoir on trigonometric series was published by R. Dedekind in 1867 and the first edition of his collected works was brought out by H. Weber in 1876. The resulting book may indeed be "the most beautiful and the greatest" mathematical book of all times.

2. The memoir on trigonometric series. The historical part.

The memoir consists of three parts. The first relates the history of the question as to whether a function can be represented by a trigonometric series. The second relies on the notion of the definite integral. The third is the longest and most difficult, and consists of a study of functions expressed by sums of trigonometric series in the most general situation. We shall analyse all three of them, and reproduce the second.

Riemann divided the history into three sections : from Euler to Fourier, from Fourier to Dirichlet, and after Dirichlet. He gave a detailed account of the contributions of d'Alembert, Euler and D. Bernoulli to the problem of vibrating strings and the controversy between them : d'Alembert rejecting his own solution when arbitrary initial positions and velocities are given, Euler claiming that his construction provides the general solution in the case of arbitrary data and rejecting the generality of Bernoulli's approach, Bernoulli assuming that the motion of a vibrating string is the superposition of harmonic motions. Then Lagrange's approach, from finite to infinite, supported Euler's claim, but was rejected by d'Alembert. As far as Bernoulli's solution was concerned, it was rejected by the other three.

Then came what Riemann called a new era, with the contributions of Fourier. Riemann had a rather severe opinion on Lagrange and Poisson. He declared that Fourier was the first who understood the nature of trigonometric series in an exact and complete manner. He mentioned the unfortunate tentative conclusions of Cauchy, but, instead of merely pointing out the mistakes, as Dirichlet has done before, he was able to make the most interesting and far reaching comments thereon.

Cauchy supposed that each periodic function $f(x)$ can be extended as an analytic function $f(x+iy)$ bounded in the whole plane, and he did not notice that it is possible only when $f(x)$ is constant. Now, according to Riemann, Cauchy needed only to extend $f(x)$ as the real part of an analytic function $F(x+iy)$ defined and bounded for $y > 0$, the possibility of which can be established either through complex methods or through Fourier series methods. The proof through complex methods was actually given by Riemann in his 1851 thesis. This comment of Riemann is only a few lines long, but it inaugurates the constant interplay between complex methods and Fourier series since then.

Cauchy stated wrongly that series with equivalent terms converge or diverge at the same time. This led Riemann to the useful distinction between absolutely convergent series ("first class") and convergent but not absolutely convergent series ("second class"). He showed that the sum of a series of the second class is an arbitrary real number if the order of the terms is modified (the complex or vectorial case was considered by Paul Lévy, and is still an interesting topic in topological vector spaces). He concluded that the rules concerning finite sums can be extended to the first class but not to the second.

Riemann described the method and results of Dirichlet mentioned in the preceding chapter. The conditions of Dirichlet (piecewise monotonicity) seemed natural to him ; he declared (wrongly, as the success of fractals in physics shows today) that these conditions were satisfied by all functions encountered in nature. On the other hand he observed that applications of Fourier series are not restricted to physics, but extend to number theory (he may have thought of theta functions ; in 1859 another justification would have been the Riemann functional equation for the zeta function). Here, he said, more general functions are relevant.

3. The memoir on trigonometric series. The notion of integral.

We reproduce this short part so that the reader can judge. However, some comments may be useful. Again, there are three sections : definition of the integral, condition of integrability, and examples. These are numbered 4, 5, 6, respectively.

The definition of the proper Riemann integral is given in a few lines, in the most complete and clearest way. Then the improper Riemann integral is defined, as a limit of proper integrals, in the way it is done now in all undergraduate courses. There are other extensions given by Cauchy, Riemann says, but with no universal agreement.

In the second section, two pages long, Riemann goes far beyond most modern university courses. He gives a necessary and sufficient condition for a function to be integrable. The question is first reduced to the case of a proper integral. The function is supposed to be bounded and its interval of definition is decomposed into a finite number of subintervals. Riemann's condition is that the total measure of subintervals of a convenient decomposition where the oscillation exceeds a given positive number can be made arbitrarily small. Another celebrated necessary and sufficient condition was given by Lebesgue in his *Lecons sur l'intégration* (1903) : a function is Riemann-integrable, stricto sensu, if and only if it is bounded and it is continuous except for a set of Lebesgue measure zero. Though Lebesgue's discovery was highly original the material was provided by Riemann. Let us express the transition from Riemann's to Lebesgue's condition in modern words : the set of points of discontinuity of any function is the countable union of those sets of points where the oscillation (defined as the limit of the oscillation in intervals shrinking around the point) is $\geq \frac{1}{n}$ (n integer). Such sets are closed, Lebesgue's condition expresses that they have zero measure, while Riemann's condition expresses that they are contained in a finite union of intervals whose total measure is arbitrarily small. Both conditions are the same for closed sets. As we see, many important notions such as countable union, closed sets, compactness of bounded closed sets, sets of vanishing Lebesgue measure were implicitely needed, given the problem and the condition provided by Riemann. Of course the Riemann integral does not play the same role now than a century ago, before the thesis of Lebesgue, when it was considered as *the* integral by essence - Jordan among others shared this bias. What remains important is Riemann's approach : real analysis needs precise definitions.

The third section gives examples of integrable functions. First, an illustration for a dense set of discontinuities :

$$f(x) = \sum_{1}^{\infty} \frac{(nx)}{n^2}$$

where $(x) = x - m$ when $|\ x - m\ | < \frac{1}{2}$, $(x) = 0$ when $x = m + \frac{1}{2}$ (m, an integer). This function jumps at $x = \frac{p}{2n}$ (irreducible fraction) by $\frac{\pi^2}{16n^2}$ and is continuous at all other points. Then Riemann gives the now classical sufficient conditions for a non-bounded decreasing function on $(0,1)$ to be integrable, by comparison with $\frac{1}{x}\log\frac{1}{x}\log\log\frac{1}{x}\cdots(\log_n\frac{1}{x})^\alpha$. Finally the function

$$\frac{d}{dx}\cos\left(\exp\frac{1}{x}\right)$$

illustrates the fact that no condition on the absolute value is necessary for integrability. More detailed examples of this kind appear later in relation with trigonometric series.

4. The memoir on trigonometric series. Functions representable by such series.

This part consists of seven sections, numbered 7 to 13.

Section 7 describes the program. While previous works consider properties of functions which imply that they can be developed into Fourier series, the purpose of Riemann is to investigate properties of functions under the mere assumption that they can be expressed as sums of series

$$\Omega : A_0 + A_1 + A_2 + \cdots$$

where $A_n = a_n \sin nx + b_n \cos nx$. When Ω converges let $f(x)$ be its value. Two cases should be considered. If $f(x)$ is defined for all values of x, then A_n tends to zero, hence Riemann says a_n and b_n tend to zero. This is true, but not obvious. Actually the proof that the coefficients a_n and b_n tend to zero when the general term A_n tends to zero for all values of x was given only by Georg Cantor in his thesis. Anyway, the first case to consider is the case when a_n and b_n tend to zero (this is the content of sections 8, 9, 10) and the opposite case is to be considered separately (in sections 11, 12, 13).

Let us skip a century. Under the assumption of section 8 (the coefficients tend to zero), Ω is the Fourier series of a distribution in the sense of Schwartz. From a local point of view, a distribution on the circle is nothing but the formal derivative of some order of a continuous function. This is exactly the starting point of Riemann. Integrating Ω twice he obtains a continuous function, $F(x)$. Here are three remarkable theorems :

I. Assuming that Ω converges to $f(x)$, the ratio

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-a-\beta)}{4\alpha\beta}$$

converges to $f(x)$ as α and β tend to zero so that the ratios α/β and β/α are bounded.

II. Returning to the general case, when the coefficients tend to zero,

$$\frac{F(x+2\alpha)+F(x-2\alpha)-2F(x)}{2\alpha}$$

tends to zero as α tends to zero uniformly with respect to x. Actually the concept of uniform convergence was not yet in use, but it is what Riemann proves. This theorem is the source of a notion defined by Antoni Zygmund : F is called a *smooth* function when this property holds. The notion proves unavoidable when classes of functions are considered in relation with best approximation by trigonometric polynomials or interpolation of operators.

III. Given a test function $\lambda(x)$, the Fourier coefficients of order n of the product $F(x)\lambda(x)$ are $o(\frac{1}{n^2})$ as $n \to \infty$.

Of course, Riemann neither uses the Landau notation $o(\ )$ nor the term of test function. For him $\lambda(x)$ is a twice differentiable function, vanishing outside an interval (b, c) together with its first and second derivative, and such that the second derivative is piecewise monotonic. The content of the theorem is not changed if we consider only test functions in the sense of Schwartz, that is, C^∞ functions supported by an interval ; such test functions are familiar to us now, as well as their use in localization problems. Here again Riemann opened the way.

The crucial point in theorem I is the case $\alpha = \beta$. In this case it can be expressed as a summability result, namely that

$$\lim_{\alpha\to 0}\sum_1^\infty A_n \frac{\sin^2 n\alpha}{n^2\alpha^2} = \sum_1^\infty A_n,$$

whenever the series in the second member converges. This is the summability theorem of Riemann. It is of the same type as the Abel-Poisson summability theorem

$$\lim_{r\uparrow 1}\sum_1^\infty A_n r^n = \sum_1^\infty A_n,$$

but more subtle. The Riemann summability was considered later by Lebesgue and other for exponents other than 2.

Theorems II and III have nothing to do with the convergence of the series. They apply to distributions whose Fourier coefficients tend to zero, called pseudofunctions by Kahane and Salem in their book of 1963.

Actually theorem III characterizes pseudofunctions, and theorem III together with theorem I characterizes the functions $f(x)$ which can be expressed as sums of everywhere convergent trigonometric series : this is the content of section 9.

As a corollary of theorem III, section 10 gives the Riemann localization principle: for a pseudofunction, the convergence of the Fourier series is a local property. Moreover, Riemann gives a most important example : a bounded and integrable function is a pseudofunction (that is, the Fourier coefficients tend to zero at infinity). Extended by Lebesgue to Lebesgue-integrable functions, this is the Riemann-Lebesgue theorem. Pseudofunctions are just those distributions for which the conclusion of the Riemann-Lebesgue theorem holds, and they enjoy a number of properties of integrable functions.

In the last three sections Riemann considers the cases when a_n and b_n do not tend to zero, or when the series Ω diverges for some values of x. In both cases it proves important to consider $f(x+t)+f(x-t)$. In particular (section 12) new examples of pseudofunctions are given, namely piecewise monotonic but unbounded functions f, such that for each a (the important case is when f is unbounded near a) both $f(a\pm t)=o(\frac{1}{t})$ $(t\to 0)$ and $f(a+t)+f(a-t)$ is integrable in the extended sense from 0 to some $\epsilon>0$.

5. The memoir on trigonometric series. The final section.

Section 13 is a firework of examples.

The first example is what Yves Meyer calls a "chirp". It is used to provide an integrable function - in the extended sense - such that the Fourier coefficients do not tend to zero. Namely,

$$f(x)=\frac{d}{dx}\left(x^{\nu}\cos\frac{1}{x}\right)\qquad(0<\nu<\frac{1}{2})$$

satisfies

$$\int_0^{2\pi} f(x)\cos n(x-a)\,dx\approx\frac{1}{2}\sin(2\sqrt{n}-na+\frac{\pi}{4})\sqrt{\pi}n^{\frac{1-2\nu}{4}}.$$

How did Riemann prove it ? He considered a more general example, what we now call an oscillatory integral, namely

$$\int_0^{2\pi}\frac{d}{dx}(\varphi(x)\cos\psi(x))\cos n(x-a)\,dx,$$

where $\varphi(x)$ tends to zero and $\psi(x)$ tends to infinity as $x\to 0$. In one page of clever estimates Riemann developed what is now known as the method of stationary phase: the above integral is a sum of four terms, the most important being

$$-\frac{1}{2}\int\varphi(x)\psi'(x)\sin(\psi(x)+n(x-a))\,dx.$$

The most important part of this integral lies around the points where $\psi(x)+n(x-a)$ is a maximum or minimum.

Therefore the Fourier series of a function integrable in the extended sense is not necessarily convergent.

Is it possible to construct an everywhere convergent trigonometric series whose sum is not integrable ? To be sure, Riemann had this question in mind, but he gave no example. The now classical example is due to Fatou and was published by Lebesgue in 1906 :

$$\sum_2^\infty \frac{\sin nx}{\log n}$$

converges everywhere and its sum is neither Riemann integrable in the extended sense nor Lebesgue integrable.

Riemann gave instead quite interesting examples of discontinuous functions whose singularities are everywhere dense.

First,

$$f(x) = \sum_1^\infty \frac{(nx)}{n}$$

where (x) has the same meaning as in section 6, that is, $(x) = x$ when $|x| < \frac{1}{2}$, $(x) = 0$ when $x = \frac{1}{2}$, and $(x+1) = (x)$. Riemann stated without proof that $f(x)$ exists for all rational values of x, and then can be expressed as

$$f(x) = \sum_1^\infty \frac{d'_n - d''_n}{n\pi} \sin 2\pi nx,$$

where d'_n is the number of odd divisors and d''_n the number of even divisors of n. Actually, $f(x)$ exists almost everywhere and its oscillation on each interval is infinite so that $f(x)$ is not Riemann integrable anywhere. On the other hand, it is Lebesgue integrable; the above series is its Fourier-Lebesgue series, and much more can be said on convergence of the series and properties of the function.

Both examples of Riemann using the function (x), namely

$$\sum_1^\infty \frac{(nx)}{n^2}, \quad \sum_1^\infty \frac{(nx)}{n}$$

go far beyond the theory of trigonometric series. They give a deep insight into what discontinuous functions may look like. The first is exactly what Paul Lévy called a compensated jump function : all jumps are negative and their sum is infinite but the continuous parts of (nx) provide a shift such that the series converges. Paul Lévy considered the simpler function

$$\sum_1^\infty \frac{(2^n x)}{2^n}$$

as an illustration of what occurs frequently in the theory of stochastic processes with independent increments. It is still simpler to obtain what Jordan discovered in 1881, a function of bounded variation, discontinuous on each interval, namely

$$\sum_1^\infty \frac{(nx)}{n^3}.$$

Obviously, Jordan overlooked Riemann's memoir. In order to obtain a nowhere Riemann integrable function expressed as sum of an everywhere convergent trigonometric series, more complicated series of the same type would be needed, such as

$$\sum_{1}^{\infty} \frac{(nx)}{n} \sin \log n.$$

Actually, there is a much simpler way to perform such a construction by starting from the example of Fatou instead of the function (x).

Secondly, Riemann introduced an interesting type of lacunary trigonometric series,

$$\sum_{0}^{\infty} c_n \cos n^2 x \quad , \quad \sum_{1}^{\infty} c_n \sin n^2 x,$$

where c_n is a positive sequence which decreases to zero so that the series $\sum c_n$ diverges. Again, he considered only rational multiples of π and he used results of Gauss on quadratic residues, namely the fact that

$$\sum_{0}^{m-1} \cos n^2 x \quad \text{and} \quad \sum_{0}^{m-1} \sin n^2 x$$

vanish at some multiples of $\frac{2\pi}{m}$ and are different from zero at others, in order to obtain both convergence and divergence of the above series on every interval. This is typical of the connections that Riemann was able to make between number theory and trigonometric series. We shall return to this type of series in a moment.

A third remarkable example is given in the form of the imaginary part of a Taylor series, namely

$$f(x) = \Im m \sum_{1}^{\infty} \frac{1}{n^3} (1 - q^n) \log \left(\frac{-\log(1 - q^n)}{q^n} \right) \qquad (q = e^{ix})$$

and its first and second derivatives, $f'(x)$ and $f''(x)$. If the series is replaced by its first term, the second derivative $f''(x)$ is essentially Fatou's example. Very likely, the *raison d'être* of this construction was to have an everywhere convergent trigonometric series whose sum is not integrable and needs the double integration procedure. However Riemann wrote only a few lines about : the only observation is that $f''(x)$ exists at infinitely many points x on each interval, while $f'(x)$ is infinite at infinitely many points. It may be that Riemann was also interested in giving an example of a nowhere continuable Taylor series in the unit disc $| q | < 1$.

Together with $\sum \frac{(nx)}{n}$, this third example is typical of what Hankel called the method of condensation of singularities in 1870. This method led eventually to the theorems of Banach and Steinhaus on linear operators in the 1920's.

A final example is the series

$$\sum_{1}^{\infty} \sin(n!\pi x),$$

which converges at all rational points while the coefficients do not tend to zero, and Riemann pointed out a few irrational values of x such that convergence holds.

The feeling in reading this section is that Riemann wrote it for himself. This is why we gave such a detailed account of it.

Before closing this review we go back to Felix Klein. He observed that the memoir on trigonometric series is a singular point in Riemann's works. It creates a new field (theory of integration, construction and study of discontinuous functions) leading to foundations of analysis and theory of functions of a real variable. And, contrary to all other papers of Riemann, geometric intuition is not visible immediately. However Felix Klein could not believe that Riemann would oppose rigor to geometric intuition in analysis, and clearly he was right. The characterization of integrable functions is geometric in nature and exhibits a new kind of geometric objects, linear sets of vanishing measure in the sense of Lebesgue. The technical power of Riemann when dealing with oscillatory integrals cannot be understood without paying attention to his geometric and dynamical insight ("let us look for the places where changes of signs follow each other as slowly as possible..." "let us study the run of the integral..."). The same holds with condensation of singularities. Riemann sees the objects of real analysis and acts on them as well as he does in complex analysis or theory of numbers. Geometric intuition is really present everywhere in Riemann's works.

6. Other special trigonometric series. Riemann and Weierstrass.

According to Weierstrass, Riemann told his students that there could exist continuous functions with no derivative at any point, and gave

$$\sum_{1}^{\infty} \frac{\sin n^2 x}{n^2}$$

as a possible candidate. Weierstrass said that he was not able to prove the nowhere differentiability of this function and that instead, using the same idea, he would prove it for a series

$$\sum_{1}^{\infty} a^n \cos b^n x$$

such that $0 < a < 1$ and b is large enough. Let us say a few words on what is known now.

The Weierstrass series represent nowhere differentiable functions as soon as

$ab \geq 1$ (G. H. Hardy 1916) and there are easy ways to prove this now, in particular using wavelets as we do in the second part of the book (for a purely trigonometric approach, see e.g. Katznelson's book, chapter V). However the cases $ab > 1$ and $ab = 1$ are quite different. When $ab > 1$, say, $a = b^{-\alpha}$ with $0 < \alpha < 1$, the sum of the series satisfies a Hölder condition of order α,

$$| f(x+h) - f(x) | \leq C | h |^{\alpha}$$

and moreover the behaviour is the same at every point :

$$\forall x, \quad \overline{\lim_{h \to 0}} \frac{| f(x+h) - f(x) |}{| h |^{\alpha}} > 0.$$

Truly $f(x)$ is an "irregular" function, but very regularly irregular. On the contrary, when $ab = 1$, the situation is more subtle and not yet understood completely : the Hölder condition becomes

$$| f(x+h) - f(x) | \leq C | h | \log \frac{1}{| h |}.$$

There are "rapid points" x, where

$$0 < \overline{\lim_{h \to 0}} \frac{| f(x+h) - f(x) |}{| h | \log \frac{1}{|h|}} < \infty$$

and that is a generic situation (meaning that it holds on a countable intersection of open dense subsets of the circle, that is, "quasi everywhere" in the sense of the Baire theory). There are "average points" x, where

$$0 < \overline{\lim_{h \to 0}} \frac{| f(x+h) - f(x) |}{| h | \log \frac{1}{|h|} \log \log \log \frac{1}{|h|}} < \infty$$

and that happens almost everywhere. There are also "slow points", where

$$0 < \overline{\lim_{h \to 0}} \frac{| f(x+h) - f(x) |}{| h |} < \infty$$

(Geza Freud 1962).

The graph of the Weierstrass function (Weierstrass assumed $ab > 1 + \frac{3\pi}{2}$) is the first historical example of what Benoît Mandelbrot called a fractal. A paradoxical situation is that the Hausdorff dimension of this object is still unkown.

Let us go to the Riemann series. Its formal derivative is

$$\sum_{1}^{\infty} \cos n^2 x$$

and Riemann may have thought of the limit values of Gaussian sums

$$\sum \exp(in^2(x+iy)) \quad (y>0)$$

as y tends to 0. The argument was used by Hardy, who was able to prove that Riemann's function has no derivative at any point, except perhaps points of the form $x=\pi\frac{2p+1}{2q+1}$ (1916). Half a century later, Joseph Gerver, then a student at Columbia University, became aware of the problem and solved it : the function is differentiable indeed at $x=\pi$ and also at all other points of the above form (see Gerver 1971 ; also Queffelec 1971)

Riemann's function is popular now because computer graphics allow us to see it in different scales, and modern analytic tools are used in order to understand its local behaviour at different points. Wavelets were used in this context. The most recent analytical tool for this purpose consists of functions $x^\nu \sin\frac{1}{x^\mu}$, called chirps, and they provide remarkable results. As we have already seen, chirps were actually introduced by Riemann at the beginning of section 13 of his memoir on trigonometric series. Stephane Jaffard has performed a complete "multifractal analysis" of Riemann's function, as we shall see at the end of this book.

7. An overview on the influence of Riemann's memoir just after 1867.

En passant we have already mentioned a number of new ways opened by Riemann: relations between real and complex methods in Fourier analysis, relations between trigonometric series and number theory, definition of the integral, characterization of integrable functions and introduction of sets of zero Lebesgue measure, construction of various types of discontinuous functions, scales of magnitude and integrability, general trigonometric series, formal integration, smooth functions, pseudo-functions and the use of test functions, summability of series, oscillatory integrals, condensation of singularities. Let us have a look at their influence on real analysis after the first publication in 1867.

This publication was due to R. Dedekind. Richard Dedekind, born in 1831, was a colleague and friend of Riemann in Göttingen, and the publication of Riemann's works took several years of his life. The Dedekind construction of real numbers as a foundation of analysis is expounded in his book *"Stetigkeit und irrazionale Zahlen"* (1872). At the same time Georg Cantor gave another construction, which we shall describe in the next chapter. After Riemann's investigations the topic was in the air.

Also in 1872 and 1873 two famous examples appeared : Weierstrass's nowhere differentiable function, and du Bois Reymond's continuous function whose Fourier series diverges at a point. In 1873 Gaston Darboux wrote to his friend Houël about Riemann's memoir, just translated into French : "one of its pearls is the definition of the integral ; therefrom I constructed a number of functions without derivative (see Dugac 1976, p. 27)." Here again the topic was in the air.

Important expositions on these topics, including new notions such as the four derivatives at a point (upper, lower ; left, right), are due to Ulysse Dini : an article in *Annali di Lincei* (1877) *"Sulle funzioni finite continue de variabili reali che non* *hanno mai derivata*", mentioning contributions by H. Hankel in 1870 and P. du Bois Reymond in 1875, and two major books : *"Fondamenti per la teorica delle funzioni di una variabile reale"* (1878) and *"Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale"* (1880), whose subject is one of the most beautiful part of modern analysis according to Dini.

However Riemann's influence on the foundations of mathematics is to be found primarily in the works of Georg Cantor on trigonometric series (1870-72). The beginning of set theory is in the 1872 paper of Cantor. We shall go back to this story in the chapter on Cantor.

8. A partial view on the influence of Riemann's memoir in the twentieth century.

In Riemann's memoir three notions are strongly interrelated : general trigonometric series, differentiability properties of functions, integration. We shall limit ourselves to a brief account of some developments.

On general trigonometric series the first continuation is described in the next chapter : it is Cantor's uniqueness theorem and the study of sets of uniqueness. The second main continuation was developed by D. Menšov from 1916 to 1954 and completed by S. Konjagin in 1989. It deals with the following questions : given a function f on the circle, is it possible to find a trigonometric series that converges to f almost everywhere ? Given two functions f and g on the circle, $f \leq g$, is it possible to find a trigonometric series whose partial sums S_n satisfy $\underline{\lim}\, S_n = f$ and $\overline{\lim}\, S_n = g$ almost everywhere ? Menšov's theorem answers the first question when f is measurable and finite. Konjagin's theorem gives a necessary and sufficient condition on f and g, namely that both are measurable and take values in $[-\infty, \infty]$ and the sets of x such that $f(x) = -\infty$ and such that $g(x) = +\infty$ differ only by a set of Lebesgue measure zero.

Let us mention a very curious trigonometric series constructed by Menšov and called a "universal trigonometric series". It has the following property : given any measurable and finite function f, there exists a sequence of partial sums of the series which converges to f almost everywhere.

Differentiability properties of functions appear in theorems I, II, III in relation to trigonometric series with coefficients tending to zero. The symmetric second derivative appears in theorem I : if the series converges to ℓ at x_0, the twice integrated series represents a continuous function whose symmetric second derivative exists and is equal to ℓ at x_0. The importance of the symmetric second derivative was recognised by H. A. Schwarz (1871) : linear functions are exactly those continuous functions whose symmetric second derivative vanishes. This is a real theorem. If the symmetric second derivative

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

is replaced by another kind of generalized second derivative, such as

$$\lim_{h \to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

the result is no longer valid (think of piecewise linear functions). There is a relation between different kinds of second or higher order derivatives but that was investigated much later, by Marcinkiewicz and Zygmund (1936). Denjoy wrote a whole book on the symmetric second derivative applied to trigonometric series (1941) as an introduction to his "totalization" theory.

Theorem II is the origin of the notion of smooth function, as we have already said. It establishes that a continuous function whose Fourier coefficients are $o\left(\frac{1}{n^2}\right)$ is smooth, that is, satisfies

$$f(x+h) + f(x-h) - 2f(x) = o(h) \qquad (h \to 0)$$

uniformly with respect to x. Other examples are lacunary series

$$\sum_1^\infty a_n \cos b^n x$$

where $a_n = o(b^{-n})$; let us recall that the Hardy-Weierstrass functions correspond to $a_n = b^{-n}$ and they just fail to be smooth. Actually, smooth functions are differentiable on an uncountable dense set. The basic paper is "Smooth functions" by Antoni Zygmund (1945) and Zygmund always referred to Riemann as the initiator of the notion.

Theorem III and its consequences can be formulated in using the language of distributions, test functions and pseudofunctions, as we did. The basic fact is that the product of a test function and a pseudofunction is a pseudofunction. Before the widespread use of Laurent Schwartz's distributions the classical expression was : formal multiplication of trigonometric series. Formal multiplication was elaborated by A. Rajchman and by A. Zygmund, then his student. The first important paper of Zygmund is devoted to this topic, and is called *"Sur la théorie riemannienne des séries trigonométriques"* (1926).

Integration is essential in order to understand Fourier formulas, and this was the starting point of Riemann when he devised his theory of the integral. Of course, integration goes far beyond Fourier series but there is always an interplay between Fourier series and any notion of the integral, since Fourier series are precisely those trigonometric series whose coefficients are obtained through the integral formulas of Fourier. Fourier-Stieltjes, Fourier-Lebesgue, Fourier-Schwartz series correspond to formulas

$$c_n = \int e^{-inx} d\mu(x)$$

$$c_n = \int e^{-inx} f(x) \frac{dx}{2\pi}$$

$$c_n = < T, \frac{1}{2\pi} e^{-inx} >$$

where the second members are a Stieltjes integral, a Lebesgue integral and a Schwartz scalar product on a circle of length 2π. The same can be said of Fourier-Young and Fourier-Denjoy. Usually Fourier series means Fourier-Lebesgue series now, and that will be explained in the chapter devoted to Lebesgue. A few words on Lebesgue and Denjoy are appropriate here. Needless to say, Lebesgue was influenced by Riemann, directly and through Cantor. Lebesgue's characterization of Riemann integrable functions was the first appearance of Lebesgue null sets in analysis. The theorem of Riemann-Lebesgue on Fourier coefficients and the fact that the Lebesgue integral is able to recover the coefficients of an everywhere convergent trigonometric series when the sum is bounded were crucial tests in favour of the Lebesgue theory of integration. The influence of Riemann is also obvious on Arnaud Denjoy. The first version of the Denjoy integration ("totalisation", 1912) was enough in order to integrate the derivatives but did not allow the integration of all sums of everywhere convergent trigonometric series : the Fatou series was a counter example. The second version ("totalisation complète", 1921) permits the integration of symmetric second derivatives and the computation of the coefficients of any everywhere convergent trigonometric series, given its sum. This is expounded in an impressive series of books called *"Leçons sur le calcul des coefficients d'une série trigonométrique"* (1941-1949) and uses three main tools developed after Riemann : the Cantor theory of transfinite numbers, the Baire theory of discontinuous functions, and the Lebesgue measure and integral.

9. An excerpt from Riemann's memoir
(sections 4, 5, 6)

Bernhard Riemann : Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe. (Habilitationsschrift, 1854).

Über den Begriff eines bestimmten Integrals und den Umfang seiner Gültigkeit.

4. Die Unbestimmtheit, welche noch in einigen Fundamentalpunkten der Lehre von den bestimmten Integralen herrscht, nöthigt uns, Einiges voraufzuschicken über den Begriff eines bestimmten Integrals und den Umfang seiner Gültigkeit.

Also zuerst : Was hat man unter $\int_a^b f(x)dx$ zu verstehen ?

Um dieses festzusetzen, nehmen wir zwischen a und b der Grösse nach auf einander folgend, eine Reihe von Werthen $x_1, x_2, \cdots, x_{n-1}$ an und bezeichnen der Kürze wegen $x_1 - a$ durch δ_1, $x_2 - x_1$ durch $\delta_2, \cdots$, $b - x_{n-1}$ durch δ_n und durch ϵ einen positiven ächten Bruch. Es wird alsdann der Werth der Summe

$$S = \delta_1 f(a + \epsilon_1\delta_1) + \delta_2 f(x_1 + \epsilon_2\delta_2) + \delta_3 f(x_2 + \epsilon_3\delta_3) + \cdots$$

$$+\delta_n f(x_{n-1}+\epsilon_n\delta_n)$$

von der Wahl der Intervalle δ und der Grössen ϵ abhängen. Hat sie nun die Eigenschaft, wie auch δ und ϵ gewählt werden mögen, sich einer festen Grenze A unendlich zu nähern, sobald sämmtliche δ unendlich klein werden, so heisst dieser Werth $\int_a^b f(x)dx$.

Hat sie diese Eigenschaft nicht, so hat $\int_a^b f(x)dx$ keine Bedeutung. Man hat jedoch in mehreren Fällen versucht, diesem Zeichen auch dann eine Bedeutung beizulegen, und unter diesen Erweiterungen des Begriffs eines bestimmten Integrals ist eine von allen Mathematikern angenommen. Wenn nämlich die Function $f(x)$ bei Annäherung des Arguments an einen einzelnen Werth c in dem Intervalle (a, b) unendlich gross wird, so kann offenbar die Summe S, welchen Grad von Kleinheit man auch den δ vorschreiben möge, jeden beliebigen Werth erhalten ; sie hat also keinen Grenzwerth, und $\int_a^b f(x)dx$ würde nach dem Obigen keine Bedeutung haben. Wenn aber alsdann

$$\int_a^{c-\alpha_1} f(x)dx + \int_{c+\alpha_2}^b f(x)dx$$

sich, wenn α_1 und α_2 unendlich klein werden, einer festen Grenze nähert, so versteht man unter $\int_a^b f(x)dx$ diesen Grenzwerth.

Andere Festsetzungen von Cauchy über den Begriff des bestimmten Integrals in den Fällen, wo es dem Grundbegriffe nach ein solches nicht giebt, mögen für einzelne Klassen von Untersuchungen zweckmässig sein ; sie sind indess nicht allgemein eingeführt und dazu, schon wegen ihrer grossen Willkürlichkeit, wohl kaum geeignet.

5. Untersuchen wir jetzt zweitens den Umfang der Gültigkeit dieses Begriffs oder die Frage : in welchen Fällen lässt eine Function eine Integration zu und in welchen nicht ?

Wir betrachten zunächst den Integralbegriff im engern Sinne, d. h. wir setzen voraus, dass die Summe S, wenn sämmtliche δ unendlich klein werden, convergirt. Bezeichnen wir also die grösste Schwankung der Function zwischen a und x_1, d. h. den Unterschied ihres grössten und kleinsten Werthes in diesem Intervalle, durch D_1, zwischen x_1 und x_2 durch $D_2 \cdots$, zwischen x_{n-1} und b durch D_n, so muss

$$\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$$

mit den Grössen δ unendlich klein werden. Wir nehmen ferner an, dass, so lange sämmtliche δ kleiner als d bleiben, der grösste Werth, den diese Summe erhalten kann, Δ sei ; Δ wird alsdann eine Function von d sein, welche mit d immer abnimmt und mit dieser Grösse unendlich klein wird. Ist nun die Gesammtgrösse der Intervalle, in welchen die Schwankungen grösser als σ sind, $= s$, so wird der Beitrag dieser Intervalle zur Summe $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ offenbar $\geq \sigma s$. Man hat daher

$$\sigma s \leq \delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n \leq \Delta, \text{ folglich } s \leq \frac{\Delta}{\sigma}.$$

$\frac{\Delta}{\sigma}$ kann nun, wenn σ gegeben ist, immer durch geeignete Wahl von d beliebig klein gemacht werden; dasselbe gilt daher von s, und es ergiebt sich also:

Damit die Summe S, wenn sämmtliche δ unendlich klein werden convergirt, ist ausser der Endlichkeit der Function $f(x)$ noch erforderlich, dass die Gesammtgrösse der Intervalle, in welchen die Schwankungen $> \sigma$ sind, was auch σ sei, durch geeignete Wahl von d beliebig klein gemacht werden kann.

Dieser Satz lässt sich auch umkehren :

Wenn die Function $f(x)$ immer endlich ist, und bei unendlichem Abnehmen sämmtlicher Grössen δ die Gesammtgrösse s der Intervalle, in welchen die Schwankungen der Function $f(x)$ grösser, als eine gegebene Grösse σ, sind, stets zuletzt unendlich klein wird, so convergirt die Summe S, wenn sämmtliche δ unendlich klein werden.

Denn diejenigen Intervalle, in welchen die Schwankungen $> \sigma$ sind, liefern zur Summe $\delta_1 D_1 + \delta_2 D_2 + \cdots + \delta_n D_n$ einen Beitrag, kleiner als s, multiplicirt in die grösste Schwankung der Function zwischen a und b, welche (n. V.) endlich ist; die übrigen Intervalle einen Beitrag $< \sigma(b - a)$. Offenbar kann man nun erst σ beliebig klein annehmen und dann immer noch die Grösse der Intervalle (n. V.) so bestimmen, dass auch s beliebig klein wird, wodurch der Summe $\delta_1 D_1 + \cdots + \delta_n D_n$ jede beliebige Kleinheit gegeben, und folglich der Werth der Summe S in beliebig enge Grenzen eingeschlossen werden kann.

Wir haben also Bedingungen gefunden, welche nothwendig und hinreichend sind, damit die Summe S bei undendlichem Abnehmen der Grössen δ convergire und also im engern Sinne von einem Integrale der Function $f(x)$ zwischen a und b die Rede sein könne.

Wird nun der Integralbegriff wie oben erweitert, so ist offenbar, damit die Integration durchgehends möglich sei, die letzte der beiden gefundenen Bedingungen auch dann noch nothwendig; an die Stelle der Bedingung, dass die Function immer endlich sei, aber tritt die Bedingung, dass die Function nur bei Annäherung des Arguments an einzelne Werthe unendlich werde, und dass sich ein bestimmter Grenzwerth ergebe, wenn die Grenzen der Integration diesen Werthen unendlich genähert werden.

6. Nachdem wir die Bedingungen für die Möglichkeit eines bestimmten Integrals im Allgemeinen, d. h. ohne besondere Voraussetzungen über die Natur der zu integrirenden Function, untersucht haben, soll nun diese Untersuchung in besondere Fällen theils angewandt, theils weiter ausgeführt werden, und zwar zunächst für die Functionen, welche zwischen je zwei noch so engen Grenzen unendlich oft unstetig sind.

Da diese Functionen noch nirgends betrachtet sind, wird es gut sein, von einem bestimmten Beispiele auszugehen. Man bezeichne der Kürze wegen durch (x) den Ueberschuss von x über die nächste ganze Zahl, oder, wenn x zwischen zweien in der Mitte liegt und diese Bestimmung zweideutig wird, den Mittelwerth aus den beiden Werthen $\frac{1}{2}$ und $-\frac{1}{2}$,also die Null, ferner durch n eine ganze, durch p eine ungerade

Zahl und bilde alsdann die Reihe

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{4} + \frac{(3x)}{9} + \cdots = \sum_{1,\infty} \frac{(nx)}{nn} \; ;$$

so convergirt, wie leicht zu sehen, diese Reihe für jeden Werth von x ; ihr Werth nähert sich, sowohl, wenn der Argumentwerth stetig abnehmend, als wenn er stetig zunehmend gleich x wird, stets einem festen Grenzwerth, und zwar ist, wenn $x = \frac{p}{2n}$ (wo p, n relative Primzahlen)

$$f(x+0) = f(x) - \frac{1}{2nn}(1 + \frac{1}{9} + \frac{1}{25} + \cdots) = f(x) - \frac{\pi\pi}{16nn},$$

$$f(x-0) = f(x) + \frac{1}{2nn}(1 + \frac{1}{9} + \frac{1}{25} + \cdots) = f(x) + \frac{\pi\pi}{16nn},$$

sonst aber überall $f(x+0) = f(x)$, $f(x-0) = f(x)$.

Diese Function ist also für jeden rationalen Werth von x, der in den kleinsten Zahlen ausgedrückt ein Bruch mit geradem Nenner ist, unstetig, also zwischen je zwei noch so engen Grenzen unendlich oft, so jedoch, dass die Zahl der Sprünge, welche grösser als eine gegebene Grösse sind, immer endlich ist. Sie lässt durchgehends eine Integration zu. In der That genügen hierzu neben ihrer Endlichkeit die beiden Eigenschaften, dass sie für jeden Werth von x beiderseits einen Grenzwerth $f(x+0)$ und $f(x-0)$ hat, und dass die Zahl der Sprünge, welche grösser oder gleich einer gegebene Grösse σ sind, stets endlich ist. Denn wenden wir unsere obige Untersuchung an, so lässt sich offenbar in Folge dieser beiden Umstände d stets so klein annehmen, dass in sämmtlichen Intervallen, welche diese Sprünge nicht enthalten, die Schwankungen kleiner als σ sind, und dass die Gesammtgrösse der Intervalle, welche diese Sprünge enthalten, beliebig klein wird.

Es verdient bemerkt zu werden, dass die Functionen, welche nicht unendlich viele Maxima und Minima haben (zu welchen übrigens die eben betrachtete nicht gehört), wo sie nicht unendlich werden, stets diese beiden Eigenschaften und daher allenthalben, wo sie nicht unendlich werden, eine Integration zulassen, wie sich auch leicht direct zeigen lässt.

Um jetzt den Fall, wo die zu integrirende Function $f(x)$ für einen einzelnen Werth unendlich gross wird, näher in Betracht zu ziehen, nehmen wir an, dass dies für $x = 0$ stattfinde, so dass bei abnehmendem positiven x ihr Werth zuletzt über jede gegebene Grenze wächst.

Es lässt sich dann leicht zeigen, dass $xf(x)$ bei abnehmendem x von einer endlichen Grenze a an, nicht fortwährend grösser als eine endliche Grösse c bleiben könne. Denn dann wäre

$$\int_x^a f(x)dx > c \int_x^a \frac{dx}{x},$$

also grösser als $c\left(\log\frac{1}{x} - \log\frac{1}{a}\right)$, welche Grösse mit abnehmendem x zuletzt in's Unendliche wächst. Es muss also $xf(x)$, wenn diese Function nicht in der Nähe von

$x = 0$ unendlich viele Maxima und Minima hat, nothwendig mit x unendlich klein werden, damit $f(x)$ einer Integration fähig sein könne. Wenn andererseits

$$f(x)x^{\alpha} = \frac{f(x)dx(1-\alpha)}{d(x^{1-\alpha})}$$

bei einem Werth von $\alpha < 1$ mit x unendlich klein wird, so ist klar, dass das Integral bei unendlichem Abnehmen der unteren Grenze convergirt.

Ebenso findet man, dass im Falle der Convergenz des Integrals die Functionen

$$f(x)x\log\frac{1}{x} = \frac{f(x)dx}{-d\log\log\frac{1}{x}}, f(x)x\log\frac{1}{x}\log\log\frac{1}{x} = \frac{f(x)dx}{-d\log\log\log\frac{1}{x}}\cdots,$$

$$f(x)x\log\frac{1}{x}\log\log\frac{1}{x}\cdots\log^{n-1}\frac{1}{x}\log^{n}\frac{1}{x} = \frac{f(x)dx}{-d\log^{1+n}\frac{1}{x}}$$

nicht bei abnehmendem x von einer endlichen Grenze an fortwährend grösser als eine endliche Grösse bleiben können, und also, wenn sie nicht unendlich viele Maxima und Minima haben, mit x unendlich klein werden müssen; dass dagegen das Integral $\int f(x)dx$ bei unendlichem Abnehmen der unteren Grenze convergire, sobald

$$f(x)x\log\frac{1}{x}\cdots\log^{n-1}\frac{1}{x}\left(\log^{n}\frac{1}{x}\right)^{\alpha} = \frac{f(x)dx(1-\alpha)}{-d(\log^{n}\frac{1}{x})^{1-\alpha}}$$

für $\alpha > 1$ mit x unendlich klein wird.

Hat aber die Function $f(x)$ unendlich viele Maxima und Minima, so lässt sich über die Ordnung ihres Unendlichwerdens nichts bestimmen. In der That, nehmen wir an, die Function sei ihrem absoluten Werthe nach, wovon die Ordnung des Unendlichwerdens allein abhängt, gegeben, so wird man immer durch geeignete Bestimmung des Zeichens bewirken können, dass das Integral $\int f(x)dx$ bei unendlichem Abnehmen der unteren Grenze convergire. Als Beispiel einer solchen Function, welche unendlich wird und zwar so, dass ihre Ordnung (die Ordnung von $\frac{1}{x}$ als Einheit genommen) unendlich gross ist, mag die Function

$$\frac{d\left(x\cos e^{\frac{1}{x}}\right)}{dx} = \cos e^{\frac{1}{x}} + \frac{1}{x}e^{\frac{1}{x}}\sin e^{\frac{1}{x}}$$

dienen.

Das möge über diesen im Grunde in ein anderes Gebiet gehörigen Gegenstand genügen; wir gehen jetzt an unsere eigentliche Aufgabe, eine allgemeine Untersuchung über die Darstellbarkeit einer Function durch eine trigonometrische Reihe.

Chapter 6

CANTOR AND SET THEORY

1. Cantor.

Georg Cantor was born in 1845 at St Petersburg in a family of merchants. His family moved to Frankfurt am Main when he was eleven. He studied at Wiesbaden, then registered at the University of Zurich in order to study mathematics ; a young leading mathematician there was H. A. Schwarz. He moved to the University of Berlin where Kummer, Kronecker, Weierstrass were professors. He met Dedekind and this began one of the most beautiful friendship story in mathematics of all times.

In the biography of Adolf Fraenkel his life is divided into four periods : youth and development (1845-1871), time of creativity and achievements (1871-1884), time of declining productivity (1884-1897), old age (1897-1918). About all his papers in mathematics were written between 1870 and 1884 and a special problem on trigonometric series lead him to his main work : the foundation of set theory. The famous exchange of letters between Cantor and Dedekind began in 1872 and shows how helpful Dedekind was to Cantor in all respects. On the other hand Cantor had to endure a strong opposition, even hostility, from important mathematicians like Kronecker. He obtained a position as a professor of mathematics at the University of Halle but soon afterwards asked for a change to a chair in philosophy (1884). His last mathematical paper, *Lectures on transfinite numbers*, was published in 1897. Meanwile set theory became both better appreciated and more controversial. The mental health of Cantor declined, and he died in 1918 in a lunatic asylum.

The details of this life are impressive and tragic. The essential work was done in twelve years and revived the whole conception of mathematics. In parallel with Dedekind, Cantor showed how to build the real line from the rationals. He introduced the basic topological notions such as accumulation points and derivation of sets. He later gave the first basic results on the structure of closed sets on the line. He proved that the real numbers cannot be enumerated - and the "diagonal method" introduced for this purpose is as important as the result. As an application he had a non constructive proof of the existence of transcendental numbers. He proved that the line and the plane have the same power - this gave one of the most striking pages of the exchange of letters between Cantor and Dedekind. He constructed the so called triadic Cantor set as an example of a perfect set on the line and showed that its power is the same as the line : "the power of continuum". He tried to prove that there is no intermediate power between countable and continuum - the beginning of a long search, ended by Paul Cohen in 1963 -. He constructed the theory of cardinal numbers as a way to express the different powers and developed the arithmetics of the transfinite.

There is a contrast between this impressive work and the modest and insecure behaviour of Cantor towards other mathematicians. In his works on trigonometric series he is very eager to acknowledge his debt to Heine, Schwarz, Kronecker, Hankel, Weierstrass, and, of course, Riemann.

We shall analyse all papers of Cantor on trigonometric series and we shall quote entirely the seminal article *"Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen"*, on the extension of a theorem on trigonometric series. It is a starting point for set theory in general, and also the starting point of the subject of exceptional sets in Fourier analysis. We shall sketch the history of a century of sets of uniqueness and sets of multiplicity, give an idea on the new topological and probabilistic methods developed after 1960, and show how the general theory of sets is involved in the study of thin sets today.

2. Cantor's works on trigonometric series.

There are five papers of Cantor on trigonometric series in the period 1870-1872, then, in 1880, two remarks about a mistake of Paul Appell, and, in 1882, an expository paper on condensation of singularities.

The first two papers, dated 1870, contain the main theorems, both suggested by Riemann's memoir :

1. If $a_n \cos nx + b_n \sin nx$ tends to zero as $n \to \infty$ whatever x may be, then a_n and b_n tend to zero.

2. If the trigonometric series

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

converges to zero whatever x may be, then $a_n = b_n = 0$ for every n.

We saw that Riemann assumed implicitely that statement 1 is valid but he gave a proof that a_n and b_n tend to zero only when they are Fourier coefficients of an integrable function in the restricted sense. The proof by Cantor was not obvious. He gave a slight simplification in 1871, using an idea of Kronecker. Before explaining the proof let us go to Appell's mistake for a while. In his intended proof Appell considered

$$B_n = \sup_x (a_n \cos nx + b_n \sin nx)$$

and said *"cette valeur B_n tend également vers zéro quand* n *augmente indéfiniment"*. Cantor reacted with a one page paper in 1880 *(Bemerkung...)* observing that Appell implicitely assumed uniform convergence *(Konvergenz in gleichem Grade)*, and with a more expository paper the same year *(Fernere Bemerkung...)* giving examples of non-uniform convergence and a survey of the topic from Abel to Darboux, through Seidel and Stolz.

Cantor's proof is as follows. Writing

$$\rho_n \cos(nx + \varphi_n) = a_n \cos nx + b_n \sin nx,$$

we have to prove $\lim \rho_n = 0$. Here is a lemma : if any subsequence of (ρ_n) contains a subsequence tending to 0, then ρ_n tends to 0. Now every subsequence of (ρ_n) contains a subsequence (ρ_{n_k}) such that n_{k+1}/n_k is very big (actually, $> 3 + \epsilon$ suffices). Then some x can be exhibited, such that

$$\cos(n_k x + \varphi_{n_k}) \geq \delta > 0.$$

Hence ρ_{n_k} tends to 0, hence the conclusion. The construction of x may appear simple now, but it was the main difficulty for Cantor, and that is the reason why he wrote two articles on the subject.

The second theorem is also very natural in the Riemann approach, that is, to consider functions represented by everywhere convergent trigonometric series. It says that, for a given function, the trigonometric series is *unique* and it is called the Cantor uniqueness theorem for that reason. In other words, if a trigonometric series converges to zero everywhere, it is the null series. An attempt at proof could be to multiply the series by $\cos nx$ or $\sin nx$ and integrate but that is not correct without a further assumption, like uniform convergence. The proof consists in observing that, given a trigonometric series everywhere convergent to 0, the coefficients tend to zero (here Cantor used his first theorem), therefore the double integration of Riemann gives a continuous function $F(x)$, and Riemann's theory shows that the symmetric second derivative vanishes. Then $F(x)$ is linear, and Cantor reproduced the proof of this fact that H. A. Schwarz communicated to him. From there it follows that the coefficients vanish. This is the content of the first paper of Cantor on the uniqueness problem (*"Beweiss, dass eine für jeden Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf einzige Weise in dieser Form darstellen lässt"*, 1870).

The second paper on this question is a complement (*"Notiz zu dem Aufsatze : Beweiss..."*, 1871), with a slight simplification due to Kronecker and a first extension of the theorem : the conclusion stays valid if we assume that the trigonometric series converges to zero everywhere except maybe on a finite number of points. *"Diese Erweiterung des Satzes ist keineswegs die letzte"*, added Cantor : other extensions are possible.

The third paper is devoted to such an extension (*"Ueber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen"*, 1872). Instead of a finite set, Cantor showed that certain countable sets can be removed in the hypothesis. Again, other extensions are possible and we shall return to the question in a moment. For the time being let us reflect on this 1872 article.

3. Über die Ausdehnung.

We reproduce this article at the end of the chapter. Here is a brief analysis of its content.

Rational numbers constitute a domain A where usual operations and inequalities are defined. From domain A a domain B is constructed : it consists, in modern terms, of Cauchy sequences in A. It is a purely abstract object, but again usual operations, equalities and inequalities are defined, and there is a natural immersion of A in B. From domain B a domain C is obtained in the same way and so on.

Given two points on a line, assigning 0 to the first and 1 to the second allows us to imbed A in the line. The imbedding extends to $B, C, \cdots L, \cdots$. It is an *axiom* that these imbeddings cover the line, that is, each point of the line is associated with some b, some c, $\cdots$ some $\ell \cdots$; then a point corresponds to infinitely many b's, infinitely many c's and so on. Therefrom, "when speaking of points, we always have in view the values by which they are given". The values *(Werte)* are nothing but elements of A, B, $\cdots L$ *(Zahlengrosse* λ^{ter} *art)*.

Sets are either *Wertmenge* (sets of values) or *Punktmenge* (point sets). Accumulation points (*Grenzpunkte*, now *Häufungspunkte)* are defined, then derived sets of any order, as sets of accumulation points of the preceding set. A given set P is said sets of order ν *(der* ν^{ten} *Art)* is the derived set of order ν is finite.

Then comes *"die Ausdehnung"* : if a trigonometric series converges to zero everywhere except maybe on a set of order ν, all coefficients vanish; it is the null series.

4. Sets of uniqueness and sets of multiplicity.

Here is the uniqueness problem suggested by *"die Ausdehnung"*. Given a set E on the circle (real numbers modulo 2π), let us consider a trigonometric series which converges to zero outside E. Is it necessarily the null series ? If the answer is yes, we say that E is a set of uniqueness, in short, a U-set, and in the opposite case we say that E is a set of multiplicity, or M-set. Let us restrict the discussion to Borel sets, and mention a few landmarks.

All countable sets are U-sets (W. H. Young 1909).

All sets of positive Lebesgue measure are M-sets (implicit in Lebesgue 1906).

There exist M-sets of Lebesgue measure zero (D. Menšov 1916).

Though this is not a deep theorem, it is the most important of the theory. Until then, it was possible to believe that U-sets were exactly the negligible sets in the sense of Lebesgue, i. e. sets of zero measure. Lebesgue himself made this mistake in the first note that he wrote on trigonometric series (1902) ; though he corrected the mistake very soon afterwards the question remained open until Menšov's theorem.

The classes of U-sets and M-sets are invariant by translation (obvious) and by dilation (not obvious : a detour through trigonometric integrals is needed - see Zygmund's book, 1959, last chapter).

The triadic Cantor set is a U-set (Rajchman 1922). This is a key theorem and the beginning of an interesting story. Up to a dilation the triadic Cantor set is constructed in this way : starting with the closed interval $[0,1]$ we remove the open interval $]\frac{1}{3},\frac{2}{3}[$ and we obtain two intervals $[0,\frac{1}{3}],[\frac{2}{3},1]$; repeating this dissection on these two intervals we obtained four intervals, and so on ; at the n^{th} step we have 2^n intervals of length 3^{-n}, whose union is a set E_n ; E_n is a decreasing sequence of sets and its limit is E, the Cantor set, or Cantor dust as Denjoy said. A slight modification is to consider $0 < \xi < 1/2$, remove the interval $]\xi, 1-\xi[$ at the first step instead of $]\frac{1}{3},\frac{2}{3}[$ and proceed in the same way ; the result is named E_ξ, the Cantor set whose dissection ratio is ξ. Does the U-set character depend on ξ ?

The answer is affirmative. Rajchman's method and result extend to the case $\xi = \frac{1}{q}$, the inverse of an integer ≥ 3. On the contrary, E_ξ is a M-set when $\xi = \frac{p}{q}$, any other rational number between 0 and $\frac{1}{2}$ (Nina Bari, 1937). Then, is it possible to characterize the ξ's such that E_ξ is a U-set in a number theoretic way ?

Again, the answer is affirmative. E_ξ is a U-set if and only if $\theta = \frac{1}{\xi}$ is a positive algebraic integer whose other conjugates lie inside the unit disc in the complex plane. These numbers θ have an interesting property : θ^n tends to 0 modulo 1 (the converse may be true, it is an open problem). They were studied by Axel Thue, Charles Pisot, T. Vijayaraghavan and Raphaël Salem called them PV numbers. The theorem was guessed by Salem and finally proved by Salem and Zygmund (1955). An important by-product obtained by Salem is that the set of PV numbers is closed (1944). The theorem of Salem and Zygmund can be considered as the pearl of the U-set theory.

Let us go back to more general statements.

A countable union of closed U-set (Nina Bari 1923) is a U-set. Shortly before Nina Bari the theorem was stated by A. Zygmund in an imperfect way and immediately corrected, but the correction escaped the attention of Nina Bari (see *Comptes-Rendus de l'Académie des Sciences de Paris*, 1923). The proof is not easy, but the idea was in the air, as often happens in mathematics.

A closed set is a U-set if and only if it carries no non-zero pseudofunction. This is easy but was stated only in the 1950's (see Kahane-Salem 1963). Since pseudofunctions, that is, distributions whose Fourier coefficients tend to zero, are in duality with functions whose Fourier coefficients are summable (this is just a translation of the classical duality between spaces c_0 and ℓ^1), U-sets are related in this way with absolutely convergent Fourier series. More information thereabout can be found in Kahane's book (1970).

A large number of other results and related notions can be found in the books of Zygmund (1935 and 1959) and Nina Bari (1957). Let us just mention an extension of U-sets proposed by Zygmund. Instead of general trigonometric series, let us consider trigonometric series whose coefficients are bounded by a given sequence $\epsilon = (\epsilon_n)$ tending to zero :

$$| a_n | + | b_n | < \epsilon_n.$$

Zygmund calls a set E a U_ϵ-set when the only trigonometric series of this type that converges to zero outside E is the null series. He proved that there exist U_ϵ-sets of positive measure, and indeed as near 2π as wanted (1926). The existence of U_ϵ-sets of full measure was proved only in 1973 by Kahane and Katznelson, and more precise results, involving Hausdorff dimensions, were obtained by Bernard Connes (1976).

Let us turn to M-sets. Given a closed set E, proving that it is a M-set consists usually in constructing a pseudofunction carried by E. If that is possible with a measure, E is called a M_0-set. For example, all E_ξ which are not U-sets are actually M_0-set. However, not all M-sets are M_0-sets (I. I. Pyatetski-Shapiro 1954).

There are M_0-sets of different scales, according to how fast the Fourier coefficients of the measure can tend to 0. When $0 < \alpha < 1$ let us say that E is a M_α-sets when it carries a non-zero measure μ whose Fourier coefficients $\hat{\mu}(n)$ satisfy

$$| \hat{\mu}(n) |^2 = O(| n |^{-\alpha}).$$

This condition implies arithmetical and dimensional properties of E. For example, when p is an integer larger than $\frac{2}{\alpha}$, the algebraic sum $E + E + \cdots + E$ (p times) contains an interval. The Hausdorff dimension of E is $\geq \alpha$. The supremum of the α's such that E is a M_α-set is called the Fourier dimension of E ; it cannot exceed the Hausdorff dimension.

These definitions make sense in a euclidian space $\mathbb{R}^d$: a compact subset E of $\mathbb{R}^d$ is a M_α-set when it carries a non zero measure μ whose Fourier transform $\hat{\mu}(u) = \int \exp(-iux)\mu(dx)$ satisfies

$$| \hat{\mu}(u) |^2 = O(| u |^{-\alpha}) \qquad (0 < \alpha < d)$$

and, given E, the supremum of such α's is its Fourier dimension. It is a M_0-set when it carries a non-zero measure μ such that $\hat{\mu}(u)$ tends to 0 as $| u | \to \infty$ ($u \in \mathbb{R}^d$, $| u | = (u_1^2 + \cdots + u_d^2)^{1/2}$). There are two typical examples. If we choose $E = S^{d-1}$, the unit sphere in $\mathbb{R}^d$, its Hausdorff dimension is $d-1$ and the area measure on E shows that it is a M_{d-1}-set; its Fourier dimension is $d-1$.

In several dimensions the boundary of polyhedra are sets of uniqueness and the boundary of smooth convex bodies are sets of multiplicity in a strong sense. In both cases the Hausdorff dimension is $d-1$; the Fourier dimension is 0 in the first example and $d-1$ in the second. We may view uniqueness and multiplicity as a kind of squareness and roundness, respectively. Generally speaking, all questions on thin sets in harmonic analysis have more geometric and manageable versions in higher dimensions than on the line or the circle.

5. Two methods for thin sets in Fourier analysis.

At the end of the nineteenth century the language and notions of Cantor on *Punktmenge* became well known. Applications were found in differential equations and analytic functions. Several branches appeared: general set theory and logical problems on its foundation, measure theory, topology of point sets. For half a century the three of them were considered as parts of the new domain called set theory. This is clear in Hausdorff's book *Mengenlehre* (1914) and more even in the program of the first specialized journal in mathematics, *Fundamenta Mathematicae*, that played a catalytic role for developing both set theory on the world scene and mathematical enthusiam in Poland. The topological method that we have in view is based on the thesis of René Baire (1899) but its importance as a general tool was recognized first by the Polish school. The probabilistic method or randomization is based on the thesis of Lebesgue (1902) and the Borel notion of totally additive measure. However measure and later probability theory were developed in Russia and Poland, not in France, in the 1920's.

Both Baire's theory and probability can be applied in order to replace delicate explicit constructions by rather simple conceptual arguments. Both express that, given a convenient frame (a complete metric space or a probability space), something happens *in general*. But *in general* depends on the frame. If the frame is a complete metric space, *in general* means : on a countable intersection of open dense sets, and Baire's theory tells us that this is not empty and actually quite big. In other words, *in general* means everywhere with the possible exception of a set contained in a countable union of nowhere dense closed sets ; such a set was named *set of the first category* by Baire and later *meager* by Bourbaki. If a property holds everywhere out of a meager set we say that it is *generic* or *quasi sure*: that is the first meaning of *in general*.

On the other hand, if we are dealing with a probability space, *in general* means on a set of probability one, that is, everywhere outside a set of probability zero. When a property holds on a set of probability one it is called almost sure.

We have already encountered the very different behaviour of Hardy-Weierstrass lacunary series ; quasi-everywhere and almost-everywhere. We shall see later in this book important applications of probabilistic methods and ideas, some of them introduced in the 1930's. For the time being let us see what both methods provide about U-sets and M-sets. The initiators were Raphaël Salem for probabilistic methods (1950) and Robert Kaufman for the use of Baire's theory (1967).

6. Baire's method.

The closed sets on the circle constitute a complete metric space when they are equipped with the Hausdorff metric (the distance between two sets being less than ϵ if the distance of every point of each set to the other is less than ϵ). In this context Baire's theory applies : quasi all closed sets are U-sets. Actually much more is true: quasi surely each continuous function of modulus one defined on the set can be approximated uniformly by a sequence of imaginary exponentials e^{inx}. This is the definition of a Kronecker set, a thin set *par excellence*. Quasi all closed sets are Kronecker sets.

There are many variations on this theme. We can start with a smaller space of closed sets E, on assuming a lacunary condition (for some sequence δ_n tending to 0, E is contained in intervals of length δ_n separated by intervals of length larger than $n\delta_n$), and a more stringent condition of proximity (the distance between E and E' is less than ϵ if there is a diffeomorphism of the circle carrying E onto E' and a fixed given point onto itself, so that its derivative is 1 on E and between $1-\epsilon$ and $1+\epsilon$ on the circle). The result holds true : quasi all E's are Kronecker sets. Since neighbouring sets locally look the same, this shows that a Kronecker set looks like any lacunary set. Since there are lacunary sets of Hausdorff dimension one, there are also Kronecker sets of Hausdorff dimension one.

On the other hand, Kronecker sets are very thin in many respects. They are independent over the rationals, in other words, they generate a subgroup of the circle of vanishing measure. By the way, they were named in the 1960's after the theorem of Kronecker in diophantine approximation which states that independent finite sets are Kronecker sets.

Let us introduce a few notations. E being a compact subset of the circle, $C(E)$ consists of continuous functions on E, $A(E)$ of restrictions to E of sums of absolutely convergent trigonometric series, $M(E)$ of complex measures supported by E, $PM(E)$ of pseudomesures (distributions with bounded Fourier coefficients) supported by E, $PF(E)$ of pseudofunctions supported by E. We already know that $PF(E) = \{0\}$ means that E is a U-set. The equality $A(E) = C(E)$ defines Helson's sets and Helson's theorem (1954) is that this condition implies that E is not a M_0-set ; at the time it was not known if Helson sets are necessarily U-sets or not. All these spaces are Banach spaces with interesting duality properties. The dual of $C(E)$ is $M(E)$, the dual of $A(E)$ is contained in $PM(E)$ and it contains $M(E)$. Is $PM(E)$ the dual of $A(E)$? It was an open problem in the 1950's and we shall see later one its meaning as "problem of spectral synthesis". When it holds, E is called a set of spectral synthesis. When E is a Kronecker set, we have $PF(E) = \{0\}$, $A(E) = C(E)$, $PM(E) = M(E)$: E is a U-set, a Helson set, and a set of spectral synthesis in a very strong sense. Of course, the Fourier dimension of E is zero.

7. Randomization.

Here is a general idea : Baire's method amplifies singularities and provides kind of squareness, randomization smoothes singularities and provides kind of roundness.

As an illustration of the second point let us consider a question that Arne Beurling asked Raphaël Salem. Given $0 < \alpha < 1$, is it possible to construct a compact set E such that both its Hausdorff and Fourier dimensions equal α ? No explicit condition is known up to now. The Hausdorff dimension condition means that some $\mu \in M(E)$ satisfies

$$0 < \sum_{-\infty}^{\infty} |\hat{\mu}(n)|^2 |n|^{\beta-1} < \infty$$

for every $\beta < \alpha$, and the Fourier dimension condition that for every $\gamma < \alpha$ some non-zero $\mu \in M(E)$ satisfies

$$|\hat{\mu}(n)|^2 = O(|n|^{-\gamma}).$$

Therefore the Hausdorff dimension always dominates the Fourier dimension, and equality means that E is a M_γ set as much as is allowed by the Hausdorff dimension. Salem answered the question affirmatively by considering generalized Cantor sets constructed according to a specific pattern (a dissection into several intervals) and different ratios of dissection, $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ at different steps. If the pattern is well chosen and the ξ_n are chosen randomly in convenient intervals, the random set that we obtain satisfies the requirement almost surely (1950).

The method of Salem finds a more natural context in the probabilistic point of view. In many circumstances random processes provide Salem sets (meaning that the Hausdorff and Fourier dimensions are the same). For example, consider the linear Brownian motion function $B(t,\omega)$ ($t \in \mathbb{R}^+$, $\omega \in \Omega$, the probability space), and any compact t-set K, with Hausdorff dimension $\frac{\alpha}{2}$. Then the image of K by $B(t,\omega)$ is almost surely a Salem set of dimension α.

The method easily gives also rationally independent M_0-sets, by applying $B(t,\omega)$ to a conveniently chosen lacunary t-set. Randomization is a pretty good way to obtain M-sets (see Kahane 1968 or 1985).

R. Kaufman used a more subtle randomization in order to replace a very delicate construction of T. Körner (1972). Here is the result : any M-set contains a Helson set of multiplicity. Let us dwell on the meaning of a Helson set of multiplicity.

A Helson set cannot be a M_0-set (Helson's theorem). Therefore, there exist M-sets that are not M_0-sets. This is far from obvious, but had been proved by I. I. Piatetskii-Shapiro previously (1954).

For a Helson set E the dual of $A(E)$ is $M(E)$ and $M(E)$ contains no pseudofunction $\neq 0$ (Helson's theorem again). If moreover E is a set of multiplicity it carries a non zero pseudofunction. Therefore $PM(E)$ is not the dual of $A(E)$, that is, E is not a set of spectral synthesis. We shall discuss this question in Chapter 10.

8. Another look on Baire's theory.

The game which consists in replacing an elaborate construction by an existence theorem based on Baire's theory was played first and quite systematically by Polish mathematicians in the 1920's. Here are a few examples. Plane indecomposable continua are rather strange objects : continua (non-empty closed connected sets) which cannot be obtained as a union of two continua. They were introduced and investigate by Janiszewski in the 1910's. Actually, considering the space of all plane continua equipped with the Hausdorff metric, quasi all continua are indecomposable (Mazurkiewicz). Other strange objects were introduced by Sierpinski (1915), as curves consisting entirely of branching points ; after B. Mandelbrot one can call them Sierpinski carpets - they look like moth-eaten carpets. Now, considering continuous mapping of a given interval in the plane, quasi-all images are Sierpinski carpets (an idea of Knaster, proved by Mazurkiewicz). Finally, the first approach of the famous Banach-Steinhaus theorem on linear operators was the condensation of singularities; it was replaced by the use of Baire's theorem, at the suggestion of Saks, by Banach and Steinhaus themselves. As we have already observed, the Banach-Steinhaus theorem is the clearest way to establish the existence of continuous functions whose Fourier series diverge at a given point. In the hands of Polish mathematicians of the 20's Baire's theory was a very good way to tame monsters. The systematic use of separable complete metric spaces for this purpose explains why Bourbaki named them Polish spaces.

We had only a glance at the use of Baire's theory for exhibiting thin sets in Fourier analysis - namely, Kaufman's procedure for Kronecker sets. It is possible also to get Helson curves in $\mathbf{T}^2$, and p-dimensional Helson manifolds in $\mathbf{T}^{2p+1}$, with the property that each continuous function on the manifold can be expressed as $g_1(x_1) + g_2(x_2) + \cdots + g_{2p+1}(x_p)$, all g_j belonging to $A(\mathbf{T})$. Quasi all manifolds of the type $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_p$ (algebraic sums) where the Γ_j are increasing curves in $\mathbb{R}^{2p+1}$ (that is, all coordinates are increasing functions of the parameter) have this property. This provides a proof of the Kolmogorov superposition theorem, according to which continuous functions of p variables can be expressed as superpositions of continuous functions of one variable and their sums (Kolmogorov's solution of the 13th problem of Hilbert) (see details and comments in Kahane 1980).

A really new look is given by T. Körner in his 1995 paper. Besides rather difficult theorems on Helson curves it contains two simple but important observations. First, consider the space of ordered pairs (E,μ) with E a closed M_0-set in T and μ a probability measure carried by E such that $\hat{\mu}(n) \to 0$ ($|n| \to \infty$). Using the metric

$$\rho((E_1,\mu_1),(E_2,\mu_2)) = d(E_1,E_2) + \| \mu_1 - \mu_2 \|_{PM},$$

where $d(\ ,\ )$ is the Hausdorff distance, we have a Polish space. Now, given any $h : [0,\infty[\to [0,\infty[$ continuous and strictly increasing with $h(0) = 0$, then E has Hausdorff h-measure 0 quasi-surely, that is, E can be covered by intervals $I_1, I_2, \cdots I_n, \cdots$ such that $\sum h(|I_n|)$ is arbitrarily small. The existence of such sets is a theorem of O. S. Ivašev-Musatov (1968). Secondly, consider the space of ordered pairs (E,μ) with E a closed M_α-set and μ a probability measure carried by E such that $|n|^\alpha \hat{\mu}(n)$ is bounded. Using now the metric

$$\rho((E_1,\mu_1),(E_2,\mu_2)) = d(E_1,E_2) + \sup_n (|n|^\alpha |\hat{\mu}_1(n) - \hat{\mu}_2(n)|),$$

then E has Hausdorff dimension α quasi-surely. Therefore, E is a Salem set quasi-surely. This is a new and simple way to obtain Salem sets.

9. Recent results and new methods from general set theory.

In 1985 R. Kaufman published a seminal paper where methods of general set theory were applied in the space of compact subsets of the circle, $\mathcal{K}(\mathbf{T})$, equipped with the Hausdorff metric. About $\mathcal{U}$, the class of closed U-sets, here is the result: $\mathcal{U}$ is coanalytic (that is, its complement in $\mathcal{K}(\mathbf{T})$, $\mathcal{M}$, is analytic in the sense of Lusin) and $\mathcal{U}$ is not borelian (therefore, not analytic either).

The logician R. Solovay proved this at the same time, independently. His study was never published but was known among logicians and inspired much work immediately. The first reference book in the matter is due to A. Kechris and A. Louveau (1987). We begin with results of G. Debs and J. Saint Raymond of the same year.

$\mathcal{U}$ has the structure of a σ-ideal in $\mathcal{K}(\mathbf{T})$, that is, it is hereditary for inclusion (each part of a U-set is a U-set) and stable under compact countable unions (Nina Bari's theorem). Solovay introduced the notion of a basis of a σ-ideal (B is a basis of a σ-ideal I if I is the smallest σ-ideal containing B) and asked if $\mathcal{U}$ has a borelian basis. The main result of Debs and Saint Raymond is the negative answer to this question. In a way that ruins any hope of devising a complete explicit procedure to obtain all U-sets. On the other hand it is an existence theorem : whatever may be a system of borelian conditions, there exists a U-set which cannot be expressed as a countable union of compact sets satisfying these conditions. The proof involved the existence of Helson sets of multiplicity. It also needed extension theorems for σ-ideals which are now a good tool in the general descriptive set theory.

Another example of a σ-ideal in $\mathcal{K}(\mathbf{T})$ was studied previously by A. Kechris and A. Louveau in their book (1987) : it is the class $\mathcal{U}_0$ consisting of U_0-sets (sets of uniqueness in the wide sense), that is, sets which do not carry any non zero measure whose Fourier coefficients tend to zero. The complement of $\mathcal{U}_0$ in $\mathcal{K}(\mathbf{T})$ is $\mathcal{M}_0$, the class of all M_0-sets. Like $\mathcal{U}$, $\mathcal{U}_0$ is coanalytic and not borelian, and it is a σ-ideal in $\mathcal{K}(\mathrm{T})$. Unlike $\mathcal{U}$, it has a borelian basis.

Let us just mention two other questions where general set theory proves to be a useful tool.

First, a theorem of Russell Lyons (1984), with a new proof by A. Louveau based on an unpublished result of G. Mokobodzki (1987). Let us define $\mathcal{R}$ as the class of measures on T whose Fourier coefficients tend to zero (Rajchman measures). Then $\mathcal{U}_0$ is exactly the class of compact subsets of T that elements of $\mathcal{R}$ annihilate : we may write $\mathcal{U}_0 = \mathcal{R}^{\perp}$. Conversely (Lyons's theorem), $\mathcal{R}$ consists exactly of those measures that annihilate all elements of $\mathcal{U}_0$: we may write $\mathcal{R} = \mathcal{U}_0^{\perp}$. The original proof of R. Lyons used a special kind of sets, W, previously introduced by Yu. A. Šreider, and showed that $\mathcal{R} = W^{\perp}$, hence $\mathcal{R} = \mathcal{R}^{\perp\perp}$. Mokobodzki's lemma is of independent interest. It states that $\mathcal{C} = \mathcal{C}^{\perp\perp}$ whenever $\mathcal{C}$ is a cone of probability measures which is norm-closed in the Banach space M of bounded complex measures and weak*-analytic (the weak * topology on M is defined by duality with continuous functions with compact support).

Secondly, a curious problem of R. Kaufman (1976 and 1990). If a M_0-set E is transformed by every diffeomorphism of the circle into a M_0-set, we write $E \in \mathcal{M}_0(C^1)$. If a Rajchman measure μ is transformed by every diffeomorphism of $\mathbf{T}$ into a Rajchman measure, we write $\mu \in \mathcal{R}(C^1)$. Kaufman proved that E_ξ (the Cantor set with dissection ratio ξ) is a $\mathcal{M}_0(C^1)$ set whenever $1/\xi$ is not a Pisot-Vijayaraghavan number (1976). The proof consists in showing that the natural measure on E_ξ, given equal weight to equal portions, belongs to $\mathcal{R}(C^1)$, and it is difficult to imagine any other possible proof. However, Kaufman recently proved that there exists sets in $\mathcal{M}_0(C^1)$ which do not carry any non-zero measure belonging to $\mathcal{R}(C^1)$ (1990). Though very specific, this kind of Fourier analysis has a quite close relation with logic.

10. The first paper in the theory of sets.

Georg Cantor : Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen. [*Math. Annalen Bd. 5, S. 123-132 (1872).*]

Im folgenden werde ich eine gewisse Ausdehnung des Satzes, dass die trigonometrischen Reihendarstellungen eindeutig sind, mitteilen.

Dass zwei trigonometrische Reihen

$$\frac{1}{2}b_0 + \sum(a_n \sin nx + b_n \cos nx) \quad \text{und} \quad \frac{1}{2}b'_0 + \sum(a'_n \sin nx + b'_n \cos nx),$$

welche für jeden Wert von x konvergieren und dieselbe Summe haben, in ihren Koeffizienten übereinstimmen, habe ich im "Journal f. d. r. u. angew. Math. Bd. 72, S. 139" [hier II 2, S. 80] nachzuweisen versucht ; in einer auf diese Arbeit sich beziehenden Notiz have ich a. a. O. ferner gezeigt, dass dieser Satz auch erhalten bleibt, wenn man für eine endliche Anzahl von Werten des x entweder die Konvergenz oder die Uebereinstimmung der Reihensummen aufgibt.

Die *hier* beabsichtigte Ausdehnung besteht darin, dass für eine *unendliche* Anzahl von Werten des x im Intervalle $(0 \cdots 2\pi)$ auf die Konvergenz oder auf die Uebereinstimmung der Reihensummen verzichtet wird, ohne dass die Gültigkeit des Satzes aufhört. Zu dem Ende bin ich aber genötigt, wenn auch zum grössten Teile nur andeutungsweise, Erörterungen voraufzuschicken, welche dazu dienen mögen, Verhältnisse in ein Licht zu stellen, die stets auftreten, sobald Zahlengrössen in endlicher oder unendlicher Anzahl sind : dabei werde ich zu gewissen Definitionen hingeleitet, welche hier nur zum Behufe einer möglichst gedrängten Darstellung des beabsichtigten Satzes, dessen Beweiss im § 3 gegeben wird, aufgestellt werden.

§ 1.

Die rationalen Zahlen bilden die Grundlage für die Feststellung des weiteren Begriffes einer Zahlengrösse ; ich will sie das Gebiet A nennen (mit Einschluss der Null).

Wenn ich von einer Zahlengrösse im weiteren Sinne rede, so geschieht es zunächst in dem Falle, dass seine durch ein Gesetz gegebene unendliche Reihe von rationalen Zahlen

$$a_1, \ a_2, \cdots a_n, \cdots \tag{1}$$

vorliegt, welche die Beschaffenheit hat, dass die Differenz $a_{n+m} - a_n$ mit wachsendem n unendlich klein wird, was auch die positive ganze Zahl m sei, oder mit anderen Worten, dass bei beliebig angenommenem (positiven, rationalen) ϵ eine ganze Zahl n_1 vorhanden ist, so dass $| \, a_{n+m} - a_n \, | < \epsilon$, wenn $n \geq n_1$ und wenn m eine beliebige positive ganze Zahl ist.

Diese Beschaffenheit der Reihe (1) drücke ich in den Worten aus : *"Die Reihe (1) hat eine bestimmte Grenze b."*

Es haben also diese Worte zunächst keinen anderen Sinn als den eines Ausdruckes für *jene* Beschaffenheit der Reihe, und aus dem Umstande, dass wir mit der Reihe (1) ein besonderes Zeichen b verbinden, folgt, dass bei verschiedenen derartigen Reihen auch verschiedene Zeichen $b, b', b'', \cdots$ zu bilden sind.

Ist eine zweite Reihe

$$a'_1, \ a'_2, \cdots a'_n, \cdots \tag{1'}$$

gegeben, welche eine bestimmte Grenze b' hat, so findet man, dass die beiden Reihen (1) und (1') eine von den folgenden 3 Beziehungen stets haben, die sich gegenseitig ausschliessen : entweder 1. wird $a_n - a'_n$ unendlich klein mit wachsendem n oder 2. $a_n - a'_n$ bleibt von einem gewissen n an stets grösser als eine positive (rationale) Grösse ϵ oder 3. $a_n - a'_n$ bleibt von einem gewissen n an stets kleiner als eine negative (rationale) Grösse $-\epsilon$.

Wenn die erste Beziehung stattfindet, setze ich

$$b = b',$$

bei der zweiten $b > b'$, bei der dritten $b < b'$.

Ebenso findet man, dass eine Reihe (1), welche eine Grenze b hat, zu einer rationalen Zahl a nur eine von den folgenden 3 Beziehungen hat. Entweder 1. wird $a_n - a$ unendlich klein mit wachsendem n, oder 2. $a_n - a$ bleibt von einem gewissen n an immer grösser als eine positive (rationale) Grösse ϵ oder 3. $a_n - n$ bleibt von einem gewissen n an immer kleiner als eine negative (rationale) Grösse $-\epsilon$.

Aus diesen und den gleich folgenden Definitionen ergibt sich als *Folge*, dass, wenn b die Grenze der Reihe (1) ist, alsdann $b - a_n$ mit wachsendem n unendlich klein wird, womit *nebenbei* die Bezeichnung "Grenze der Reihe (1)" für b eine gewisse Rechtfertigung findet.

Die Gesamtheit der Zahlengrössen b möge durch B bezeichnet werden.

Mittels obiger Festsetzungen lassen sich die Elementaroperationen welche mit rationalen Zahlen vorgenommen werden, ausdehnen auf die beiden Gebiete A und B zusammengenommen.

Sind nämlich b, b', b'' drei Zahlengrössen aus B, so dienen die Formeln

$$b \pm b' = b'', \quad bb' = b'', \quad \frac{b}{b'} = b''$$

als Ausdruck dafür, dass zwischen den den Zahlen b, b', b'' entsprechenden Reihen

$$\begin{array}{l} a_1,\ a_2, \cdots \\ a'_1,\ a'_2, \cdots \\ a''_1,\ a''_2, \cdots \end{array}$$

resp. die Beziehnungen bestehen

$$\lim(a_n \pm a'_n - a''_n) = 0, \qquad \lim(a_n a'_n - a''_n) = 0,$$

$$\lim\left(\tfrac{a_n}{a'_n} - a''_n\right) = 0 \quad [\textit{für } a'_n \neq 0],$$

wo ich auf die Bedeutung des lim-Zeichens nach dem Vorhergehenden nicht näher einzugehen brauche. Ähnliche Definitionen werden für die Fälle aufgestellt, dass von den drei Zahlen eine oder zwei dem Gebiete A angehören.

Allgemein wird sich daraus jede mittels einer endlichen Anzahl von Elementaroperationen gebildete Gleichung

$$F(b, b', \cdots, b^{(\rho)}) = 0$$

als der Ausdruck für eine bestimmte Beziehung ergeben, welche under den Reihen stattfindet, durch welche die Zahlengrössen $b, b', b'', \cdots b^{(\rho)}$ gegeben sind(1).

Das Gebiet B ergab sich aus dem Gebiete A ; es erzeugt nun in analoger Weise in Gemeinschaft mit dem Gebiete A ein neues Gebiet C.

Liegt nämlich eine unendliche Reihe

$$b_1,\ b_2, \cdots b_n, \cdots \tag{2}$$

von Zahlengrössen aus den Gebieten A und B vor, welche nicht sämtlich dem Gebiete A angehören, und hat diese Reihe die Beschaffenheit, dass $b_{n+m} - b_n$ mit wachsendem n unendlich klein wird, was auch m sei, eine Beschaffenheit, die nach den vorangegangenen Definitionen begrifflich etwas ganz Bestimmtes ist, so sage ich von dieser Reihe aus, dass sie eine bestimmte Grenze c hat.

Die Zahlengrössen c konstituieren das Gebiet C.

Die Definitionen des Gleich-, Grössser- und Kleinerseins, sowie der Elementaroperationen sowohl unter den Grössen c, wie auch zwischen ihnen und den Grössen der Gebiete B und A werden dem früheren analog gegeben.

Während sich nun die Gebiete B und A so zueinander verhalten, dass zwar jedes a einem b, nicht aber umgekehrt jedes b einem a gleichgesetzt werden kann, stellt es sich hier heraus, dass sowohl jedes b einem c, wie auch umgekehrt jedes c einem b gleichgesetzt werden kann.

Obgleich hierdurch die Gebiete B und C sich gewissermassen gegenseitig decken, ist es bei der hier dargelegten Theorie (in welcher die Zahlengrösse, zunächst an sich im allgemeinen gegenstandslos, nur als Bestandteil von Spätzen erscheint, welchen Gegenständlichkeit zukommt, des Satzes z. B., dass die entsprechende Reihe die Zahlengrösse zur Grenze hat) wesentlich, an dem begrifflichen Unterschiede der beiden Gebiete B und C festzuhalten, indem ja schon die Gleichsetzung zweier Zahlengrössen b, b' aus B ihre Identität nicht einschliesst, sondern nur eine bestimmte Relation ausdrückt, welche zwischsen den Reihen stattfindet, auf welche sie sich beziehen.

Aus dem Gebiete C und den vorhergehenden geht analog ein Gebiet D, aus diesen ein E hervor usw.; durch λ solcher Übergänge (wenn ich den Übergang von A zu B als den ersten ansehe) gelangt man zu einem Gebiete L von Zahlengrössen. Dasselbe verhält sich, wenn man die Kette der Definitionen für Gleich-, Grösser- und Kleinersein und für die Elementaroperationen von Gebiet zu Gebiet vollzogen denkt, zu den vorhergehenden, mit Ausschluss von A so, dass eine Zahlengrösse ℓ stets gleichgesetzt werden kann einer Zahlengrösse $k, i, \cdots c, b$ und umgekehrt.

---

(1) Wenn z. B. eine Gleichung μ^{ten} Grades $f(x) = 0$ mit ganzzahligen Koeffizienten eine reelle Wurzel ω besitzt, so heisst dies im allgemeinen nichts anderes, als dass eine Reihe

$$a_1,\ a_2, \cdots a_n, \cdots$$

von der Beschaffenheit der Reihe (1) vorliegt, für deren Grenze das Zeichen ω gewählt ist, welche ausserdem die Eigenschaft hat

$$\lim f(a_n) = 0.$$

Auf die Form solcher Gleichsetzungen lassen sich die Resultate der Analysis (abgesehen von wenigen bekannten Fällen) zurückführen, obgleich (was hier nur mit Rücksicht auf jene Ausnahmen berührt sein mag) der Zahlenbegriff, soweit er hier entwickelt ist, den Keim zu einer in sich notwendigen und absolut unendlichen Erweiterung in sich trägt.

Es scheint sachgemäss, wenn eine Zahlengrösse im Gebiete L gegeben ist, sich des Ausdruckes zu bedienen : *sie ist als Zahlengrösse, Wert oder Grenze* λ^{ter} *Art gegeben*, woraus ersichtlich ist, dass ich mich der Worte *Zahlengrösse, Wert* und *Grenze* im allgemeinen in gleicher Bedeutung bediene.

Eine mittels einer endlichen Anzahl von Elementaroperationen aus Zahlen ℓ, $\ell', \cdots \ell^{(\rho)}$ gebildete Gleichung $F(\ell, \ell', \cdots \ell^{(\rho)}) = 0$ erscheint bei der hier angedeuteten Theorie genau genommen als der Ausdruck für eine bestimmte Beziehung zwischen $\rho + 1$, *im allgemeinen* λ *fach unendlichen Reihen rationaler Zahlen* ; es sind dies die Reihen, welche aus den einfach unendlichen auf die sich die Grössen ℓ, ℓ', $\cdots \ell^{(\rho)}$ zunächst beziehen, hervorgehen, indem man in ihnen die Elemente durch ihre entsprechenden Reihen ersetzt, die entstehenden, im allgemeinen zweifach unendlichen Reihen ebenso behandelt und diesen Prozess so lange fortführt, bis man nur rationale Zahlen vor sich sieht.

Es sei mir vorbehalten, auf alle diese Verhältnisse bei einer andern Gelegenheit ausführlicher zurückzukommen. *Wie* die in diesem § auftretenden Festsetzungen und Operationen mit Nutzen der Infinitesimalanalysis dienen können, darauf einzugehen ist hier gleichfalls nicht der Ort. Auch das folgende, wo der Zusammenhang der Zahlengrössen mit der Geometrie der geraden Linie dargelegt wird, beschränkt sich fast nur auf die notwendigen Sätze, aus welchen, wenn ich nicht irre, as übrige mittels rein logischer Beweisführung abgeleitet werden kann. Zum Vergleiche mit § 1 und § 2 sei das 10.Buch der "Elemente des Euklides" erwähnt, welches für den darin behandelten Gegenstand massgebend bleibt.

§ 2.

Die Punkte einer geraden Linie werden dadurch begrifflich bestimmt, dass man unter Zugrundelegung einer Masseinheit ihre Abszissen d. h. ihre Entfernungen von einem festen Punkte o der geraden Linie mit dem $+$ oder $-$ Zeichen angibt, je nachdem der betreffende Punkt in dem (vorher fixierten) positiven oder negativen Teile der Linie von o aus liegt.

Hat diese Entfernung zur Masseinheit ein rationales Verhältnis, so wird sie durch eine Zahlengrösse des Gebietes A ausgedrückt ; im andern Falle ist es, wenn der Punkt etwa durch eine Konstruktion *bekannt* ist, immer möglich, eine Reihe

$$a_1,\ a_2, \cdots a_n, \cdots \tag{1}$$

anzugeben, welche die in § 1 ausgedrückte Beschaffenheit und zur fraglichen Entfernung eine solche Beziehung hat, dass die Punkte der Geraden, denen die Entfernungen a_1, $a_2, \cdots a_n, \cdots$ zukommen, dem zu bestimmenden Punkte mit wachsendem n unendlich nahe rücken.

Dies drücken wir so aus, dass wir sagen : *Die Entfernung des zu bestimmenden Punktes von dem Pukte o ist gleich b, wo b* die der Reihe (1) entsprechende Zahlengrösse ist.

Hierauf wird nachgewiesen, dass das Grösser-, Kleiner- und Gleichsein von bekannten Entfernungen in Übereinstimmung ist mit dem in § 1 definierten Grösser-, Kleiner- und Gleichsein der entsprechenden Zahlengrössen, welche die Entfernungen angeben.

Dass nun ebenso auch die Zahlengrössen der Gebiete C, $D, \cdots$ befähigt sind, bekannte Entfernungen zu bestimmen, erbigt sich ohne Schwierigkeit. Um aber den in diesem § dargelegten Zusammenhang der Gebiete der in § 1 definierten Zahlengrössen mit der Geometrie der geraden Linie vollständig zu machen, ist nur noch ein *Axiom* hinzuzufügen, welches einfach darin besteht, dass auch umgekehrt zu jeder Zahlengrösse ein bestimmter Punkt der Geraden gehört, dessen Koordinate gleich ist jener Zahlengrösse, und zwar in dem Sinne gleich, wie solches in diesem § erklärt wird [(1)].

Ich nenne diesen Satz ein *Axiom,* weil es in seiner Natur liegt, nicht allgemein beweisbar zu sein.

Durch ihn wird denn auch nachträglich für die Zahlengrössen eine gewisse Gegenständlichkeit gewonnen, von welcher sie jedoch ganz unabhängig sind.

Dem Obigen gemäss betrachte ich einen Punkt der Geraden als bestimmt, wenn seine Entfernung von o mit dem gehörigen Zeichen versehen, als Zahlengrösse, Wert oder Grenze λ^{ter} Art gegeben ist.

–oOo–

Wir wollen nun, unserm eigentlichen Gegenstande näher tretend, Beziehungen betrachten, welche auftreten, sobald Zahlengrössen in endlicher oder unendlicher Anzahl gegeben sind.

Nach dem Vorhergehenden können die Zahlengrössen den Punkten einer Geraden zugeordnet gedacht werden. Der Anschaulichkeit wegen (nicht dass es wesentlich zur Sache gehörte) bedienen wir uns dieser Vorstellung im folgenden und haben, wenn wir von Punkten sprechen, stets Werte im Auge, durch welche sie gegeben sind.

Eine gegebene endliche oder unendliche Anzahl von Zahlengrössen nenne ich der Kürze halber eine *Wertmenge* und dem entsprechend eine gegebene endliche oder unendliche Anzahl von Punkten einer Geraden eine *Punktmenge.* Was im folgenden von Punktmengen ausgesprochen wird, lässt sich dem gesagten gemäss unmittelbar auf Wertmengen übertragen.

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(1) Es gehört also zu jeder Zahlengrösse ein bestimmter Punkt, einem Punkte kommen aber unzählig viele gleiche Zahlengrössen als Koordinaten im obigen Sinne zu ; denn es folgt, wie schon oben angedeutet wurde, aus rein logischen Gründen, dass gleichen Zahlengrössen *nicht* verschiedene Punkte entsprechen können und dass ungleichen Zahlengrössen als Koordinaten *nicht* ein und derselbe Punkt zukommen kann.

Wenn in einem endlichen Intervalle eine Punktmenge gegeben ist, so ist mit ihr im allgemeinen eine zweite Punktmenge, mit dieser im allgemeinen eine dritte usw. gegeben, welche für die Auffassung der Batur der ersten Punktmenge wesentlich sind.

Um diese abgeleiteten Punktmengen zu definieren, haben wir den Begriff *Grenzpunkt* [*"Häufungspunkt"*] *einer Punktmenge* vorauszuschicken.

Unter einem "Grenzpunkt einer Punktmenge P" verstehe ich einen Punkt der Geraden von solcher Lage, dass in jeder Umgebung desselben unendlich viele Punkte aus P sich befinden, wobei es vorkommen kann, dass er ausserdem selbst zu der Menge gehört. Unter "Umgebung eines Punktes" sei aber hier ein jedes Intervall verstanden, welches den Punkt *in seinem Innern* hat Darnach ist es leicht zu beweisen, dass eine aus einer undendlichen Anzahl von Punkten bestehende ["beschränkte"] Punktmenge stets zum wenigsten *einen* Grenzpunkt hat.

Es ist nun ein bestimmtes Verhalten eines jeden Punktes der Geraden zu einer gegebenen Menge P, entweder ein Grenzpunkt derselben oder kein solcher zu sein, und es ist daher mit der Punktmenge P die Menge ihrer Grenzpunkte *Begrifflich* mit gegeben, welche ich mit P' bezeichnen und *"die erste abgeleitete Punktmenge* von P" nennen will.

Besteht die Punktmenge P' nicht aus einer bloss endlichen Anzahl von Punkten, so hat sie gleichfalls eine abgeleitete Punktmenge P'', ich nenne sie *die zweite abgeleitete von P*. Man findet durch ν solcher Übergänge den Begriff der ν^{ten} abgeleiteten Punktmenge $P^{(\nu)}$ von P.

Besteht beispielsweise die Menge P aus allen Punkten der Geraden, denen rationale Abszissen zwischen 0 und 1, die Grenzen ein- oder ausgeschlossen, zukommen, so besteht die abgeleitete Menge P' aus *allen* Punkten des Intervalles $(0 \cdots 1)$, die Grenzen 0 und 1 mit eingeschlossen. Die folgenden Mengen $P'', P''', \cdots$ stimmen hier mit P' überein. Oder, besteht die Menge P aus den Punkten, welchen die Abszissen $1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots$ zukommen, so besteht die Menge P' aus dem *einen* Punkte 0 und hat selbst Abgeleitete.

Es kann eintreffen, und dieser Fall ist es, welcher uns hier ausschliesslichinteressiert, dass nuch ν Übergängen die Menge $P^{(\nu)}$ aus einer endlichen Anzahl von Punkten besteht, mithin selbst keine abgeleitete Menge hat ; in diesem Falle wollen wir die ursprüngliche Punktmenge P *von der* ν^{ten} Art nennen, woraus folgt, dass alsdann $P', P'', \cdots$ von der $\overline{\nu - 1}^{\text{ten}}$, $\overline{\nu - 2}^{\text{ten}} \cdots$ Art sind.

Es wird also bei dieser Auffassungsweise das Gebiet aller Punktmengen bestimmter Art als ein besonderes Genuss innerhalb des Gebietes aller denkbaren Punktmengen betrachtet, von welchem Genus die sogenennten Punktmengen $^{\text{ter}}$ Art eine besondere Art ausmachen.

Ein Beispiel einer Punktmenge ν^{ter} Art bietet schon ein einzelner Punkt dar, wenn seine Abszisse als Zahlengrösse ν^{ter} Art, welche gewissen, leicht festzustellenden Bedingungen genügt, gegeben ist. Löst man nämlich alsdann diese Zahlengrösse in die Glieder $(\nu - 1)^{\text{ter}}$ Art der ihr entsprechenden Reihe auf, diese Glieder wieder in die sie konstituierenden Glieder $\overline{(\nu - 2)}^{\text{ter}}$ Art usw. so erhält man zuletzt eine unendliche Anzahl rationaler Zahlen ; denkt man sich die diesen Zahlen entsprechende

Punktmenge, so ist dieselbe von der ν^{ten} Art[1].

Nach diesen Vorbereitungen sind wir nun imstande, den beabsichtigten Satz im folgenden § kurz anzugeben und zu beweisen.

§ 3.

THEOREM. *Wenn eine Gleichung besteht von der Form*

$$0 = C_0 + C_1 + \cdots + C_n + \cdots, \tag{1}$$

wo $C_0 = \frac{1}{2}d_0$; $C_n = c_n \sin nx + d_n \cos nx$, *für all Werte von* x *mit Ausnahme derjenigen, welche den Punkten einer im Intervalle* $(0 \cdots 2\pi)$ *gegebene Punktmenge* P *der* ν^{ten} Art *entsprechen, wobei* ν *eine beliebig grosse ganze Zahl bedeutet, so ist*

$$d_0 = 0, \ c_n = d_n = 0.$$

Beweiss. In diesem Beweise hat man, wie durch den Fortgang ersichtlich wird, wenn von P die Rede ist, nicht bloss die gegebene Menge ν^{ter} Art der Ausnahmepunkte im Intervalle $(0 \cdots 2\pi)$, sondern diejenige Menge im Auge, welche auf der ganzen, unendlichen Linie aus der periodischen Wiederholung jener hervorgeht.

Betrachten wir nun die Funktion

$$F(x) = C_0 \frac{xx}{2} - C_1 - \frac{C_2}{4} - \cdots - \frac{C_n}{nn} - \cdots$$

Aus der Natur einer Punktmenge ν^{ter} Art ergibt sich leicht, dass ein Intervall $(\alpha \cdots \beta)$ vorhanden sein muss, in welchem kein Punkt der Menge P liegt; für alle Werte von x in diesem Intervalle wird also wegen der vorausgesetzten Konvergenz unserer Reihe (I) sein

$$\lim(c_n \sin nx + d_n \cos nx) = 0,$$

mithin ist einem bekannten Satze gemäss (Math. Ann. Bd. 4, S. 139) [hier II 4, S. 87]

$$\lim c_n = 0, \qquad \lim d_n = 0.$$

Die Funktion $F(x)$ hat also (siehe Riemann : Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe, § 8) folgende Eigenschaften:

1. sie ist stetig in der Nähe eines jeden Wertes von x,

---

(1) Dass dies nicht stets der Fall ist, möchte vielleicht noch ausdrücklich hervorgehoben zu werden verdienen. Im allgemeinen kann die auf jene Weise aus einer Zahlengrösse ν^{ter} Art hervorgehende Punktmenge sowohl von neiderer wie auch von höherer als der ν^{ten} Art oder selbst gar nicht von bestimmter Art sein.

2. es ist $\lim \frac{F(x+\alpha)+F(x-\alpha)-2F(x)}{\alpha\alpha} = 0$, wenn $\lim \alpha = 0$, für alle Werte von x mit Ausnahme der den Punkten der Menge P entsprechen den Werte,

3. es ist $\lim \frac{F(x+\alpha)+F(x-\alpha)-2F(x)}{\alpha} = 0$, wenn $\lim \alpha = 0$, für jeden Wert von x ohne Ausnahme.

Ich will nun zeigen, dass $F(x) = cx + c'$ ist.

Dazu betrachte ich zuerst irgend ein Intervall $(p \cdots q)$, in welchem nur eine endliche Anzahl von Punkten der Menge P liegt : diese Punkte seien $x_0, x_1, \cdots x_r$, ihrer Aufeinanderfolge nach geschrieben.

Ich behaupte, dass $F(x)$ im Intervalle $(p \cdots q)$ *linear* ist; denn $F(x)$ ist wegen der Eigenschaften 1. und 2. eine lineare Funktion in jedem der Intervalle, in welche $(p \cdots q)$ durch die Punkte $x_0, x_1, \cdots x_r$ geteilt wird; da nämlich in deines dieser Intervalle Ausnahmepunkte fallen, so gelten hier die im Aufsatze (siehe Journal f. d. r. u. angew. Math. Bd. 72, S. 139) angewandten Schlüsse; es bleibt daher nur übrig, die Identität dieser linearen Funktionen nachzuweisen.

Ich will dies für je zwei benachbarte tun und wähle dazu die in den beiden Intervallen $(x_0 \cdots x_1)$ und $(x_1 \cdots x_2)$.

In $(x_0 \cdots x_1)$ sei $F(x) = kx + \ell$.

In $(x_1 \cdots x_2)$ sei $F(x) = k'x + \ell'$.

Wegen 1. ist $F(x_1) = kx_1 + \ell$; ferner ist für hinreichend kleine Werte von α

$$F(x_1 + \alpha) = k'(x_1 + \alpha) + \ell' \;;\; F(x_1 - \alpha) = k(x_1 - \alpha) + \ell.$$

Wegen 3. hat man also

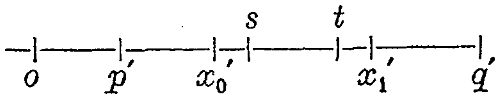
$$\lim \frac{(k'-k)x_1 + \ell' - \ell + \alpha(k'-k)}{\alpha} = 0, \text{ für } \lim \alpha = 0,$$

was nicht anders möglich ist, als wenn [vgl. unsere Anmerkung zu **II**, 3]

$$k = k', \quad \ell = \ell'.$$

(A) *"Ist $(p \cdots q)$ irgend ein Intervall, in welchem nur eine endliche Anzahl von Punkten der Menge P liegt, so ist $F(x)$ in diesem Intervalle linear."*

Weiter betrachte ich irgend ein Intervall $(p' \cdots q')$, welches nur eine endliche Anzahl von Punkten $x'_0, x'_1, \cdots x'_r$ der ersten abgeleiteten Menge P' enthält, und behaupte zunächst, dass in jedem der Teilintervalle, in welche $(p \cdots q)$ durch die Punkte $x'_0, x'_1, \cdots$ zerfällt, die Funktion $F(x)$ linear ist, z. B. in $(x'_0 \cdots x'_1)$.



Denn jedes dieser Teilintervalle enthält zwar im allgemeinen unendlich viele Punkte aus P, so dass das Resultat (A) nicht unmittelbar auf dasselbe Anwendung findet;

dagegen enthält jedes Intervall $(s \cdots t)$, welches ganz innerhalb $(x'_0 \cdots x'_1)$ fällt, nur eine endliche Anzahl von Punkten aus P (weil sonst zwischen x'_0 und x'_1 noch andere Punkte der Menge P' fallen würden), und die Funktion ist also in $(s \cdots t)$ wegen (A) linear. Indem man aber die Endpunkte s und t den Punkten x'_0 und x'_1 beliebig nahe bringen kann, wird ohne weiteres geschlossen, dass die stetige Funktion $F(x)$ auch linear ist in $(x'_0 \cdots x'_1)$.

Nachdem dies für jedes der Teilintervalle von $(p' \cdots q')$ nachgewiesen ist, erhält man durch dieselben Schlüsse wie diejenigen, welche das Resultat (A) erzielten, folgendes :

(A') *"Ist $(p' \cdots q')$ irgend ein Intervall, in welchem nur eine endliche Anzahl von Punkten der Menge P' liegt, so ist $F(x)$ in diesem Intervalle linear."*

Der Beweiss geht in diesem Sinne fort. *Steht nämlich einmal fest, dass $F(x)$ eine lineare Funktion ist in irgend einem Intervalle $(p^{(k)} \cdots q^{(k)})$, welches nur eine endliche Anzahl von Punkten aus der k^{ten} abgeleiteten Punktmenge $P^{(k)}$ von P enthält, so folgert man ebenso wie bei dem Übergange von (A) zu (A')weiter, dass $F(x)$ auch eine lineare Funktion ist in irgend einem Intervalle $(p^{(k+1)} \cdots q^{(k+1)})$, welches nur eine endliche Anzahl von Punkten der $(k+1)^{\text{ten}}$ abgeleiteten Punktmenge $P^{(k+1)}$ in sich fasst.*

Wir schliessen so durch eine *endliche* Anzahl von *Übergängen*, dass $F(x)$ in jedem Intervalle, welches nur eine endliche Anzahl von Punkten der Menge $P^{(\nu)}$ enthält, *linear* ist. Nun ist aber die Menge P von der ν^{ten} Art, wie vorausgesetzt wurde, es enthält mithin überhaupt ein beliebig in der Geraden angenommenes Intervall $(a \cdots b)$ nur eine endliche Anzahl Punkte aus $P^{(\nu)}$. Es ist also $F(x)$ linear in jedem willkürlich angenommenen Intervalle $(a \cdots b)$, und daraus folgt, wie leicht zu sehen, für $F(x)$ die Form: $F(x) = cx + c'$ für alle Werte des x. Nachdem dies dargetan ist, geht der Beweiss in der nämlichen Weise weiter wie in der schon zweimal zitierten Abhandlung von dem Momente an, wo darin ebenfalls für $F(x)$ die lineare Form nachgewiesen ist.

Dem hier bewiesenen Satze kann auch die folgende Fassung gegeben werden :

"Eine unstetige Funktion $f(x)$, welche für all Werte von x, welche den Punkten einer im Intervalle $(0 \cdots 2\pi)$ gegebenen Punktmenge P der ν^{ten} Art entsprechen von Null verschieden oder unbestimmt, für alle übrigen Werte des x aber gleich Null ist, kann durch eine trigonometrische Reihe nicht dargestellt werden."

Chapter 7

THE TURN OF THE CENTURY AND FEJÉR'S THEOREM

1. Trigonometric series as a disreputable subject.

Considering how mathematical analysis evolved during the nineteenth century we see two main streams : functions of a complex variable on one part, differential and partial differential equations on the other. There was spectacular progress in the theory of functions of a complex variable, the "theory of functions" *par excellence.* Its use by Hadamard and de La Vallée Poussin in order to prove the prime number theorem (1896) might look like a coronation : analytic functions dominate the whole realm of mathematics. Such was the situation in the year 1900.

In the second rank we see functions of several variables, enriched by the various problems coming from mechanics and physics, leading to important geometrical notions. Of course differentiability of every order was supposed, since nature seemed to obey differential and partial differential equations.

Far behind lay the functions of one real variable, a strange and insecure country. Despite the initial contributions of Riemann and Weierstrass (whose theorem on polynomial approximation of continuous functions was celebrated immediately), despite the books of Dedekind and Dini and the new conceptions of Cantor, it was not yet considered as a proper domain of mathematics. In this uncivilized country monsters were discovered in the 1870's : continuous functions without derivative at any point, continuous functions whose Fourier series diverge at some point. Hermite wrote to Stieltjes that he turned away with fright and horror from that lamentable ulcer which is a continuous function with no derivative at all. Poincaré in *l'Enseignement Mathématique* complained that examples were not constructed any more in order to illustrate theorems and theories, but just for the purpose of showing that our predecessors were wrong. Gaston Darboux, who wrote an important memoir on discontinuous functions in 1875, turned to geometry quite prudently.

Between 1880 and 1900 there were only a few works on trigonometric series and they did not attract much attention. Fourier series did not appear to mathematicians as a reliable and convenient tool ; there were too many strange things about them. Maybe there are continuous functions whose Fourier series diverge everywhere, just as there are nowhere differentiable functions (this was still an open question in 1965, before Carleson's theorem). Is it even possible that the Fourier series converges but does not represent the function, as it is the case for Taylor series of some functions admitting derivatives of all order ? This last question seems to have been raised by Minkowski and was still considered as an open problem at the time of Fejér's theorem, in 1900.

Emile Picard's *Traité d'analyse,* which appeared in 1891, is rather instructive in this respect. The problem of finding a harmonic function in a domain when boundary values are given (the Dirichlet problem) is treated for the sphere, in the section of the book devoted to functions of several variables, before being treated for the circle. This is not because things are worse for the circle than for the sphere. But the solution for the circle belongs to the chapter "Fourier series" and this is not a subject to begin with.

Let us consider more closely the pages devoted by Emile Picard to the Dirichlet problem for the circle ; in 1900 Fejér knew them well and refers to them in his note. Picard presents the method of Schwarz (1872), which is based on the properties of the "Poisson kernel"

$$\frac{1-r^2}{1-2r\cos t+r^2}=1+2\sum_1^\infty r^n\cos nt.$$

As an application he shows that a function which is continuous on the circle and has the Fourier series

$$\sum_0^\infty (a_n\cos nt+b_n\sin nt)$$

can be expressed as uniform limit of trigonometric polynomials of the form

$$\sum_{n\leq N(r)} (a_n\cos nt+b_n\sin nt)r^n$$

giving therefore a new proof of the Weierstrass approximation theorem.

Somewhat later, in 1893, Ch. de la Vallée Poussin also used the Poisson kernel to establish the Parseval formula (that is, in modern notation,

$$\int f\bar{g}=\sum \hat{f}(n)\,\overline{\hat{g}(n)}\,)$$

which is essentially equivalent to the totality of the trigonometric system. In 1901 Adolf Hurwitz stated the same formula as a lemma to his solution of the isoperimetric problem by means of Fourier series, saying that he would prove it later. We shall return to this story in the following chapters.

There were actually a few interesting results on Fourier series but results and problems were not related to each other. The problem of Minkowski could have been solved easily using the method of Schwarz presented in Picard's book but Picard did not know about the problem and Minkowski was apparently unaware of this part of the works of Schwarz and Picard. Schwarz's method also yielded the formula obtained by Ch. de la Vallée Poussin but de la Vallée Poussin was not aware of it. For certain Hurwitz did not know the paper of de la Vallée Poussin : this is acknowledged in an article of Hurwitz in 1903. Thus all these were isolated works on a marginal subject.

Fejér wrote at the beginning of his thesis that nothing new appeared on Fourier series between 1880 and 1900. Though this is not true completely, still Fourier series appeared as a stagnant subject, out of fashion.

2. The circumstances of Fejér's theorem.

On December 10, 1900, a note of Leopold Fejér *"Sur les fonctions intégrables et bornées"* appeared in the *Comptes-Rendus de l'Académie des Sciences de Paris.* It contains what are called now the Fejér sums, Fejér kernel, Fejér summation process and the famous theorem of Fejér : Fejér's sums

$$\sigma_n = \frac{1}{n}(S_0 + S_1 + \cdots + S_{n-1})$$

approximate the given function f at each point x where $f(x+0)$ and $f(x-0)$ exist and $f(x) = \frac{1}{2}(f(x+0) + f(x-0))$, and uniformly when f is continuous on the circle.

Leopold Fejér (Fejér Lipot) was then an unknown twenty year old Hungarian mathematician. But his theorem became famous quickly. It was used for proving the totality of the trigonometric system in the 1903 article of Hurwitz on the isoperimetric problem, extended to Lebesgue-integrable functions (Fejér-Lebesgue theorem), extended to other kernels useful in approximation theory (Ch. de la Vallée Poussin), applied to a completely new proof of the Dirichlet-Jordan theorem (Hardy), made more precise by means of the notion of absolute summability (Hardy and Littlewood), extended to the so-called (C, α) processes of summation for all $\alpha > 0$ (Marcel Riesz ; Fejér's case is $\alpha = 1$) and, above all, its simplicity made it accessible to any mathematics student. We already mentioned some of these points in the chapter on Dirichlet and we shall return to the others in a moment.

Let us trace the circumstances of Fejér's discovery. The idea of assigning a sum to a divergent series by means of some summation process was well known in 1900. Lebesgue traced it back to d'Alembert, namely for the series

$$\frac{1}{2} + \sum_{1}^{\infty} \cos nt \qquad (0 < t < 2\pi)$$

and the process of taking the arithmetic means of partial sums. It became a significant topic of Abel's investigations, then of those of Poisson, Frobenius, Hölder, Cesáro and Borel. Abel proved the famous theorem which asserts that

$$\lim_{r \uparrow 1} \sum_{0}^{\infty} a_n r^n = \sum_{0}^{\infty} a_n$$

whenever the righthand member exists in the usual sense. Poisson considered the left hand side as the generalized sum of the series, whether the series converges or not. Frobenius generalized Abel's theorem by showing that

$$\lim_{r \uparrow 1} \sum_{0}^{\infty} a_n r^n = \lim_{n \to \infty} \frac{1}{n}(S_0 + S_1 + \cdots + S_{n-1})$$

whenever the right hand side exists ; then Hölder generalized this theorem of Frobenius by iterating the process of arithmetic means. Cesàro generalized another theorem of Abel, on multiplication of series, and introduced for this purpose the processes that we now denote by (C, k). In 1900 Cesàro's investigations were fairly recent (1890). From 1895 on Emile Borel defined new summation processes and applied them to the analytic continuation of functions defined by a Taylor series. Fejér was aware of Borel's investigations. He knew the Frobenius theorem.

In order to prove the Dirichlet problem for the circle in the form

$$f(r\cos t, r\sin t) = \frac{a_0}{2} + \sum_1^\infty (a_n \cos nt + b_n \sin nt) r^n$$

when the function prescribed at the boundary has the Fourier series

$$\frac{a_0}{2} + \sum_1^\infty (a_n \cos nt + b_n \sin nt)$$

it was tempting to make use of Abel's theorem. However the example of P. du Bois Reymond excluded any hope to obtain a solution for general continuous boundary functions in this way. Could one apply the theorem of Frobenius ? This seems to have been the starting point of Fejér.

According to J. Horváth (oral communication) Fejér knew of this problem when visiting Berlin in 1900. He obtained a solution and wrote it in Budapest within a few days at the end of October. He observed immediately that the method he used was more important than the new solution of the Dirichlet problem. In the note that he sent to Paris the solution of the Dirichlet problem appears as one of the consequences of his theorem. The proof is based on the kernel

$$K_n(x) = \frac{1 - \cos nx}{1 - \cos x}$$

and it is strongly related to Schwarz's solution for the Dirichlet problem, which makes use of the Poisson kernel

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

Fejér indeed did *not* introduce any new summation process for divergent series; on the contrary, he used the most evident of them. It was *not* he who introduced positive kernels in investigating Fourier series. The Poisson kernel was well known and its application to Fourier series could be found in Picard's treatise long before 1900. Fejér did *not* solve a difficult conjecture by sophisticated methods.

What he did is much more than that. He gave a clear, simple, applicable treatment in a field where the strange and the bizarre prevailed according to the general feeling of mathematicians. By coupling summation processes and Fourier series he found a convenient frame for both theories. Since that time the summation process of Riemann, the method of Schwarz for the Dirichlet problem on the circle, the process already introduced by Fourier for computing temperatures inside a heated body, the newly introduced process of Weierstrass in order to study temperatures as a function of time (leading to series

$$\sum \left(a_n \cos nx + b_n \sin nx) e^{-tn^2}\right)$$

appeared as expressing the same principle, most simply presented in Fejér's note : on one hand, regularization of the function by means of a convenient kernel, on the other hand, a summation process for the Fourier series. The role of positive kernels was emphasized, and developed in many works of Fejér himself later. The two pages long note of Fejér completely changed the position of trigonometric series in mathematics. It also gave impetus to the general study of summation methods.

3. A few applications and continuations of Fejér's theorem.

We shall measure the immediate deep effect of Fejér's theorem by describing applications and continuations made by Hurwitz, Lebesgue, Young, de la Vallée Poussin, Hardy, Fejér himself, Hardy and Littlewood, Marcinkiewicz and Zygmund. We shall not dwell on other processes of summation of the same type, though there was an enormous literature thereabout between 1910 and 1940.

As we have already said, Hurwitz announced a proof of a statement equivalent to the totality of the trigonometric system and his proof was never published. In the definitive developed article on Fourier series applied to the isoperimetric problem (1903) Hurwitz simply used the theorem of Fejér.

The extension by Lebesgue was published in a memoir in *Mathematische Annalen* 61 (1905) and expounded in his book of 1906. Here it is: writing

$$\begin{cases} \sigma_n - f = \dfrac{1}{n\pi} \displaystyle\int_0^{\pi/2} \left(\dfrac{\sin nt}{\sin t}\right)^2 \varphi(t)dt \\ \varphi(t) = f(x+2t) + f(x-2t) - 2f(x) \end{cases}$$

$\sigma_n - f$ tends to zero at every point x (known now as Lebesgue point) such that

$$\lim_{t\to 0} \frac{1}{t} \int_0^t |\varphi| = 0,$$

therefore almost everywhere. This relies on theorems on integration which are the subject of the beginning of the book. We shall return to it in the chapter on Lebesgue.

The Fejér kernel can be written as

$$K_n(x) = \sum_{-n}^{n} \left(1 - \frac{|m|}{n}\right) e^{imx}.$$

A linear combination of K_n with positive coefficients yields a positive function whose Fourier coefficients c_m (in the complex form) are even ($c_m = c_{-m}$), positive, convex for $m \geq 0$ and decreasing to zero as $m \to \infty$. Conversely, each sequence c_m of this form is a sequence of Fourier coefficients of a positive and integrable function. This is a theorem of W. H. Young (1913). The important point is that the c_m can decrease to zero as slowly as we want. This is a very good tool in the study of $L^1(\mathbf{T})$. In particular it is the way to prove that each function f in $L^1(\mathrm{T})$ (considered as a convolution algebra) can be factored in the form $f = g * h$, $g \in L^1(\mathrm{T})$, $h \in L^1(\mathrm{T})$ (Salem 1939). There is an analogue in $L^1(\mathbb{R})$: an even function $c(t)$ ($t \in \mathbb{R}$) which is positive, convex for $t \geq 0$ and decreasing to zero at infinity is the Fourier transform of a positive and integrable function ; if moreover $c(0) = 1$ the function is a probability density, so that $c(t)$ is a characteristic function in the sense of probability theory. Such functions $c(t)$ are called Pólya functions (Pólya 1949).

In 1908 Ch. de la Vallée Poussin introduced kernels of the type

$$k_n(\cos\frac{x}{2})^{2n}$$

as a tool in order to approximate a function together with its derivatives of any order. Today however what we call the de la Vallée Poussin kernel has the form

$$V_n = 2K_{2n} - K_n,$$

and its coefficients form a trapezoid on $[-2n, 2n]$. Although V_n is not a positive kernel any more, it has two advantages compared to K_n. First, it is very well adapted to approximation theory (it was introduced by Ch. de la Vallée Poussin in his 1919 lectures on the approximation of functions of one real variable) : the approximation by the de la Vallée Poussin sums of order $2n$ is about the same as the best approximation by trigonometric polynomials of order n. The second advantage is that the differences of the trapezoidal sequences of coefficients

$$\hat{V}_{2^{j+1}} - \hat{V}_{2^j}$$

form a partition of unity with supports $[-2^{j+2}, -2^j] \cup [2^j, 2^{j+2}]$ going rapidly to infinity. Thus the decomposition provided by the kernel difference $V_{2^{j+1}} - V_{2^j}$ is strongly connected with the dyadic decomposition of the Fourier series

$$\sum_{n\in\mathbb{Z}} = \sum_{j=0}^{\infty} \sum_{2^j \leq |n| < 2^{j+1}} c_n e^{int} \qquad (c_0 = 0)$$

which is the core of the Littlewood-Paley theory developed in the 1930's. A large part of the theory of approximation by trigonometric polynomials consists of theorems of the form

$$\| (V_{2^{j+1}} - V_{2^j}) * f \|_E = O(\omega_j) \Leftrightarrow f \in F$$

where E and F are Banach spaces and ω a sequence tending to zero. For instance

$$\| (V_{2^{j+1}} - V_{2^j}) * f \|_C = O(2^{-j\alpha}) \Leftrightarrow f \in \Lambda_\alpha$$

where C is the space of continuous functions on $\mathbf{T}$, Λ_α is the space of functions on $\mathbf{T}$ satisfying a Hölder condition of order α for $0 < \alpha < 1$ (alternatively denoted by Lip α), that is,

$$\sup_{h,t}(|\, h\,|^{-\alpha}|\, f(t+h) - f(t\,|) < \infty,$$

Λ_1 is what Zygmund calls Λ_* (1959, I p. 43) and $f \in \Lambda_\alpha$ means $f' \in \Lambda_{\alpha-1}$ for $\alpha > 1$. We shall return to the dyadic decomposition later in this book.

Ch. de la Vallée Poussin was not the first to introduce such linear combinations of Fejér kernels. By considering

$$\big((N+1)K_{(N+1)n} - NK_{Nn}\big) * f$$

and comparing with the partial sum S_{Nn} we obtain results on ordinary convergence when we assume an extra condition on the coefficients. What we want is

$$\sum_{Nn \le |m| < (N+1)n} |\, c_m\,| < \epsilon_N, \qquad \lim_{N\to\infty} \epsilon_N = 0.$$

This is guaranteed if

$$(H) \qquad c_m = O\left(\frac{1}{|\, m\,|}\right) \qquad (|\, m\,| \to \infty)$$

and also if

$$(F) \qquad \sum |\, m\,||\, c_m\,|^2 < \infty.$$

Condition (H) goes back to Hardy (1910). It is satisfied if f has bounded variation, giving a new proof of the Dirichlet-Jordan theorem.

Condition (F) is due to Fejér (1913). He used it to prove that if a function $f(z) = \sum_0^\infty c_n z^n$ is holomorphic in the disc $|\, z\,| < 1$ and continuous on $|\, z\,| \le 1$, and if the image of the disc has bounded area on the Riemann surface spanned by $f(z)$, then the Taylor series converges uniformly on the closed disc. This is the case in particular when $f(z)$ yields a conformal mapping of the disc $|\, z\,| < 1$ on the interior of a simple Jordan curve. This theorem was used by Pál, Bohr and Salem to show that for every continuous real function g on $\mathbf{T}$ there is a homeomorphism h of $\mathbf{T}$ such that the Fourier series of the function $g(h(t))$ converges uniformly (see Salem 1944). Only in the 1970's was the result extended to complex functions g, by purely real methods (Saakian 1979 ; Kahane-Katznelson 1983).

Let us turn to another theorem of Taylor series in the unit disc, by Marcinkiewicz and Zygmund (1941). We begin with a simple example : if $f(z) = \frac{1}{1-z}$ the partial sums S_n diverge at every point of the boundary $|\, z\,| = 1$ and the Fejér sums σ_n converge everywhere at the boundary except at $z = 1$. Here is the theorem : if, given a Taylor series $f(z) = \sum_0^\infty c_n z^n$ in the unit disc $|\, z\,| < 1$, the Fejér sums σ_n converge on a subset E of the circle $|\, z\,| = 1$ whose linear measure is positive, then, almost everywhere on E, the accumulation points of S_n have a "circular structure", i.e. they form a union of circles centered at σ_n. Is it true that, in some sense, the S_n are equally distributed on this circular set ? The question is still unresolved.

We end this review by the absolute summability theorem of Hardy and Littlewood, which can be written in the form

$$(HL) \qquad \frac{1}{n}\sum_{0}^{n-1} | S_j(x) | \leq C \| f \|_\infty$$

for each bounded function f and each n and x. This answers questions of the type: is it possible that the absolute values $| S_j(x) |$ tend to infinity at some point x when the given function f is bounded ? Fejér's theorem provides a negative answer when f is real. In the complex case the Hardy-Littlewood theorem is needed. Here are a few comments on this question. Can one find a continuous function f and a sequence x_n such that $S_n(x_n) \to \infty$? This is possible, even so that $S_n(x_n)/\log n$ tends to zero arbitrarily slowly (E. Busko 1968). One can ask the same question for a sequence of partial sums S_{n_j}. For $n_j = 2^j$ there are continuous functions f such that $S_{n_j}(0) \to \infty$, even such that $S_{n_j}(0)/\sqrt{\log n_j}$ tends to 0 arbitrarily slowly. Conversely, whatever the increasing sequence n_j, the inequalities

$$\frac{1}{k}\sum_{j=1}^{k} | S_{n_j}(x) | \leq C \| f \|_\infty \sqrt{\log n_k}$$

are valid, showing the role of $\sqrt{\log n_k}$ in these estimates. For what sequences n_j does the analogue of (HL), namely

$$\frac{1}{k}\sum_{j=1}^{k} | S_{n_j}(x) | \leq C \| f \|_\infty$$

hold ? Surprisingly this question can be answered completely under the simple condition that n_j is a convex sequence ; the necessary and sufficient condition is

$$\overline{\lim_{j\to\infty}} \frac{\log n_j}{\sqrt{j}} < \infty$$

(Long Rui-lin 1981).

There is an almost everywhere version of (HL) for integrable functions. The strongest result in that direction is a theorem of Gogoladze (1988) : for each $f \in L^1$ and $A > 0$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{0}^{n-1} (\exp(A \mid S_j(x) - f(x) \mid) - 1) = 0$$

holds almost everywhere (see Rodin 1992 for reference and context).

Chapter 8

LEBESGUE AND FUNCTIONAL ANALYSIS

1. Lebesgue.

Henri Lebesgue was born in 1875 and died in 1941. His father was a worker in a printing house and died when Henri was quite young. Like Fourier, Lebesgue was a poor orphan and a bright boy. He got grants until he entered *Ecole Normale Supérieure* in 1894, taught in high schools, got his doctorate in 1902, taught at the Universities of Rennes and Poitiers, was appointed to the Sorbonne in Paris in 1910, to the College de France in 1921, and elected to the Academy of Sciences in 1922.

Though he worked on a number of questions coming from analysis, his name is associated mainly with the Lebesgue measure and integral, whose theory was completed in the years 1902-1906. The main landmarks are his doctoral thesis on Integrals, Length, Area in 1902, his book on *Integration* in 1904 and his book on *Trigonometric Series* in 1906. Both books were written on the basis of lectures that he was invited to give at the *Collège de France*, the so called *"cours Peccot"*. The Peccot grants and lectures had an important effect on mathematics in France at that time. They allowed a few young doctors to address younger people and also contributed to their living expenses. From the "cours Peccot" point of view, 1904, 1905, 1906 were exceptional vintage years. In 1904 Lebesgue lectured on integration, in 1905 René Baire (who got his doctorate in 1899 and had a great influence on Lebesgue) on discontinuous functions, in 1906 Lebesgue again on trigonometric series. The resulting books were published in a collection edited by Emile Borel. The lectures of Baire, *Leçons sur les fonctions discontinues*, were written by Arnaud Denjoy, then a young student. The lectures of Lebesgue were attended by Pierre Fatou, and some results of Fatou were published only in the book of Lebesgue on trigonometric series. Fatou wrote his thesis in 1907, on trigonometric series and Taylor series, as a direct continuation of Lebesgue's book.

We shall not dwell on the impact of the Lebesgue integral in mathematics ; this is the subject of other books, in particular T. Hawkins's "Lebesgue's theory of integration" (1970). We shall summarize the works of Lebesgue and Fatou on trigonometric series and show how they interact with the theory of integration and with the beginning of functional analysis. We shall sketch the history of the Fatou-Parseval relation and the Riesz-Fischer theorem, where square integrable functions on the circle and square summable sequences appear as isomorphic notions, therefore L^2 as a model of Hilbert space. Then we describe the L^p and H^p spaces, the duality theorems, the conjugacy theorems as prefiguration of the theory of linear operators and Banach spaces. The "Lebesgue constants" play a special role in the theory of orthogonal series and suggested a series of modern investigations - we think of the so-called Littlewood conjecture, proved in 1981, in particular. The Lebesgue approximation theorems contributed to the linkage between approximation theory and functional analysis.

2. Lebesgue and Fatou on trigonometric series (1902-1906).

The work of Lebesgue on trigonometric series extends from 1902 to 1910 and can be divided into two parts. The main part deals with convergence and divergence of Fourier-Lebesgue series. It includes three notes in the *Comptes-Rendus* (1902, 1905 twice) and two articles (1903, 1905). The whole of it is contained and expounded in the book *Leçons sur les séries trigonométriques* (1906). The second part consists of two articles in 1909 and 1910, on singular integrals and trigonometric approximation.

The thesis of Fatou, *Séries trigonométriques et séries de Taylor* (1906), deserves to be considered together with Lebesgue's works. Fatou contributed to Fourier series, in particular absolute convergence of Fourier series, in subsequent papers (1913). At the present time he is mainly known among mathematicians by the beautiful work on iteration of analytic mappings that he developed in 1919-1920. He was an astronomer, never taught in a university, got a full position as an astronomer only in 1928 at the age of 50, and died one year later.

The main purpose of Lebesgue is described in his 1903 article : *"je vais appliquer la notion d'intégrale à l'étude du développement trigonométrique des fonctions non intégrables au sens de Riemann"* (I shall apply the notion of integral to study trigonometric expansions of non Riemann integrable functions). The main result, already announced in 1902, is that an everywhere convergent trigonometric series whose sum is a bounded function is the Fourier-Lebesgue series of that function. The same holds when *everywhere* is replaced by *everywhere outside a reducible set in the sense of Cantor* (countable set whose derivative of order ν is finite for some ν). Lebesgue wrote wrongly that *everywhere* can be replaced by *almost everywhere* (everywhere except a set of zero measure) in 1902 and corrected the mistake in 1903; the counter example came only in 1916 as we have already said (Menšov).

For the convergence problem Lebesgue used the notation

$$\pi(S_n - f) = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)t}{\sin t}\varphi(t)dt$$

$$\varphi(t) = f(x+2t) + f(x-2t) - 2f(x) = \psi(t)\sin t.$$

Here are the most general results. If i) $\int^x \mid \varphi \mid$ has a vanishing derivative at $x = 0$, ii) $\int_\delta^{\frac{\pi}{2}} \mid \psi(t+\delta) - \psi(t) \mid dt$ tends to 0 together with δ, then S_n converges to f (Lebesgue convergence theorem). If only i) holds, then $\sigma_n = \frac{1}{n}(S_0 + \cdots + S_{n-1})$ converges to f (Fejér-Lebesgue theorem).

A particular case of the convergence theorem can be deduced from the Riemann-Lebesgue theorem (the Fourier coefficients of a Lebesgue integrable function tend to zero) : if ψ is Lebesgue-integrable, then convergence holds (Dini condition). As a consequence the Riemann localization principle (convergence at a point depends only on the values of the function in a neighborhood of that point) holds for Lebesgue integrable functions.

All convergence theorems have a uniform version, where uniform conditions are assumed and result in uniform convergence. That is an observation of Fatou, given in Lebesgue's book. We have already mentioned in chapter 5 Fatou's example of an everywhere convergent trigonometric series whose sum is not Lebesgue integrable. Lebesgue gives more elaborated examples : Lebesgue but not Riemann integrable functions whose Fourier series converge everywhere, continuous functions whose Fourier series converge everywhere but not uniformly, continuous functions whose Fourier series diverge at a point (see chapter 4).

The so-called Lebesgue constants (L^1 norm of Dirichlet kernels) appear in this context, together with the estimate

$$\int |D_n| \approx A \log n.$$

Lebesgue went back to Dirichlet and other kernels when he studied what he called singular integrals (1909) and trigonometric approximation (1910), as we shall see in a moment.

Let us turn to Fatou.

The most celebrated theorem in the thesis of Fatou is about boundary values of bounded analytic functions : given a bounded analytic function in the unit disc, it has a radial and a even non-tangential limit at almost every point of the boundary, and this limit is different from zero almost everywhere. Radial limit of $f(z)$ at $e^{i\theta}$ means

$$\lim_{r\uparrow 1} f(re^{i\theta})$$

and non-tangential limit means

$$\lim_{r\uparrow 1, |\varphi-\theta|\leq\lambda(1-r)} f(re^{i\varphi})$$

for all $\lambda > 0$.

Fatou considered also bounded harmonic functions in the unit disc and proved the existence of a non tangential limit almost everywhere at the boundary. The boundedness condition can be weakened, as he showed, into what we call now an H^2 condition, namely $\sum(a_n^2 + b_n^2) < \infty$, when the harmonic function is written as

$$\sum_0^\infty r^n(a_n \cos nt + b_n \sin nt) \qquad (z = re^{it}).$$

Conjugate harmonic functions are defined as real and imaginary parts of an analytic function. The above harmonic function has

$$\sum_1^\infty r^n(b_n \sin nt - a_n \sin nt)$$

as conjugate. The term "conjugate series" also applied to trigonometric series

$$\sum_0^\infty (a_n \cos nt + b_n \sin nt)$$

$$\sum_1^\infty (b_n \cos nt - a_n \sin nt).$$

Fatou proved that Λ_α (the class of continuous functions which satisfy a Hölder condition of order α – then named a Lipschitz condition) is stable under conjugacy.

Among the numerous results on Taylor series $\sum_0^\infty c_n z^n$ let us mention the simplest convergence theorem : if c_n tends to zero, the series converges at every regular point of the circle of convergence.

We shall discuss the Fatou-Parseval formula

$$\frac{1}{\pi} \int |f|^2 = 2a_0^2 + \sum_1^\infty (a_n^2 + b_n^2)$$

in a moment. Fatou showed that it holds under the mere condition that $|f|^2$ is Lebesgue integrable (we say that f is square-integrable), and he was very near showing that, given the a_n and b_n such that the series of the squares converges, they are Fourier coefficients of a square integrable function (the Riesz-Fischer theorem) - he considered only the case when $n\,a_n$ and $n\,b_n$ tend to zero, which leads to convergence theorems.

The Fatou lemma

$$\int f \leq \underline{\lim} \int f_n$$

$(f = \lim f_n,\ f_n \geq 0)$ appears in this context.

3. Trigonometric series and the Lebesgue integral.

In Lebesgue's thesis (1902) and in his book on integration (1904) the Lebesgue integral is defined for bounded measurable functions. The modern notion of an integrable function (Lebesgue says *"sommable"*) appears in connection with trigonometric series only. Actually the first chapter of Lebesgue's book on trigonometric series contains the definition which is classical now and states the main properties : $\int (f+g) = \int f + \int g$ and f is the derivative of $F(x) = \int_a^x f$ for almost every x.

An important corollary is introduced in the first chapter also : when f is Lebesgue integrable, the strong differentiability property of the integral, that is

$$\lim_{t \to 0} \frac{1}{t} \int_0^t |f(x+s) - f(x)|\, ds = 0$$

holds almost everywhere.

The dominated convergence theorem is stated for uniformly bounded functions. Finally the continuity of the translation in the space L^1 is given in the form :

$$J(f,\delta) = \int_{\alpha}^{\beta} | f(x+\delta) - f(x) | \, dx$$

tends to zero together with δ.

All these are necessary tools for Fourier series. For example, in order to express the Dini-Lebesgue convergence theorem, integrability should be defined for general and not only for bounded functions. The strong differentiability property is needed for the Fejér-Lebesgue theorem. Continuity of the translation is needed also in a number of places, for example the Riemann-Lebesgue theorem, whose proof can be written in modern notations

$$\left| \int_{\mathbb{T}} f(t) e^{-2\pi i n t} dt \right| = \left| \frac{1}{2} \int_{\mathbb{T}} (f(t) - f(t + \frac{1}{2n})) e^{-2\pi i n t} dt \right| \le \frac{1}{2} \int_{\mathbb{T}} | f(t) - f(t + \frac{1}{2n}) | \, dt.$$

These remarks hold also for the Fatou lemma, introduced in relation with the Fatou-Parseval formula, and ready to be used for the completeness of L^p spaces.

Before discussing L^p and other function spaces let us concentrate on L^2 and the Riesz-Fischer theorem.

4. Fatou-Parseval and Riesz-Fischer.

In modern wording the Bessel inequality, the Parseval formula and the Riesz-Fischer theorem are related to a L^2 space and an orthonormal system (u_n) in that L^2 space. Given $f \in L^2$, its Fourier coefficients are the coordinates

$$c_n = \int f \, \bar{u}_n .$$

The Bessel inequality is

$$\sum | c_n |^2 \le \int | f |^2 .$$

The Parseval formula applies when (u_n) is total, in other words, an orthonormal basis in L^2. It reads

$$\sum | c_n |^2 = \int | f |^2 .$$

Finally the Riesz-Fischer theorem states that, given $(c_n) \in \ell^2$, there exists $f \in L^2$ whose Fourier coefficients are the c_n ; it is an easy consequence of the completeness of L^2, and for that reason "L^p is complete" is sometimes designated as a theorem of F. Riesz.

However, we should remember that infinite dimensional spaces and function spaces were hardly introduced in 1907. Hilbert spaces were not yet defined and their elementary geometric properties not yet investigated, orthonormal systems were just a wide collection of examples, completeness as a topological notion did not exist, L^p spaces were not born. The sentence "L^2 is complete" is a short and excellent way to express the substance of a long demonstration. But, from a historical point of view, the demonstrations of Riesz and Fischer came first, and the now basic concepts of L^2-spaces, Hilbert spaces, Cauchy sequences, complete spaces followed.

If we consider the space $L^2(0,1)$ and the trigonometric system written in the complex form, $u_n = e^{2\pi i n x}$ $(n \in \mathbb{Z})$, the Bessel inequality expresses the simple fact that the u_n constitute an orthonormal system, and the Parseval equality the more subtle fact that (u_n) $(n \in \mathbb{Z})$ is total, that is, linear combinations of the u_n are dense in $L^2(0,1)$.

Fatou's contribution to the Parseval formula was to give the appropriate framework and prove that it holds when f is square summable. He also gave credit to Parseval, who wrote a memoir, dated 1806, at a time when nobody was able to really prove anything like totality of the trigonometric system. Prior to Fatou, the Parseval formula was called by A. Hurwitz *"Fundamentalsatz der Fourierschen Konstanten"* (Mathematische Annalen, 1903) and E. Fischer had just published *"Zwei neue Beweise für den Fundamentalsatz der Fourierschen Konstanten"* (Monatshefte für Mathematik und Physik 1904). Proofs were given for bounded integrable functions. For Hurwitz the *Fundamentalsatz* was needed in his approach to the isoperimetric problem (1901) as we already saw and will explain in chapter 12. Hurwitz was not aware of the note of Stekloff (1902) where the Fundamentalsatz was extended to orthogonal expansions related to a general Sturm-Liouville equation, nor of the previous work of Liapounoff (1896) that Stekloff mentions, nor of the paper of de la Vallée Poussin (1893) where Parseval's formula is proved for a wide class of functions f, nor of the works of Ossian Bonnet (1850 and 1879), Schwarz (1872), P. du Bois Reymond (1879), Picard (1891), where Poisson's formula is used and from which the totality of the trigonometric system follows, at least for continuous functions, nor of the theorem of Weierstrass on trigonometric approximation (1885). In his note in 1901 he announced a proof. Then he read the Fejér theorem (1900) and simply used that theorem in his 1903 article.

5. Riesz-Fischer and the beginning of Hilbert spaces.

The theory of the Hilbert space ℓ^2 (square summable sequences) was built by David Hilbert in connection with the theory of integral equations, to which he devoted a series of papers published in the *Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen* between 1904 and 1910, under the title *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*. The notion of the ℓ^2-space and the spectral theory of bounded quadratic forms were published in 1906, in the fifth and sixth communications. All these papers were reproduced in book form in 1912 and had a tremendous influence in analysis.

The Schmidt orthogonalization and the Bessel inequality appeared in the dissertation of Erhard Schmidt, *Entwicklung willkürlicher Funktionen nach vorgeschriebener Systemen* (1905), published in 1907 in *Mathematische Annalen*. The notion of an abstract Hilbert space was developed much later, in the years 1927-1930, with the work of John von Neumann inspired by quantum mechanics and resulting in the book *Mathematische Grundlagen der Quantummechanik* (1932).

Some basic definitions in abstract spaces, such as neighbourhood, separability, compactness (in the sense of sequential compactness) appeared already in the doctoral thesis of Maurice Fréchet (1906), which anticipated a number of further developments in functional analysis.

The notes of F. Riesz and E. Fischer appeared in 1907 in the *Comptes-Rendus*. The first (11 March) was entitled *"Sur les systèmes orthogonaux de fonctions"*, by Frédéric Riesz, and it was inspired by Hilbert's approach of integral equations through orthogonal expansions. F. Riesz treated the particular case of trigonometric series first, then extended the result to general orthonormal systems, using Fatou-Parseval. The second, *Sur la convergence en moyenne*, by Ernst Fischer, is dated April 1907, but refers to a lecture delivered by Fischer at the Mathematic Society in Brünn on the 5th of March. The proof is direct and introduces the essential concepts (scalar product, norm, triangular inequality, Cauchy sequences in L^2). In a third note (May 27) Ernst Fischer discussed some applications and emphasized the role of the Lebesgue integral in that question. Moreover he pointed out the geometric aspect :

> *"pour la théorie de l'approximation en moyenne, les ensembles dénombrables $\varphi_1, \varphi_2, \cdots$ ne sont aucunement le vrai point de départ ; je développerai cette théorie dans une autre occasion en m'appuyant sur une espèce de géométrie des fonctions."*

F. Riesz reacted immediately (fourth note) saying that he also had the idea of a new kind of analytical geometry.

F. Riesz told the story of this joint discovery in volume 1 of *Annales de l'Institut Fourier* (1949), and called the Riesz-Fischer theorem a permanent roundtrip ticket *(un billet aller et retour permanent)* between the spaces L^2 (square integrable functions) and ℓ^2 (square summable sequences). He also pointed out that it was the triumph of the Lebesgue notion of integration.

Besides its interest in Fourier series the theorem established L^2 as another model of Hilbert space. Fischer foresaw the geometry of Hilbert spaces. This played a crucial role in the development of functional analysis.

6. L^p, ℓ^q, functions and coefficients.

The Riesz-Fischer theorem gives an exact characterization of Fourier coefficients of L^p functions and also of functions represented by trigonometric series with coefficients in ℓ^p when $p = 2$. There is no such characterization when $p \neq 2$. Instead, there is a theorem due to W. H. Young (1912) and F. Hausdorff (1923) for the trigonometric system and extended by F. Riesz (1923) to more general orthogonal systems. Here is F. Riesz's version. We are given a measure space such that $\int 1 = 1$ (that is, a probability space), for example $(0,1)$ equipped with the Lebesgue measure, and an orthonormal system (u_n) such that $| u_n(t) | \leq M$ for all t and all n. Suppose $1 < p \leq 2$, and $\frac{1}{p} + \frac{1}{p'} = 1$ (then p and p' are called conjugate exponents). If $f \in L^p$

and $c_n = \int f\bar{u}_n$, then

$$\left(\sum | c_n |^{p'}\right)^{1/p'} \leq M^{\frac{2-p}{p}} \left(\int | f |^p\right)^{1/p}.$$

If $(c_n) \in \ell^p$ and $f = \sum c_n u_n$ (convergent series in L^2), then

$$\left(\int | f |^{p'}\right)^{1/p'} \leq M^{\frac{2-p}{p}} \left(\sum | c_n |^p\right)^{1/p}.$$

When $p = 2$ we recover the Riesz-Fischer theorem. For $p = 1$, $p' = \infty$ the inequalities hold on replacing the first members by $\sup | a_n |$ and $\sup | f | = \| f \|_\infty$. Actually the modern way to get the inequalities when $1 < p < 2$ is to start from the extreme cases and to use an interpolation theorem (namely, the theorem of Marcel Riesz (1926), called sometimes the Riesz-Thorin theorem because of the simple and beautiful proof given by Thorin in 1948). Marcel Riesz was a younger brother of Frederic Riesz.

For the complex trigonometric system, say, $e_n(t) = \exp(2\pi int)$ $(n \in \mathbb{Z})$, we have $M = 1$, $c_n = \int f\bar{e}_n$. In short, the mappings $f \to (c_n)$ and $(c_n) \to f$ are norm-decreasing from L^p to $\ell^{p'}$ and from ℓ^p to $L^{p'}$ $(1 \leq p \leq 2)$. This is the Hausdorff-Young theorem. In case $p = 1$ a little more can be said : the mapping $f \to (c_n)$ is norm decreasing from L^1 to c_0, space of sequences tending to 0 at infinity (closed subspace of ℓ^∞), and the mapping $(c_n) \to f$ is norm decreasing from ℓ^1 to C, space of continuous functions (closed subspace of L^∞).

Let us write $\mathcal{F}$ for the Fourier transform ($f \to (c_n)$ or $(c_n) \to f$). Then $\mathcal{F}L^1 \subset c_0$, $\mathcal{F}\ell^1 \subset C$, $\mathcal{F}L^p \subset \ell^{p'}$, $\mathcal{F}\ell^p \subset L^{p'}$ $(1 < p < 2)$. All these inclusions are strict. This can be seen in considering bounded linear forms on the first space which do not extend to the second. For example

$$c_n \to \sum_1^\infty \frac{1}{n}(c_n - c_{-n})$$

is a linear form defined on $\mathcal{F}L^1$, because

$$f \to \int f \sum_1^\infty \frac{\sin 2\pi nt}{n}$$

is a linear form on L^1, and it does not extend to c_0. On the other hand the value of p' is best possible as is easy to check.

The Riesz-Fischer theorem suggests two questions, on functions f and Fourier coefficients (c_n). 1) We know that $\sum | c_n |^2 < \infty$ implies $f \in L^2$. Is there a less restrictive condition on the $| c_n |$ which implies that the c_n are Fourier-Lebesgue coefficients ? 2) We know that $(c_n) \in \ell^2$ when $f \in C$. Is it possible to get a better result, of the type : the Fourier coefficients of a continuous function belong to a strict sublattice of ℓ^2 ? A lattice is defined by the fact that multiplication by bounded factors is allowed.

Both questions have negative answers. The solution to the first was given by Littlewood, using a clever construction. The best proof is a theorem of Paley and Zygmund (1930) : if $\sum | c_n |^2 = \infty$, then almost surely $\sum \pm c_n e^{2\pi i n t}$ is not a Fourier series. The solution to the second is due to De Leeuw, Kahane and Katznelson (1978): if $\sum | d_n |^2 < \infty$, there exists a continuous function f whose Fourier coefficients c_n satisfy $| c_n | \geq | d_n |$ for all n. Both theorems illustrate the use of probability methods in Fourier analysis, that will be explained in the next chapter. There are several variations and new ideas around the second theorem, by considering special classes of continuous functions (see Kislyakov 1988).

In this section we used L^p and ℓ^q with the current meanings of these notations. They did not become universal before the mid thirties. In his thesis (1920), S. Banach used S (summable functions) instead of L^1 and S^p instead of L^p. In his celebrated book *Théorie des opérations linéaires* (1932) the notation for L^p is $(L^{(p)})$. The modern notations are used, for example, by Zygmund and by Kaczmarz and Steinhaus in their books *Trigonometrical series* and *Theorie der Orthogonalreihen*, both published in 1935 in the same collection *Monografje matematyczne* as Banach's book.

Of course, much more can be said when we consider special classes of continuous functions. For example, using the formula that we gave in section 3 for the proof of the Riemann-Lebesgue theorem, we see that any estimate on $\int | f(t+\delta) - f(t) | \, dt$ gives an estimate on c_n. Using derivation of functions and series we can extend these estimates to the case $f \in \Lambda_\alpha^p$, meaning that f is differentiable $[\alpha]$ times and that $f^{[\alpha]}$ satisfies a Hölder condition of order $\alpha - [\alpha]$ in the norm of L^p. The simplest and most complete statements, apart from the Riesz-Fischer theorem and its corollaries, concern infinitely differentiable functions and analytic functions. Here they are.

A continuous function f defined on $\mathbf{T}$ is C^∞ if and only if $\lim_{|n|\to\infty} | n^A c_n | = 0$ whatever A may be ; then, according to L. Schwartz, we write $(c_n) \in s$ and we say that (c_n) is a rapidly decreasing sequence. Moroever f is analytic if and only if $\lim_{|n|\to\infty} | \rho^{|n|} c_n | = 0$ for some $\rho > 1$; then we say that (c_n) is an exponentially decreasing sequence.

7. $\mathbf{L^p}$, $\mathbf{H^p}$, conjugate functions.

We have already discussed conjugate harmonic functions in the unit disc (real and imaginary parts of an analytic function), conjugate trigonometric series (real and imaginary parts of a Taylor-type trigonometric series, that is, $\sum_0^\infty c_n z^n$, $z = e^{2\pi i t}$), conjugate functions (whose Fourier series are conjugate). We have stated Fatou's theorem, that Λ_α is stable under conjugacy. The same is true for λ_*, the Zygmund smooth functions, and can be seen in many different ways (see *Trigonometric series*, I, chap. 3 and 7).

We can define the conjugate of the series $\sum c_n e_n$ as $\sum c'_n e_n$ with $c'_n = -i c_n \operatorname{sign} n$ when $n \neq 0$ and $c'_0 = 0$. From the Riesz-Fischer theorem it follows that L^2 is stable under conjugacy, and that

$$\| \tilde{f} \|_2 \leq \| f \|_2$$

$\tilde{f}$ denoting the conjugate of f ; equality holds when $\int f = 0$.

The situation for L^p $(1 < p < \infty)$ is expressed by a theorem of Marcel Riesz: L^p is stable under conjugacy and

$$\| \tilde{f} \|_p \leq A_p \| f \|_p \, .$$

Here again there are several proofs. The original proof (reproduced in *Trigonometric Series*, I, chap. 7) uses H^p, the class of analytic functions $F(z)$ in the unit disc such that

$$\| F \|_p = \sup_{0<r<1} \left(\int_{\mathbb{T}} | F(re^{2\pi it}) |^p \, dt \right)^{1/p} < \infty.$$

H^p can be identified with the class of boundary values $f(t) = F(e^{it})$ which exist as non-tangential limits almost everywhere, according to an extension of Fatou's theorem. In this way H^p is identified with the subspace of L^p consisting of functions whose Fourier series is of the Taylor type. The quickest and most recent proof is due to S. Pichorides (1990).

Fatou's theorem extends to the Nevanlinna class N, which consists of analytic functions $F(z)$ in $| z |< 1$ such that

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+ | F(re^{2\pi it}) | \, dt < \infty \qquad ,$$

($\log^+$ is the positive part of log). Surprisingly, an analytic function $F(z)$ in $| z |< 1$ belongs to N if and only if it is the quotient of two bounded analytic functions. The extension of Fatou's theorem follows.

The conjugate of a Fourier-Lebesgue series is not necessarily a Fourier-Lebesgue series ; this can be seen with series $\sum c_n e_n$ where $c_n = c_{-n}$ and c_n $(n \geq 0)$ is a convex sequence which decreases slowly to zero. However, if f is a real integrable function, $f \sim \sum c_n e_n$, the analytic function

$$F(z) = \exp\left(2 \sum_1^\infty c_n z^n \right) \qquad (| z |< 1)$$

belongs to the Nevanlinna class, therefore $F(z)$ has nontangential limits almost everywhere at the boundary, therefore the same holds for $2 \sum_1^\infty c_n z^n$. The real part of the limit is the given f, the imaginary part defines $\tilde{f}$. In general, $\tilde{f}$ is not integrable. However the distribution of $\tilde{f}$ is controlled by an important inequality, due to A. Kolmogorov (1925) :

$$\lambda(\{t : | \tilde{f}(t) | > x\}) \leq C \| f \|_1 / x$$

for all $x > 0$, λ denoting the Lebesgue measure. This is expressed by saying that conjugacy maps L^1 into "weak L^1".

Starting from $L^1 \to$ weak L^1 and $L^2 \to L^2$ is it possible to recover $L^p \to L^p$ for $1 < p < 2$. Here the Riesz-Thorin interpolation theorem is inefficient. The good tool is a theorem of interpolation of J. Marcinkiewicz (1939). From $L^p \to L^p$ when $1 < p < 2$ the case $p > 2$ follows easily. We shall state and apply these interpolation theorems in chapter 11.

Conjugacy can be expressed in many different ways. We started from conjugate trigonometric series and conjugate harmonic functions. Starting from $f \in L^1(\mathrm{T})$ a purely real approach is the Hilbert transform

$$\tilde{f}(t) = -\lim_{\epsilon \to 0} \int_{\epsilon < |s| \leq \frac{1}{2}} \frac{f(t-s)}{\mathrm{tg}\ \pi s} ds$$

and a complex approach is the Cauchy transform

$$\frac{1}{2\pi i} \int \frac{f}{e^{2\pi i t} - z} dt$$

where either $|z| < 1$ or $|z| > 1$. The Hilbert transform is the prototype of singular integrals involved in Fourier analysis in euclidean spaces and in partial differential equations. The Cauchy transform can be considered for functions defined on certain curves in the plane and appears as a key question for deciding the $L^2 \to L^2$ character of a class of operators. Moreover, interpolation theory for operators developed enormously after Thorin and Marcinkiewicz. See for example the book of Stein and Weiss.

8. Functionals.

The idea of functionals (functions whose variables are functions) goes back to Hadamard, Pincherle, Volterra, and a general frame was provided by Maurice Fréchet in his thesis *"Sur quelques points du calcul fonctionnel"* (1906). Linear functionals on L^2 and L^p ($p \neq 2$) were characterized by M. Fréchet (1907) and F. Riesz (1909, 1910). The main general theorems on linear functionals came in the years 1924-1927 and they are associated with the names of Stephan Banach, Hans Hahn and Hugo Steinhaus. The book of Banach *Théorie des opérations linéaires* (1932) made them popular. Those of Zygmund (1935) and Kaczmarz and Steinhaus (1935), already mentioned in section 6, used linear functionals and linear operators widely. Though the idea of functionals came from many parts of analysis (integral equations and Hilbert methods in particular), let us emphasize the role of Lebesgue through Lebesgue's integral and trigonometric series.

The first words of Banach in his book are : *nous admettons que le lecteur connaît la théorie de la mesure et de l'intégrale de Lebesgue* (the reader is supposed to know the theory of Lebesgue's measure and integral). After a brief review of integration, inequalities, convergence, Banach quotes a theorem of Lebesgue in the following manner (we abbreviate the notations) : given a sequence (x_n) in L^1, a necessary and sufficient condition for $\lim \int \alpha x_n = 0$ whenever $\alpha \in L^\infty$ is that the conditions: i) $\int | x_n | = O(1)$; ii) $\lim \int_H x_n = 0$ uniformly with respect to n, whenever the measure of H tends to 0 ; iii) $\lim \int_I x_n = 0$ when $n \to \infty$ whatever the interval I, hold simultaneously. This is typical and Banach announces other theorems of the same kind later in the book.

Now, where is this theorem of Lebesgue coming from ? It comes from the study of what Lebesgue called *singular integrals*, of the form

$$I_n = \int f(t)\varphi(t-x,n)dt,$$

which occur in Fourier series when a process of summation is used (1909). The purpose of Lebesgue was to give necessary and sufficient conditions on the kernels $\varphi(x,n)$ for I_n to converge to f when f is in a given class of functions. The idea of weak convergence of a sequence of linear functionals was in germ in this particular question on Fourier series.

Actually Lebesgue already considered the linear functionals $f \to S_n(0)$ defined on continuous functions in order to construct continuous functions whose Fourier series diverges at 0, as we already explained in chapter 4. In modern terms, the norms of these functionals are the "Lebesgue constants" L_n and Lebesgue observed that they increase like $\log n$. According to Lebesgue (*Leçons...* p. 86) the existence of divergent Fourier series of continuous functions derives from the "remark" that $L_n \to \infty$. The Banach-Steinhaus theorem (1927) provides the derivation in a beautiful way.

The role of these linear forms was recognized widely, not only for trigonometric series, but also for general orthogonal series as we already mentioned in chapter 4. Given a complete orthonormal system (u_n) in L^2 $(n \in \mathbb{N})$, the "Lebesgue functions" are defined as

$$L_n(x) = \int \left| \sum_{m=0}^{n} u_m(x)\,\overline{u_m(t)} \right| dt$$

(we assume $\int dt = 1$). Estimates and interpretations are discussed in the book of Kaczmarz and Steinhaus (1935) (pp. 173-177). Here is the main result : if $L_n(x) = O(\lambda_n)$ $(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots)$ when x belongs to a given set E then all series $\sum a_n u_n(x)$ such that $\sum a_n^2 \lambda_n < \infty$ converge almost everywhere on E (Kaczmarz 1929). The case of the trigonometric system was treated previously by Kolmogorov and Seliverstov, and independently by Plessner, and it remained the best result on almost everywhere convergence of Fourier series until Carleson proved that $\sum a_n^2 < \infty$ suffices (1966).

It is natural to ask whether or not the Lebesgue constants can be decreased by a reordering of the exponentials e^{inx}. In other words, is

$$\int \mid e^{in_1x} + \cdots + e^{in_kx} \mid dx \geq \int \mid e^{ix} + e^{2ix} + \cdots + e^{ikx} \mid dx$$

whenever $n_1 < n_2 < \cdots < n_k$? This question is still unanswered. However we know that

$$\int \mid e^{in_1x} + \cdots + e^{in_kx} \mid dx > C \log k$$

for an absolute constant C. This is a theorem obtained by S. Konjagin on one part, and C. McGehee, L. Pigno and B. Smith on the other in 1980, and settles a long standing problem of Littlewood.

9. Approximation.

Approximation of functions by polynomials or trigonometric polynomials was already an important mathematical topic in the 19th century. The theory of best approximation was introduced by P. Tchebycheff (1821-1894) and some deep theorems on best approximation were obtained before any general theorem on approximation. The basic approximation theorem, which asserts that continuous functions on an interval can be approximated uniformly by polynomials or trigonometric polynomials, is due to Weierstrass (1885) and we have already mentioned it as an ancestor of the Fejér theorem. Lebesgue had another and simple proof (1898).

The order of best approximation of a periodic function by trigonometric polynomials, defined as

$$E_n(f) = \inf_P \sup_x \mid f(x) - P(x) \mid$$

for all trigonometric polynomials P of order $\leq n$, became an important topic in the 1910's. The first methods and results were provided by Charles de la Vallée Poussin (1908) and Lebesgue (1909, 1910). De la Vallée Poussin introduced the kernel that we mentioned in chapter 7. Lebesgue pointed out the relation between $E_n(f)$ and

$$R_n = \sup_x \mid f(x) - S_n(x) \mid$$

where $S_n(x)$ denotes the n^{th} partial sum of the Fourier series, that is

$$E_n(f) \leq \frac{R_n}{A \log n + B}$$

for absolute constants A and B (1909) and he established a very close connection between $E_n(f)$ and the modulus of continuity

$$\omega(\delta) = \sup_{|x-y|<\delta} \mid f(x) - f(y) \mid$$

namely

$$E_n(f) = O\left(\frac{1}{\log n}\right) \Leftrightarrow \omega(\delta) = O\left(\frac{1}{\log \frac{1}{\delta}}\right) \qquad (1910).$$

Then came the thesis of Dunham Jackson (1911), a fundamental memoir of Serge Bernstein (1912), and a series of contributions of Charles de la Vallée Poussin culminating in his book *Leçons sur l'approximation des fonctions d'une variable réelle* (1919), which contains references to all the works quoted above, and also a series of characterizations of spaces of functions through best approximation, for example

$$f \in \Lambda_\alpha \Leftrightarrow \omega(\delta) = O(\delta^\alpha) \Leftrightarrow E_n(f) = O(\frac{1}{n^\alpha}) \qquad (0 < \alpha < 1)$$

and the extensions of this result already mentioned in chapter 7.

Estimates for the best approximation had a number of offsprings. The abstract setting was provided by Fréchet (1906) : a compact set in a metric space can be covered by a finite number of balls of radius ϵ. The idea of using the minimal number of such balls, $N(\epsilon)$, in functional analysis, is due to A. Kolmogorov (1956) ; $\log N(\epsilon)$ is called the metric entropy of the compact set under consideration. It has important applications for proving negative results of the type : two given Banach spaces are not isomorphic, or functions of n variables in a given class cannot be obtained by superpositions of functions of less than n variables in another class. Statements and references can be found in the book of G. G. Lorentz, *Approximation of functions* (1966).

On the other hand, many important spaces of functions can be defined through approximation properties. Examples appear in the second part of this book.

Chapter 9

LACUNARITY AND RANDOMNESS

1. A brief history

We have seen that Riemann and Weierstrass had the idea of using lacunary trigonometric series in order to construct strange functions. Weierstrass's example of a continuous and nowhere differentiable function is

$$\sum_1^\infty a^n \cos b^n x$$

where b is an odd integer ≥ 3, and a a positive number such that $a < 1$ and $ab > 1+\frac{3\pi}{2}$ (1872). In 1892 Poincaré and Hadamard used the same idea in order to exhibit Taylor series whose circle of convergence is a natural boundary. Poincaré considered the series

$$\sum_1^\infty 2^{-n} z^{3^n} \qquad (|\, z \,| < 1)$$

and Hadamard a more general example, namely

$$\sum_1^\infty a_n z^{\lambda_n}$$

such that the radius of convergence is finite and $\neq 0$, and (λ_n) is an increasing sequence of integers such that

$$\frac{\lambda_{n+1}}{\lambda_n} > q > 1 \qquad (n = 1, 2, \cdots).$$

This is known as Hadamard's lacunarity condition. It plays quite an important role in many directions. However, if we want non continuation across the circle of convergence, the weaker condition

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 0$$

is sufficient (Fabry, 1896) and it is the best possible condition on the λ_n (Pólya, 1942).

Emile Borel had another idea. In 1896 and 1897 he published a note and a short paper with the purpose of proving that *in general* the circle of convergence of a Taylor series is a natural boundary. Actually he anticipated the notion of a random Taylor series and the notion of an almost sure event. The attempt at proof contains the first occurrence of what Borel named countable probabilities later. The first correct statement and proof were given only in 1929, by H. Steinhaus; his approach had a decisive effect on the study of random functions.

Lacunary trigonometric series appear in a natural way in connection with the Riemann-Lebesgue theorem. We know that the Fourier coefficients tend to zero. Is there a sequence $\rho_n \downarrow 0$ such that the Fourier coefficients of any integrable function are $O(\rho_n)$ $(n \to \infty)$? The negative answer is provided easily by the lacunary series

$$\sum \sqrt{\rho_{n_j}} e^{i n_j x}$$

where the n_j are chosen so as to insure absolute convergence. The question was raised by Lebesgue and treated by him, in 1910, in a slightly different manner.

In the 1920's three interrelated topics were considered in relation with convergence almost everywhere : Rademacher series (Rademacher 1922), lacunary trigonometric series (Kolmogorov 1924), series of independent random variables (Khintchine and Kolmogorov 1925). It seems that Kolmogorov was led to the last topic through the more sophisticated theory of lacunary trigonometric series. This theory developed later on with Sidon (1927), Banach (1930), Zygmund (1930 and 1932) and it so happened that most theorems on sums of independent random variables were stated first for lacunary trigonometric series. This is true in particular for the integrability properties of the partial sums (boundedness in L^p) and the summability almost everywhere, established by Zygmund for lacunary trigonometric series before being stated by Paley and Zygmund for Rademacher series. Part of the influence of the Lebesgue integration theory on probability theory went through lacunary trigonometric series.

Random trigonometric series arose in the series of papers of Paley and Zygmund entitled : *On some series of functions* (1930 and 1932). They considered also random Taylor series, their inspiration came from Rademacher and Steinhaus. In 1933, Norbert Wiener joined Paley and Zygmund in order to study Gaussian trigonometric series, motivated by the Wiener theory of Brownian motion. When Wiener wrote the definitive exposition of this theory (Paley-Wiener 1934) the starting point was the Steinhaus approach and the Brownian motion was introduced by its Fourier expansion.

This series of papers contained a lot of important results, but proved still more important through the methods which they introduced and the problems which they raised. The methods led to the books of Kahane (1968 and 1985), Marcus and Pisier (1981), Ledoux and Talagrand (1991) ; in particular they were one of the main sources of the current work on probabilities in Banach spaces. The main problems left open in 1934 were of two kinds : 1) give explicit conditions for a random trigonometric series to represent almost surely a continuous function ; 2) investigate the relations between Rademacher Fourier series, Steinhaus Fourier series, Gaussian Fourier series with the same coefficients, and the same for Taylor series. After the contributions of Salem and Zygmund (1954) and Billard (1963), a decisive step was taken by Marcus and Pisier (1978), and only a few questions on Taylor series remain unsolved.

Random trigonometric series have an intimate relation with lacunary trigonometric series : they enjoy the same kind of properties, they can be used for creating the same kind of objects, and randomness is also a tool in order to study lacunarity ; Sidon sets of integers, a generalisation of Hadamard sequences, is a field of application of probabilistic methods.

Lacunarity has other aspects. First, it appears in the context of orthogonal series, and the corresponding abstract setting was elaborated already in the book of Kaczmarz and Steinhaus (1935). Secondly, lacunarity has been linked with quasi-analyticity, as pointed out by S. Mandelbrojt (1935) and, in a different form, in the books of Paley and Wiener (1934) and N. Levinson (1940). We shall be more specific at the end of this chapter.

2. Rademacher, Steinhaus and Gaussian series.

Originally Rademacher functions were defined on the interval $[0,1]$. The first Rademacher function is

$$r_1(t) = \begin{cases} 1 & \text{when } 0 < t < \frac{1}{2} \\ -1 & \text{when } \frac{1}{2} < t < 1 \\ 0 & \text{when } t = 0, \frac{1}{2} \text{ or } 1 \end{cases}$$

and it is convenient to extend it by periodicity :

$$r_1(t+1) = r_1(t).$$

The n^{th} Rademacher function $(n = 2, 3, \cdots)$ is

$$r_n(t) = r_1(2^{n-1}t).$$

A Rademacher series is

$$\sum_1^\infty a_n r_n(t).$$

Now we can forget about the real variable aspect and consider the r_n as independent random variable with the same distribution, namely

$$P(r_n = 1) = P(r_n = -1) = \frac{1}{2},$$

where P is the probability (that is Lebesgue measure in the previous definition) on a probability space Ω ($[0,1]$ previously). From this viewpoint, the Rademacher series is nothing but the random series $\sum \pm a_n$ endowed with the natural probability.

Convergence can be considered in many ways : 1) almost sure convergence ; 2) convergence in a function space, usually $L^p(\Omega)$ $(0 \leq p \leq \infty)$; 3) convergence of the distribution laws.

Almost sure convergence means convergence outside a set of probability 0. Convergence in $L^\infty(\Omega)$ is almost sure uniform convergence. The $L^p(\Omega)$ $(1 \le p < \infty)$ are Banach spaces with norms

$$\| X \|_p = (E(| X |^p))^{1/p}$$

where $E(\ )$ denotes the expectation, that is, the integral on Ω with respect to P. The $L^p(\Omega)$ $(0 \le p < 1)$ are complete metric spaces with a distance $d(X,Y)$ defined by $d(X,Y) = E(| X - Y |^p)$ when $0 < p < 1$ and $d(X,Y) = E(| X - Y | \wedge 1)$ when $p = 0$ ($\wedge$ means infimum). $L^0(\Omega)$ is the space of all random variables and convergence in $L^0(\Omega)$ is called *convergence in probability.* The spaces $L^p(\Omega)$ decrease as p increases, and convergence in $L^p(\Omega)$ implies convergence in $L^q(\Omega)$ when $p > q$. Almost sure convergence implies convergence in probability. Therefore convergence in probability is the weakest convergence when we consider only 1) and 2).

The distribution law μ of a random variable X is the image of P through X, defined by

$$\int h(x)d\mu(x) = E(h(X))$$

for all continuous and bounded functions h. Its Fourier transform is

$$\hat{\mu}(u) = \int e^{iux} d\mu(x) = E(e^{iuX})$$

and is called the characteristic function of X, because it characterizes its distribution law μ. Given a sequence of random variables (X_n), with distribution laws μ_n, and a random variable X, with distribution μ, we say that the μ_n converge to μ if $\int h\, d\mu_n$ converges to $\int h\, d\mu$ for every bounded and continuous function h. And a necessary and sufficient condition is that $\hat{\mu}_n(u)$ converges to $\hat{\mu}(u)$ for all real u. In probability theory convergence of the distribution laws is named *convergence in law.* Convergence in probability implies convergence in law.

For Rademacher series $\sum_1^\infty a_n r_n$ convergence in $L^\infty(\Omega)$ means $\sum_1^\infty | a_n | < \infty$. *Almost sure convergence, convergence in* $L^p(\Omega)$ $(0 \le p < \infty)$, *convergence in law occur simultaneously, namely when* $\sum_1^\infty | a_n |^2 < \infty$.

In this theorem the most difficult part is to show that almost sure convergence holds when $(a_n) \in \ell^2$. This was settled in the 1920's (references can be found in Kaczmarz-Steinhaus). The easiest is convergence in $L^2(\Omega)$, since (r_n) is an orthonormal system. Convergence in $L^p(\Omega)$ $(p < \infty)$ is easy, because

$$E(e^{u\sum a_n r_n}) = \prod E(e^{u a_n r_n}) = \prod \operatorname{Ch} u a_n$$

$$\le \exp\left(\frac{1}{2}u^2 \sum a_n^2\right)$$

and

$$\frac{u^{2k}}{(2k)!} | \sum a_n r_n |^{2k} \le Ch(u \sum a_n r_n) \qquad (k = 1, 2, \cdots).$$

Therefore $\sum a_n r_n$ is a Cauchy sequence in $L^{2k}(\Omega)$. Convergence in law means that the infinite product

$$\prod_1^\infty \cos u\, a_n$$

converges to a characteristic function and that implies $(a_n) \in \ell^2$.

Let us add three remarks.

First, when $(a_n) \in \ell^2$, not only $S = \sum_1^\infty a_n r_n$ belongs to all $L^p(\Omega)$ $(p < \infty)$, but also, for all real u,

$$E(e^{uS}) \leq \exp(\frac{1}{2}u^2 \parallel S \parallel_2^2).$$

This implies, for all $\lambda > 0$,

$$P(\mid S \mid > \lambda) \leq 2\exp(-\frac{1}{2}\frac{\lambda^2}{\parallel S \parallel_2^2}),$$

therefore

$$E(e^{\mu |S|^2}) < \infty$$

when $\mu \parallel S \parallel_2^2 < \frac{1}{2}$, therefore (dropping if necessary a finite number of terms of the series) the same holds for all $\mu > 0$.

Secondly, when $(a_n) \notin \ell^2$, not only divergence has a positive probability, but divergence is almost sure. This is a consequence of the zero-one law of Kolmogorov: the probability of an event which depends on $(r_n)_{n\geq 1}$ only through the "tail" $(r_n)_{n\geq N}$, whatever N, is necessarily 0 or 1.

Thirdly, the theorem holds if we replace "convergence" by "summability" with respect to a given process of summation.

A Steinhaus sequence is defined as $(e^{2\pi i\omega_n})_{n\geq 1}$ where the ω_n are independent random variables on $[0,1]$ whose distribution law is the Lebesgue measure. A Steinhaus series is

$$\sum_1^\infty a_n e^{2\pi i\omega_n}.$$

All that we have said about Rademacher series holds for Steinhaus series.

A normal sequence is defined as $(\xi_n)_{n\geq 1}$ where the ξ_n are independent random variables with the same distribution, named normal standard, meaning

$$E(e^{iu\xi_n}) = e^{-\frac{1}{2}u^2} \qquad (u \in \mathbb{R}).$$

A Gaussian series is

$$\sum_1^\infty a_n \xi_n.$$

All that we said on Rademacher series holds for Gaussian series, except that

$$E(e^{\mu|S|^2}) < \infty$$

holds when μ is small enough and does not hold when μ is large.

3. Hadamard series, Riesz products and Sidon sets.

A Hadamard trigonometric series is

$$(H) \qquad \sum_{1}^{\infty} a_n \cos(\lambda_n x + \varphi_n)$$

where (λ_n) satisfies the Hadamard lacunarity condition

$$\frac{\lambda_{n+1}}{\lambda_n} > q > 1.$$

It enjoys a number of properties of Rademacher (or Steinhaus, or Gaussian) series. The proofs are slightly more difficult and often need the decomposition of the given series into a finite number of series (say ν) for which q is replaced by q^ν. In this way q can be supposed large enough ; usually the condition $q > 3$ is adequate.

Here are the main results.

If $(a_n) \notin \ell^2$, the series (H) diverges almost everywhere and is not a Fourier series.

If $(a_n) \in \ell^2$, (H) converges almost everywhere and its sum S satisfies

$$e^{\mu|S|^2} \in L^1$$

for all $\mu > 0$.

If $(a_n) \notin \ell^1$, (H) is not the Fourier series of a bounded function.

Let us prove the last statement when we assume $q > 3$. We introduce the Riesz product (named from F. Riesz 1918)

$$\prod_{1}^{\infty} (1 + \epsilon_n \cos(\lambda_n x + \varphi_n)) , \qquad \epsilon_n = \frac{a_n}{| a_n |}.$$

The partial products are positive and, due to the lacunarity condition, they can be written as

$$1 + \sum \epsilon_n \cos(\lambda_n x + \varphi_n) + R$$

where R is a trigonometric polynomial whose frequencies do not contain 0 nor any λ_n. Therefore the infinite product represents a positive measure μ whose expansion has the same form. If (H) is the Fourier series of a bounded function f, then

$$\int f \, d\mu = \frac{1}{2} \sum_{1}^{\infty} | a_n |$$

therefore $(a_n) \in \ell^1$, which is what we had to prove.

Since this is a theorem of Sidon, Sidon sets of positive integers are defined as the sets (λ_n) for which $(a_n) \notin \ell^1$ implies that (H) is not the Fourier series of a bounded function. It is slightly more elegant to consider Sidon sets on $\mathbb{Z}$. Let Λ be a subset of $\mathbb{Z}$. It is a Sidon set if

$$
\begin{aligned}
&\alpha) \quad C_\Lambda = A_\Lambda \\
&\beta) \quad L^\infty_\Lambda = A_\Lambda \\
&\gamma) \quad \ell^\infty(\Lambda) = \hat{M}(\Lambda) \\
&\delta) \quad c_0(\Lambda) = \hat{L}_1(\Lambda)
\end{aligned}
$$

and those conditions are equivalent. C_Λ and L^∞_Λ denote the spaces of continuous and bounded functions respectively with frequencies in Λ, and

$$A_\Lambda = \left\{ \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda x}, \ \sum_{\lambda\in\Lambda} | a_\lambda | < \infty \right\}.$$

$\ell^\infty(\Lambda)$, $c_0(\Lambda)$, $\hat{M}(\Lambda)$, $\widehat{L^1}(\Lambda)$ denote the spaces of restrictions to Λ of bounded sequences, sequences tending to zero at infinity, sequences of Fourier-Stieltjes coefficients, sequences of Fourier-Lebesgue coefficients respectively. The equivalence is an easy game on duality in Banach spaces, and this was done by Banach. In Kahane 1958, these sets were named Banach-Szidon. Sidon sets became a popular subject with the book of Walter Rudin, *Fourier analysis on groups* (1962).

Sidon trigonometric series

$$(S) \qquad \sum_{\lambda\in\Lambda} a_\lambda e^{i\lambda x} \qquad (\Lambda : \text{ Sidon set})$$

behave like Rademacher series as far as boundedness and absolute convergence are concerned. They enjoy another property of Rademacher series : if $(a_n) \in \ell^2$, the sum of the series belongs to all L^p and moreover it satisfies

$$\int \exp(\mu \mid S \mid^2) dx < \infty$$

for some $\mu > 0$. Writing L^ψ for the class of functions f such that

$$\int \exp(\mu \mid f \mid^2) dx < \infty$$

for some $\mu > 0$, any of the conditions α) to δ) implies

$$(\epsilon) \qquad L^\psi_\Lambda = L^2_\Lambda$$

where the subscript Λ is a way to denote the subspaces of functions with frequencies in Λ. This was discovered by W. Rudin in 1960 (see Rudin 1962).

A remarkable theorem of Gilles Pisier (1978) establishes that (ϵ) is another definition of Sidon sets. The proof relies heavily on random trigonometric series as we shall see in a moment.

4. Random trigonometric series.

Let us begin with Rademacher series in a Banach space :

$$\sum_{1}^{\infty} r_n u_n,$$

where (r_n) is a Rademacher sequence and (u_n) a sequence of vectors in a Banach space X. We are interested in almost sure convergence or almost sure summability in X and properties of the sum when it exists. In general there is no hope to have a simple necessary and sufficient condition on the u_n for a. s. convergence. However a general theory is available.

First, the zero-one law applies when we consider tail properties of such a random series. In particular, either a. s. convergence or a. s. divergence holds.

Secondly almost sure summability implies almost sure convergence. Here summability refers to any given process of summation. We are given an infinite matrix (a_{mn}) $(m, n \geq 1)$ with the only assumption

$$\forall n, \qquad \lim_{m \to \infty} a_{nm} = 1$$

and we say that the series $\sum_1^\infty v_n$ $(v_n \in X)$ is (a_{nm})-summable to s if each series $\sum_1^\infty a_{nm} v_n$ converges and

$$\lim_{m \to \infty} \sum_{1}^{\infty} a_{nm} v_n = s \qquad (\text{in } X).$$

Such summability processes appear when we deal with trigonometric series (Fejér, Poisson) or Taylor series (analytic continuation of a Taylor series $\sum_0^\infty a_n z^n$ across an arc of the circle of convergence to a point Z at the exterior of the circle of convergence can be considered as a summation process for the series $\sum_0^\infty a_n Z^n$). Now, assuming almost sure convergence, the sum of the series is a random vector S. The distribution of S has many interesting properties. The most important fact is that

$$E(\| S \|) < \infty ;$$

in other words, $\| S \| \in L^1(\Omega)$. Actually more can be said : $\| S \|$ belongs to all $L^p(\Omega)$ $(p < \infty)$ and, more precisely,

$$\| S \| \in L^{\psi}(\Omega).$$

The story of these estimates goes from Kahane 1964 to Kwapien 1976 (see Kahane 1985). A consequence is the so-called Khintchin inequality

$$(E(\| S \|^q))^{1/q} \leq K_{pq}(E(\| S \|^p))^{1/p}$$

when $0 < p < q < \infty$, and K_{pq} depends on p and q only (see Marcus-Pisier 1981). The estimate

$$K_{12} = \sqrt{2}$$

is quite recent (1994). Its very ingenious proof will be given in chapter 11.

Finally the integrability property of $\| S \|$ provides a contraction principle, namely that, given any bounded sequence (λ_n), almost sure convergence of $\sum_1^\infty r_n u_n$ implies almost sure convergence of $\sum_1^\infty r_n \lambda_n u_n$. There are several variations on this theme (Kahane 1968, Marcus-Pisier 1981, p. 45, Ledoux-Talagrand 1991, theorem 4.12). A simple application is that, for any complex Banach space X and any sequence (u_n), the probability of convergence is the same for the Rademacher series $\sum_1^\infty r_n u_n$ and the Steinhaus series

$$\sum_1^\infty e^{2\pi i \omega_n} u_n$$

and that a. s. convergence of the Gaussian series

$$\sum_1^\infty \xi_n u_n$$

implies a. s. convergence of the corresponding Rademacher and Steinhaus series.

Rademacher series in a Banach space play a basic role in the relation between geometry and probability in Banach spaces (Hoffmann-Jorgensen 1974 and 1977, Maurey and Pisier 1976, Garling 1977, Pisier 1989, Ledoux and Talagrand 1991, where a huge bibliography is given). The origin of this theory is the series of papers of Paley and Zygmund (1930-1932) and more precisely the study of random trigonometric series, to which we shall turn now.

Let us restrict ourselves to three types : the Rademacher, Steinhaus and Gaussian trigonometric series, defined as

$$(R) \qquad \Re e \sum_0^\infty c_n r_n(\omega) e^{int}$$

$$(S) \qquad \Re e \sum_0^\infty c_n e^{2\pi i \omega_n} e^{int}$$

$$(T) \qquad \sum_0^\infty | c_n | (\xi_n \cos nt + \xi_n' \sin nt)$$

where (ξ_n) and (ξ_n') are two independent normal sequences. For these random series almost sure convergence at a given t, almost sure convergence almost everywhere, almost sure convergence in one $L^p(\mathbf{T})$ $(1 \le p < \infty)$ almost sure convergence in all $L^p(\mathbf{T})$ $(1 \le p < \infty)$, almost sure convergence in $L^\psi(\mathbf{T})$, are equivalent to $(c_n) \in \ell^2$. Since a. s. convergence in $L^1(\mathbf{T})$ is the same as a. s. Cesàro summability in $L^1(\mathbf{T})$ and every Fourier-Lebesgue series is Cesàro summable in $L^1(\mathbf{T})$, (R) fails a. s. to be a Fourier-Lebesgue series when $\sum_1^\infty | c_n |^2 = \infty$; we have already announced this result in chapter 8, section 6.

What is the situation when we consider a. s. convergence in $C(\mathbf{T})$? For every of the series (R), (S), (T), a. s. convergence in $C(\mathbf{T})$ is the same as a. s. convergence everywhere. It is also the same as being a. s. the Fourier series of a continuous function. Necessary conditions and sufficient conditions appear in the works of Paley-Zygmund, Paley-Wiener-Zygmund, Salem-Zygmund, Billard. The simplest conditions involve the average energy in dyadic blocks of frequencies. Writing

$$s_j = \left(\sum_{2^j < n \leq 2^{j+1}} | c_n |^2 \right)^{1/2}$$

a. s. convergence in $C(\mathbf{T})$ implies $\sum s_j < \infty$, and $\sum s_j < \infty$ together with s_j decreasing implies a. s. convergence in $C(\mathbf{T})$ for each of the series (R), (S), (T).

The contraction principle implies that a. s. convergence in $C(\mathbf{T})$ is the same for (R) and (S) : this is due to Billard and was the origin of the principle.

A most remarkable fact is that a. s. convergence in $C(\mathbf{T})$ is the same for (R), (S) and (T). This is due to Marcus and Pisier and had far reaching consequences since it connected Rademacher trigonometric series to the better known Gaussian series (T), which represent all Gaussian and stationary processes on the circle $\mathbb{R}/2\pi\mathbb{Z}$. In particular necessary and sufficient conditions for a. s. convergence in $C(\mathbf{T})$ can be expressed through the correlation function $\sum | c_n |^2 \cos nt$ (see Marcus-Pisier 1981 or Kahane 1985).

5. Application of random methods to Sidon sets.

Until now we have given only one type of examples of Sidon sets on $\mathbb{Z}$, namely Hadamard sets. One can prove easily that finite unions of Hadamard sets are also Sidon sets and, using translations of Hadamard blocks, that there are other Sidon sets. Here is the widest class of Sidon sets known until now : it consists of finite unions of quasi-independent sets, Λ being called quasi-independent if there is no linear relation with coefficients ± 1 between non zero elements of Λ. The problem of whether this characterizes Sidon sets is open.

If Λ is a Sidon set, that is $C_\Lambda = A_\Lambda$, the norms in both spaces are equivalent, therefore

$$\| Q \|_A \leq K \| Q \|_C$$

for some constant $K = K(\Lambda)$ and all trigonometric polynomials Q with spectrum in Λ. That implies that Λ is lacunary in some sense. The best way to see it is to consider a random trigonometric polynomial

$$Q(t) = \sum_{\lambda \in \Lambda} \pm c_\lambda e^{i\lambda t}.$$

We may have an estimate of the form

$$P\left(\| Q \|_C \leq B(\sum | c_\lambda |^2)^{1/2} \right) > 0.$$

For example, according to Salem and Zygmund 1954, we can take $B = \sqrt{\log N}$, N being the degree of Q. Let us choose $c_\lambda = 0$ or 1 so that

$$\sum | c_\lambda | = \sum | c_\lambda |^2 = \nu,$$

the number of terms in Q. Then a convenient choice of $\pm$ gives

$$\nu \leq K\, B\, \nu^{1/2}.$$

Using the estimate of Salem and Zygmund we see that the number of points of Λ in $[-N, N]$ cannot exceed $K^2 \log N$. Using an estimate of the same kind for random trigonometric polynomials in several variables we have a more precise condition : if Λ is a Sidon set, there exists a constant K' such that Λ contains at most $[K' s \log \nu]$ elements of the form

$$a + n_1\rho_1 + n_2\rho_2 + \cdots + n_s\rho_s$$

whenever $a, \rho_1, \cdots, \rho_\nu$ are real numbers and $n_1, n_2, \cdots, n_s$ are integers such that $| n_1 | + | n_2 | + \cdots + | n_s | \leq \nu$. No better necessary condition is known and the question whether this characterizes Sidon sets is open (Kahane 1958).

The first breakthrough in the theory of Sidon sets was made by S. Drury in 1970, when he proved that a finite union of Sidon sets is a Sidon set ; his method was to introduce $\Omega = \mathbf{T}^{\mathbb{Z}}$ and use harmonic analysis on Ω as well as $\mathbf{T}$.

The next step was taken by D. Rider 1975. Rider gave a new characterization of Sidon sets, which can be expressed as

$$(\varsigma) \qquad C_{a.s.\ \Lambda} = A_\Lambda.$$

Here $C_{a.s.}$ is the space of functions

$$f \sim \sum \hat{f}_n e^{int}$$

such that, for almost all changes of signs,

$$\sum \pm \hat{f}_n e^{int}$$

represents a continuous function (if this is true for all changes of sign, then $f \in A$). Actually, Rider considered Steinhaus and not Rademacher series, but we know that this is the same.

Finally, using (ς), G. Pisier proved in 1978 that

$$(\epsilon) \qquad L^\psi_\Lambda = L^2_\Lambda,$$

the Rudin condition, is also a characterization of Sidon sets (see Marcus-Pisier 1981). Both (ς) and (ϵ) prove that the class of Sidon sets is stable under finite union, the Drury theorem.

New characterizations of Sidon sets were given by Pisier (1983) and Bourgain (1985). Bourgain proved the implications

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)$$

where

(1) : Λ is a Sidon set, that is (α)

(2) : Λ has the Rudin property, that is (ϵ)

(3) : Λ has the Pisier property, meaning that there exists a $\delta > 0$ such that each finite subset A of Λ contains a quasi independent subset B such that $\mid B \mid > \delta \mid A \mid$ ($\mid \; \mid$ is the cardinal)

(4) : Λ has the Bourgain property, meaning that there exists a $\delta > 0$ such that, given $(a_\lambda)_{\lambda \in \Lambda}$ vanishing outside a finite set, there exists a quasi independent $A \subset \Lambda$ such that

$$\sum_{\lambda \in A} \mid a_\lambda \mid \geq \delta \sum_{\lambda \in \Lambda} \mid a_\lambda \mid .$$

Pisier had already proved (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). (1) $\Rightarrow$ (2) was Rudin's theorem. Bourgain's proof uses random sets of integers in order to get (2) $\Rightarrow$ (3) and a clever argument for (3) $\Rightarrow$ (4). The last implication (4) $\Rightarrow$ (1) is easy through Riesz products.

The subject of Sidon sets can be viewed also as a Banach space property of C_Λ. The definition $C_\Lambda = A_\Lambda$ implies that C_Λ is isomorphic to the space ℓ^1. Conversely, if C_Λ is isomorphic to ℓ^1, Λ is a Sidon set (Varopoulos 1976). There are other characterizations, involving the notion of finite cotype, in Bourgain-Milman 1985. Finite cotype comes from Rademacher series in a Banach space X : X has cotype q ($q \geq 2$) if

$$\left(\sum \| u_j \|^q \right)^{1/q} \leq C E \| \sum r_j u_j \|$$

for some C and all finite sequences (u_j) in X. It is known that ℓ^1 has cotype 2, and Bourgain-Milman's theorem is that Λ is a Sidon set if and only if C_Λ has a finite cotype.

Sidon sets are still a mine of open problems and they are the closest link between Fourier series and the rapidly moving field of Probability and Banach spaces. This was the reason to spend some time on them.

6. Lacunary orthogonal series. $\Lambda(s)$ sets.

In the book of Kaczmarz and Steinhaus (1935) one chapter is devoted to lacunary orthogonal series. They observe that the Rademacher system (r_n) has the property that

$$\left(\int \mid \sum a_k r_k \mid^p \right)^{1/p} \leq \left(\frac{1}{p} + 1 \right)^{1/2} \left(\sum a_k^2 \right)^{1/2}$$

for all $p > 0$ and finite sequences a_k. These are the original Khintchin inequalities and they can be derived from the estimate

$$\int e^{u\sum a_k r_k} \leq e^{\frac{1}{2}u^2\sum a_k^2}$$

which we have already encountered. They define an orthonormal system $\{\varphi_n\}$ as lacunary of order p ($p > 2$) when each φ_n belongs to L^p and when the norms of linear combinations of the φ_n in L^2 and L^p are equivalent. Duality between L^p and $L^{p'}$ when $\frac{1}{p} + \frac{1}{p'} = 1$ provides an equivalent definition, involving the $L^{p'}$-norm of a function f and the ℓ^2-norm of the sequence of coefficients $\int f\bar{\varphi}_n$. Going back to the system $\{e^{int}\}_{n\in\mathbb{Z}}$ and to a subset Λ of $\mathbb{Z}$, Rudin (1960) defined Λ to be a $\Lambda(s)$ set ($s > 1$) if and only if the norms

$$\int |P|, \qquad (\int |P|^s)^{1/s}$$

are equivalent for all finite sums $P(t) = \sum_{\lambda\in\Lambda} a(\lambda)e^{i\lambda t}$. Sidon sets are $\Lambda(s)$ sets for all s. The most interesting class is $\Lambda(2)$. Already Sidon (1932) had proved that, if the number of solutions of $\lambda_1 \pm \lambda_2 = n$ ($n \in \mathbb{Z}\backslash\{0\}$ given, $\lambda_1 \in \Lambda$, $\lambda_2 \in \Lambda$) is bounded uniformly with respect to n, Λ is a $\Lambda(2)$ set ; actually it is a $\Lambda(4)$ set. For a long time there seemed to be no way to obtain a $\Lambda(2)$ set except through $\Lambda(4)$. Finally Bourgain (1988) proved that the classes $\Lambda(s)$ are strictly increasing, through a clever combination of probabilistic arguments and the geometry of Banach spaces. Therefore some $\Lambda(2)$ sets are not $\Lambda(4)$. A challenge is $\Lambda = \{n^2\}_{n\in\mathbb{N}}$; it is not $\Lambda(4)$ and it is still unknown whether it is $\Lambda(2)$ or not.

7. Local and global properties of random trigonometric series.

Let us consider a Rademacher trigonometric series

$$(R) \qquad \sum_{-\infty}^{\infty} c_n r_n(\omega) e^{int}$$

together with a property like continuity, analyticity, square-integrability. Roughly speaking, if (R) represents a function which enjoys such a property somewhere, it enjoys the same property everywhere. Here are the precise statement and proof.

We denote by $\mathcal{P}$ a property of periodic functions and by $f \in \mathcal{P}(I)$ the fact that f enjoys this property on an interval I. If the length of I exceeds the period of f, this means that f enjoys the property everywhere and we write $f \in \mathcal{P}$. Then $\mathcal{P}(I)$ and $\mathcal{P}$ can be considered as classes of functions. We assume that 1) $\mathcal{P}(I)$ is a linear space and contains all trigonometric polynomials ; 2) $\mathcal{P}(I)$ is invariant under multiplication by e^{it} ; 3) $\mathcal{P}(I_1) \cap \mathcal{P}(I_2)$ contains $\mathcal{P}(I)$ wherever $I \subset I_1 \cup I_2$. These definitions extend to formal trigonometric series. Then, assuming $(R) \in \mathcal{P}(I(\omega))$ for some random interval $I(\omega)$ when ω belongs to a set of positive probability, $(R) \in \mathcal{P}$ a. s.

Let us sketch the proof. The assumption holds when $I(\omega)$ is restricted to rational intervals, then when $I(\omega)$ is one rational interval, I. Using the zero-one law, $(R) \in \mathcal{P}(I)$ a. s.. First suppose $|I| > \pi$ and consider

$$(R') \qquad \sum_{-\infty}^{\infty} (-1)^n c_n r_n(\omega) e^{int}.$$

Since $\{r_n(\omega)\}$ and $\{(-1)^n r_n(\omega)\}$ have the same distribution, $(R') \in \mathcal{P}(I)$ a. s., therefore $(R) + (R') \in \mathcal{P}(I)$ and $(R) - (R') \in \mathcal{P}(I)$ a. s.. The last series being either π-periodic or π-periodic after multiplication by e^{it}, we have $(R) + (R') \in \mathcal{P}$, $(R) - (R') \in \mathcal{P}$, therefore $(R) \in \mathcal{P}$ a. s.. If only $|I| > \frac{2\pi}{N}$ we have to consider the N series

$$N \sum_{n \equiv k(N)} c_n r_n(\omega) e^{int}$$

instead of $(R) + (R')$ and $(R) - (R')$ and the same conclusion holds.

The proof and conclusion extend if we replace (R) by any random series

$$(Z) \qquad \sum_{-\infty}^{\infty} Z_n e^{int}$$

where the Z_n are independent symmetric random variables. As a corollary, let us consider a random Taylor series

$$\sum_{0}^{\infty} Z_n z^n$$

whose coefficients Z_n are chosen in this way (say, *à la Borel*). According to the zero-one law, the radius of convergence, ρ, is constant a. s. ; let us assume $0 < \rho < \infty$. Then, as Borel said, the circle of convergence is a natural boundary a. s. ; for, were it to contain regular points with a positive probability, it would consist entirely of regular points a. s., which is impossible.

We now turn to a Steinhaus trigonometric series

$$(S) \qquad \sum_{-\infty}^{\infty} c_n e^{2\pi i \omega_n} e^{int}$$

and assume $(c_n) \in \ell^2$, so that (S) is the Fourier series of a square integrable random function $F(t)$. Now, the distribution of $(F(t))_{t \in \mathbb{R}}$ and the distribution of any of its translates are the same. Thus the behaviour of F in the neighborhood of any given point t_0 controls its global behaviour. For example, one can prove that, given $0 < \alpha < 1$, if

$$F(t) - F(t_0) = O(|t - t_0|^{\alpha}) \qquad (t \to t_0)$$

with a positive probability, then $F \in \Lambda_{\alpha'}$, for all $\alpha' < \alpha$ a. s.; i.e.

$$\sup_{t,t'} \frac{|F(t) - F(t')|}{|t - t'|^{\alpha'}} < \infty \text{ a.s.}$$

(see Kahane's book 1985, exerc. 4, p. 108 and theorem 2, p. 86). The same result holds for Gaussian trigonometric series. It is not known for series (R).

8. Local and global properties of lacunary trigonometric series.

Given $\Lambda \subset \mathbb{Z}$ let us consider L^1_Λ, the space of integrable functions f with spectrum in Λ, i.e.

$$f \sim \sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda t}.$$

We shall consider two problems raised by S. Mandelbrojt in 1935. First, assume that f has a property $\mathcal{P}$ on an arbitrary small interval ; does it imply that f has the same property everywhere ? Second, assume that f is small in the neighborhood of a point; does this imply that f vanishes everywhere ?

The first problem can be considered from a slightly more general viewpoint, introduced by Paley and Wiener already in 1934, and developed by Levinson in 1940: the "gap and density" viewpoint. Given Λ and $\mathcal{P}$, we define D $(= D(\mathcal{P}, \Lambda))$ as the $\mathcal{P}$-density of Λ if $2\pi D$ is the infimum of the lengths of the intervals I such that, whenever $f \in L^1_\Lambda$ enjoys $\mathcal{P}$ on I, it enjoys $\mathcal{P}$ everywhere (strictly speaking : a version of f enjoys $\mathcal{P}$ everywhere). Let us quote the main results in the chronological order of their first appearance.

1. $\mathcal{P}$=analyticity. The theorems of Fabry and Pólya already mentioned at the beginning of this chapter can be extended easily in this context :D is zero if and only if the ordinary density of Λ is zero. The ordinary density is

$$\lim_{t \to \pm\infty} \frac{n(t)}{t}$$

where $n(t)$ is the counting function of Λ, that is

$$n(t) = \mathrm{Card}(\Lambda \cap [0, t]) \qquad t > 0$$

$$n(t) = -\mathrm{Card}(\Lambda \cap [t, 0]) \qquad t < 0,$$

whenever the limit exists. In general there is no explicit formula known for D. According to a theorem of Pólya,

$$D \leq \text{maximal density of } \Lambda,$$

meaning the infimum of the densities of the sets Λ' which contain Λ and have a density. In other words,

$$\text{maximal density} = \text{outer ordinary density}.$$

2. $\mathcal{P}$ =square integrability. Here the question goes back to Paley and Wiener, and $2\pi D$ was named the pseudoperiod - the notion extends to subsets of $\mathbb{R}$ readily. Now

$$D = \text{outer uniform density}.$$

Outer has the same meaning as above, but uniform density is stronger than ordinary density : "Λ has uniform density Δ" means

$$n(t) - \Delta t = O(1) \qquad (t \to \pm\infty)$$

while "Λ has ordinary density Δ" means

$$n(t) - \Delta t = o(t) \qquad (t \to \pm\infty).$$

The same formula holds when $\mathcal{P}$ expresses any good local property of the Fourier transform (see f. e. Kahane's book 1970).

3. $\mathcal{P}$ =vanishing. This is the uniqueness problem : $2\pi D$ is the supremum of the lengths of the intervals where a function with spectrum in Λ can vanish without being identically zero. In other words, $2\pi(1-D)$ is the infimum of the lengths of the intervals J which carry a function $\not\equiv 0$ orthogonal to the e^{int} $(n \in \mathbb{Z}\backslash\Lambda)$ in $L^2(J)$. It is therefore related to the problem of totality of a sequence (e^{int}) in $L^2(J)$, also considered by Paley and Wiener, then by Levinson, and solved by A. Beurling and P. Malliavin (1967). The result of Beurling and Malliavin is very deep and involves a density intermediate between uniform and ordinary. Let us say that Λ has B.-M. density Δ if

$$\int_{\mathbb{R}} \frac{| n(t) - \Delta t |}{1+t^2} dt < \infty.$$

Then the totality problem uses the outer B.-M. density, and consequently the uniqueness problem involves the inner B.-M. density (supremum of the B.-M. densities of the sets contained in Λ and having a B.-M. Density) :

$$D = \text{inner B.} - \text{M. density}.$$

4. $\mathcal{P}$ =continuity. The problem was studied by Yves Meyer (1973). The main general result is

$$\text{outer uniform density} \leq D \leq \liminf_{m\to\infty} \frac{q_m}{m},$$

q_m being the number of distinct residues of elements of Λ modulo m. Let us observe that the right member is the infimum of the densities of the periodic sets which contain Λ ; it can be named the outer strict density, if we define strict density Δ by the condition

$$n(t) - \Delta(t) \text{ periodic}.$$

Here is an example where more can be said (theorem 6 in Yves Meyer). Suppose that, for any $\lambda \in \Lambda$ and $m \in \mathbb{N}^+$ there exists a sequence $(\lambda_k)_{k\geq 1}$ in Λ such that : α) $\lambda_k \equiv \lambda$ (mod. m) for all k ; β) the sequence $(\alpha\lambda_k)_{k\geq 1}$ is equidistributed modulo 1 whatever α irrational. Then

$$D = \text{outer strict density}.$$

9. Local and global properties of Hadamard trigonometric series.

Now we suppose that Λ is symmetric and that its positive part satisfies the Hadamard lacunarity condition. Then all densities considered above vanish.

Here are a few easy results, obtained by transfering an information on a local behaviour into an estimate on Fourier coefficients. We always assume $f \in C_\Lambda$.

1. If, given t_0 and $0 < \alpha < 1$,

$$f(t) - f(t_0) = O(| t - t_0 |^\alpha) \qquad (t \to t_0),$$

then $f \in \Lambda_\alpha$ (Hölder class of order α).

2. If, given t_0, $n \in \mathbb{N}^+$ and $0 < \alpha < 1$,

$$f(t) = P_n(t, t_0) + O(| t - t_0 |^{n+\alpha}) \qquad (t \to t_0)$$

where $P_n(\cdot, t_0)$ is a polynomial of degree $\leq n$, then f is n times differentiable, and

$$f(t) = \sum_{m=0}^{n} \frac{(t-s)^m}{m!} f^{(m)}(s) + O(| t - s |^{n+\alpha})$$

uniformly with respect to s and t (in short, $f \in \Lambda_{n+\alpha}$).

These statements are the only ones that we know where the Hadamard lacunary condition is both sufficient and necessary.

Let us turn to the second problem of Mandelbrojt. The solution involves a special function $f_0 \in C_\Lambda$, whose Laplace transform is the reciprocal of a canonical product associated with Λ :

$$\int_0^\infty f_0(t) e^{-wt} dt = \begin{cases} \left(\prod_{\lambda \in \Lambda, \lambda > 0} (1 - w^2 \lambda^{-2}) \right)^{-1} & \text{if } 0 \notin \Lambda \\ \left(w \prod_{\lambda \in \Lambda, \lambda > 0} (1 - w^2 \lambda^{-2}) \right)^{-1} & \text{if } 0 \in \Lambda \end{cases}$$

Then f_0 is the smallest function of L^1_Λ near O in the following sense : if $f \in L^1_\Lambda$ and

$$\liminf_{\alpha \to 0} \frac{\int_0^\alpha | f |}{\int_0^\alpha | f_0 |} < \infty,$$

then f is a multiple of f_0 ; if $\underline{\lim} = 0$, then $f = 0$. For a proof, see M. Izumi, S.I. Izumi, J.-P. Kahane (1965).

Another type of uniqueness was considered by Zygmund (1948). Here is Zygmund's theorem : if $f \in L^1_\Lambda$ vanishes on a set of positive Lebesgue measure, then $f = 0$ a.e.. It is possible to replace sets of positive measure by Cantor sets of a special type. See references in Kahane 1964. However the statement fails for Sidon sets, as Myriam Déchamps pointed out in 1972, there is the substitute which she proved : when Λ is a Sidon set and $f \in L^1_\Lambda$ vanishes on an interval, then $f = 0$ a.e..

Chapter 10

ALGEBRAIC STRUCTURES

1. An inheritance from Norbert Wiener.

Translations of functions and convolutions were used for a long time before they were defined. The leading role in the movement which put these notions at the very basis of harmonic analysis was that of Norbert Wiener, whose centenary is celebrated in 1994. His article on Tauberian theorems (1932) and his book on the Fourier integral (1933) can be considered as a foundation for algebraic structures in Fourier series.

His main purpose was a comparison of different processes of summation of series, with application to the prime number theorem. The comparison between $\sum_1^\infty a_n$ and $\lim_{x \uparrow 1} \sum_1^\infty a_n x^n$ was initiated by Abel in one direction and Tauber in the other ; hence the name of Tauberian theorems. Wiener discovered a new and general approach, through expressions of the form

$$\lim_{x \to \infty} \int_{\mathbb{R}} K(x-y)H(y)dy$$

where H is a bounded function under investigation, and K an integrable function used as a tool in order to provide H with a kind of limit. Translations of functions and translation-invariant spaces appear in this connection. Wiener studied the closed translation-invariant subspaces of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ generated by a function (what we call its span now) and gave necessary and sufficient conditions in order to obtain the whole space. A basic tool is convolution, for which he used the German term of *Faltung*. Fourier transformation exchanges convolutions and ordinary multiplication. Writing $\mathcal{F}$ for the Fourier transformation, $L^1(\mathbb{R})$ has the same structure under convolution as $A(\mathbb{R}) = \mathcal{F}L^1(\mathbb{R})$ under multiplication : what we call now Banach algebra.

The same holds for $\ell^1(\mathbb{Z})$ and $A(\mathbf{T}) = \mathcal{F}\ell^1(\mathbb{Z})$. Locally $A(\mathbf{T})$ and $A(\mathbb{R})$ look the same, and $A(\mathbf{T})$ is somewhat more manageable because it contains a unit. The fact that the inverse of a non-vanishing function in $A(\mathbf{T})$ belongs also to $A(\mathbf{T})$ has a key role in Tauberian theorems.

The first follower of Wiener was Paul Lévy (1934). The theory of Banach algebras (then called normed rings) by I. Gelfand came a few years later (1939). At the same time André Weil gave locally compact abelian groups as a general frame for commutative harmonic analysis (1940). In particular Fourier series were defined on all compact abelian groups.

Banach algebras on one hand, structure of translation-invariant spaces on the other, became a classical way of exposition, as well as the general framework for many specific questions. It is interesting to compare the books of Loomis (1953), Rudin (1962), Edwards (1967), Katznelson (1968), Kahane (1970), Graham and McGehee

(1979), Helson (1983), Kõrner (1988 and 1993) and to observe the shift of interest from purely algebraic to more historically rooted subjects.

In this chapter we shall concentrate on some questions motivated by Wiener's work and stated by P. Lévy (1934), L. Schwartz (1948), W. Rudin (1962). We shall define and give examples of compact abelian groups, state and comment on the Wiener-Lévy theorem and its converse, consider two problems in spectral synthesis with a negative solution, and conclude with homomorphisms of Wiener algebras. We give some proofs (for example for the last question) and sketch some other.

2. Compact abelian groups.

Let G be a compact abelian group. Let us denote by γ any character of G, that is, a continuous homomorphism of G into the unit circle of the complex plane, considered as a group under multiplication. The characters constitute a multiplicative abelian group, Γ, called the dual of G. The Wiener algebra of G, denoted by $A(G)$, consists of all functions

$$f = \sum_{\gamma\in\Gamma} a_\gamma \gamma, \qquad \sum | a_\gamma | < \infty.$$

THEOREM. - *If $f = 0$, all a_γ are zero.*

The classical proof uses the Haar measure on G and Fourier formulas. Here is an elementary proof. Starting from

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\le N} z^n = \begin{cases} 0 & \text{if } \ |z|=1,\ z\neq 1, \\ 1 & \text{if } \ z=1 \end{cases}$$

we have, given $g_1, g_2, \cdots g_\nu$ in G,

$$(*) \qquad \lim_{N\to\infty} \frac{1}{(2N+1)^\nu} \sum_{|n_j|\le N, j=1,2,\cdots\nu} f(n_1 g_1 + n_2 g_2 + \cdots + n_\nu g_\nu) =$$

$$\sum_{\gamma | \gamma(g_1)=\gamma(g_2)=\cdots=\gamma(g_\nu)=1} a_\gamma .$$

If a_1, the constant term, is not zero, we choose $\gamma_1, \gamma_2, \cdots \gamma_\nu$ different from 1 so that

$$| a_1 | > \sum_{\gamma\notin\{1,\gamma_1,\gamma_2,\cdots\gamma_\nu\}} | a_\gamma |,$$

then g_j so that $\gamma_j(g_j) \neq 1$ $(j = 1, 2, \cdots \nu)$. Using $(*)$ we see that $a_1 \neq 0$ implies that f is not zero. Multiplying by $\bar\gamma$, $a_\gamma \neq 0$ implies that f is not zero. That ends the proof.

As a corollary $A(G)$ is a Banach space isomorphic to $\ell^1(\Gamma)$, when endowed with the norm

$$\| f \|_A = \sum | a_\gamma | .$$

The norm is submultiplicative :

$$\| fg \|_A \leq \| f \|_A \| g \|_A$$

and the constant 1 has norm 1. This is the structure of a Banach algebra with unit.

THEOREM. - *$A(G)$ is dense in $C(G)$, the space of continuous functions on the compact group G.*

This is a simple application of the Stone approximation theorem : G being compact, any subalgebra of $C(G)$ which distinguishes points in G is dense in $C(G)$.

EXAMPLES.

1. $G = \mathrm{T} = \mathbb{R}/2\pi\mathbb{Z}$, the additive circle group. The characters are the functions $e_n(t) = \exp(2\pi int)$ $(n \in \mathbb{Z})$ and Γ is isomorphic to the additive group $\mathbb{Z}$.

2. $G = \mathrm{T}^\nu$, the ν-dimensional torus. The characters are the functions $e_n(t) = \exp(2\pi in{\cdot}t)$ $(n \in \mathbb{Z}^\nu)$, where $t = (t_1, \cdots, t_\nu)$, $n = (n_1, \cdots, n_\nu)$, $n{\cdot}t = n_1t_1 + \cdots + n_\nu t_\nu$, and Γ is isomorphic to the additive group $\mathbb{Z}^\nu$.

3. $G = \mathbb{B}$, the Bohr group, compact closure of the real line $\mathbb{R}$ endowed with the smallest topology for which all functions $e_\lambda(t) = \exp(2\pi i\lambda t)$ $(\lambda \in \mathbb{R})$ are continuous. The characters are the e_λ extended to $\mathbb{B}$, and Γ is isomorphic to the additive group $\mathbb{R}$ (here we only refer to the algebraic structure of $\mathbb{R}$). The functions in $C(\mathbb{B})$ are well defined by their restrictions to $\mathbb{R}$, called almost periodic functions, and defined as uniform limits of finite sums $\sum a_\lambda e_\lambda$ (a_λ complex). The functions in $A(\mathbb{B})$ correspond to the absolutely convergent series

$$\sum_{\lambda \in \mathbb{R}} a_\lambda e_\lambda, \qquad \sum | a_\lambda | < \infty.$$

4. $G = \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ $(d = 2, 3, 4, \cdots)$, cyclic group of order d. Here the characters take values among the d-th roots of unity and are defined as $\gamma(g) = \exp\left(\frac{2\pi i}{d} \underline{\gamma}\, g\right)$, where $\underline{\gamma} \in \mathbb{Z}_d$. Therefore Γ is isomorphic to the additive group $\mathbb{Z}_d$.

5. $G = \mathbb{D} = \mathbb{Z}_2^{\mathbb{N}}$, the dyadic group homeomorphic to a Cantor set, sometimes called the Cantor group. The characters are

$$\gamma(g) = \exp(\pi i \,\underline{\gamma} \cdot g)$$

where $g = (g_j)_{j \in \mathbb{N}}$ with $g_j \in \mathbb{Z}_2$, $\underline{\gamma} = (\underline{\gamma}_j)_{n \in \mathbb{N}}$ where $\underline{\gamma}_j \in \mathbb{Z}_2$ and only a finite number of $\underline{\gamma}_j$ are $\neq 0$, and

$$\underline{\gamma} \cdot g = \sum_{j \in \mathbb{N}} g_j \underline{\gamma}_j.$$

The $\underline{\gamma}$ constitute an additive group $\underline{\Gamma}$ isomorphic to Γ. Here we have a specific language and notations. The functions

$$r_j = \exp(\pi i g_j)$$

are called Rademacher functions, the $\underline{\gamma}$ can be identified with finite sequences of 0 and 1, so that

$$\gamma = \prod_{j \in S} r_j$$

S being the support of $\underline{\gamma}$, and the characters can be indexed by the integers

$$n = \sum_{j \in S} 2^j$$

which run over $\mathbb{N}$ as S runs over all finite subsets of $\mathbb{N}$. Then we write

$$\gamma = w_n$$

and call w_n the n-th Walsh function. In this way $w_0 = 1, w_1 = r_0, w_2 = r_1, w_3 = r_1 r_2$, $w_4 = r_3$, etc. We can represent $A(G)$ as the algebra of absolutely convergent Walsh series

$$\sum_{n \in \mathbf{N}} a_n w_n, \qquad \sum | a_n |< \infty.$$

When $\mathbb{D}$ is endowed with the Haar measure, and therefore becomes a probability space, the r_j are independent symmetric random variables with values ± 1, that is, a Rademacher sequence in the sense of Chapter 9.

3. The Wiener-Lévy theorem.

THEOREM (Wiener-Lévy). - *Let $F(z)$ be an analytic function of z in a domain D_F of the complex plane, and $f \in A(G)$ with range in D_F. Then $F \circ f \in A(G)$.*

Let us give a proof when $G = \mathbf{T}$ (the classical case) and sketch it in the general case. Let $\rho > 0$, smaller than the distance from the range of f to the boundary of D_F, and s a trigonometric polynomial (finite sum $\sum a_n e_n$) such that $\| f - s \|_A \leq \epsilon = \frac{1}{2}\rho$; we write $f = s + r$. The functions

$$G_\omega(t) = F(s(t) + \epsilon e^{2\pi i \omega}) \qquad (\omega \in \mathbf{T})$$

are analytic on $\mathbf{T}$ and their derivatives of order $\leq m$ (given) are bounded uniformly. Therefore $G_\omega \in A(\mathbf{T})$ and the norms $\| G_\omega \|_A$ are bounded uniformly by some $B > 0$. For each t and ω

$$G_\omega(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(s(t))}{n!} \epsilon^n e^{2\pi i n \omega}.$$

Integrating $G_\omega(t)e^{-2\pi i n\omega}$ with respect to ω and taking norms in $A(\mathrm{T})$ we have

$$\left\| \frac{F^{(n)} \circ s}{n!} \epsilon^n \right\|_A \leq B.$$

Therefore the series

$$\sum_{n=0}^{\infty} \frac{F^{(n)} \circ s}{n!} r^n$$

converges in $A(\mathbf{T})$, and its sum is $F \circ f$, therefore $F \circ f \in A(G)$.

In the general case we choose $s = \sum a_\gamma \gamma$ in the same way. The γ's which appear in the sum generate a subgroup of Γ, dual of a quotient of some T^ν, and this quotient is either finite or an analytic manifold ; it is used instead of $\mathbf{T}$ in the remainder of the proof.

This proof is due to A. P. Calderón and appeared in the second edition of Zygmund's book in 1959 (vol. I, p. 246 ; see note p. 380). A new version was given by T. Coquand and G. Stolzenberg in 1991. The original proof of Wiener, who restricted himself to the case $F(z) = z^{-1}$, used a different pattern ; it relies on a fact of independent interest, namely, that a continuous function on G belongs to $A(G)$ if it coincides with some function of $A(G)$ in the neighbourhood of each point of G ; in other words, belonging to $A(G)$ is a local property. Wiener's proof is given in the first edition of Zygmund's book (1935) p. 140 and in Kahane's book (1970) p. 57. A comparison of both proofs can be found in Coquand-Stolzenberg 1991.

The original theorem of Wiener stated that $\frac{1}{f} \in A(G)$ whenever $f \in A(G)$ and f does not vanish anywhere. It was needed as a lemma in order to obtain Tauberian theorems, and Tauberian theorems were a key for a new proof of the prime number theorem of Hadamard and de la Vallée-Poussin. The best exposition of the whole subject is Wiener's book (1933), complemented by Lévy (1934).

Let us express Wiener's theorem in another way.

Given a family of functions $f_i \in A(G)$ without common zeros, the ideal which they generate contains the finite sums $\sum f_i \bar{f}_i$, therefore some non vanishing f, therefore, using Wiener's theorem, the whole of $A(G)$. Therefore a proper ideal in $A(G)$ consists of functions whose zero-sets have at least one point in common. Conversely, the functions which vanish at a given point g constitute a maximal proper ideal I_g.

THEOREM (Wiener). - *The maximal proper ideals in* $A(G)$ *are the* I_g $(g \in G)$.

PROBLEM. - Let E be a closed subset of G and I_E the closed ideal in $A(G)$ consisting of all functions $\in A(G)$ vanishing on E. Is every closed ideal an I_E ? In other words, is every proper closed ideal an intersection of maximal ideals ?

This was called Wiener's problem by L. Schwartz and solved by him in a negative way when $\mathbf{T} = \mathbf{T}^\nu$ $(\nu \geq 3)$ (1948). We shall return to this problem in section 5.

4. The converse of Wiener-Lévy's theorem.

If G is finite, $A(G) = C(G)$. If G is infinite, $A(G) \neq C(G)$; this is not difficult to prove directly ; anyway it is a consequence of the following theorem.

THEOREM (Katznelson). - *Let G be infinite, and let F be a continuous function on a closed interval I of the real line, such that $F \circ f \in A(G)$ whenever $f \in A(G)$ and the range of f is contained in I. Then F is analytic on I (that is, expandable as a Taylor series in the neighbourhood of each point of I). In brief, only analytic functions operate on $A(G)$.*

This theorem was discovered in 1958 and the proof can be found in several books (Rudin 1962, Katznelson 1965, Kahane 1970,...). The proof relies on the following facts.

1. Suppose that Φ is a continuous 2π-periodic function and f a real-valued function in $A(G)$ such that $\Phi(f+x) \in A(G)$ for each real x and $\| \Phi(f+x) \|_A \leq B$ independently of x. Then the Fourier coefficients c_n of Φ satisfy

$$c_n e^{inf} = \int_0^{2\pi} \Phi(f+x) e^{-inx} \frac{dx}{2\pi} \qquad (f = f(y), \quad y \text{ given}).$$

Hence, considering this as an equality in $A(G)$,

$$| c_n | \leq B(\| e^{inf} \|_A)^{-1}.$$

2. For each $r > 0$,

$$\sup_{f \text{ real}, \|f\|_A \leq r} \| e^{if} \|_A = e^r.$$

This holds whenever G is infinite. Let us give the proof when $G = \mathbb{D}$, using the Rademacher functions r_j. First observe that

$$\| e^{i\alpha r_j} \|_A = | \cos\alpha | + | \sin\alpha | .$$

Then

$$\| e^{i\alpha(r_0+\cdots+r_{N-1})} \|_A = (| \cos\alpha | + | \sin\alpha |)^N.$$

Choose $\alpha = \frac{r}{N}$ and let $N \to \infty$.

3. In the assumptions of the theorem, given y inside I, there exist $\alpha > 0$, $\rho > 0$ and $B > 0$ such that $\Phi(f+x) \in A(G)$ and $\| \Phi(f+x) \|_A \leq B$ $(x \in \mathbb{R})$ whenever f is real valued and $\| f \|_A \leq \rho$, and

$$\Phi(u) = F(y + \alpha \sin u).$$

This needs an argument of the type of condensation of singularities.

At this stage, applying steps 3, 1, 2, we see that the Fourier coefficients of Φ decrease exponentially. Therefore Φ is analytic, and so F is analytic inside I. The case of the endpoints is easy: if $I = [0,1]$, $F(x^2)$ is analytic on $]-1,1[$, and its Taylor expansion around 0 contains only even powers of x.

Steps 1, 2, 3 were discovered in this order, as explained in the note of Katznelson 1958.

Katznelson's theorem extends in two ways.

First, given E, a closed subset of G, we consider the "restriction algebra" $A(E)$, which consists of restrictions to E of functions belonging to $A(G)$. Thus $A(E)$ is contained in $C(E)$, the algebra of all continuous functions on E and the Wiener-Lévy theorem holds for $A(E)$. In a number of cases Katznelson's theorem is valid when we replace $A(G)$ by $A(E)$. For example, this holds whenever, given any integer n, E contains the algebraic sum of two sets of n points. Of course, it does not hold when $A(E) = C(E)$, that is, when E is a "Helson set".

Here is the so-called dichotomy conjecture : either E is a Helson set, or only analytic functions operate in $A(E)$.

Secondly, as pointed out first by P. Malliavin in 1959, a slightly different problem can be raised, namely to exhibit a function $f \in A(G)$ (or $A(E)$) such that $F \circ f \in A(G)$ (or $A(E)$) implies that F is "almost" an analytic function. The study is not difficult when G is the product of two uncountable groups, in particular when $G = \mathbb{D}$. The best result is this : suppose that I is a closed interval on the line, $(M_n)_{n \geq 0}$ a positive sequence such that $n^{-1} M_n^{1/n} \to \infty$, and E contains the algebraic sum of two perfect subsets of G ; then $A(E)$ contains a function f with range in I such that $F \circ f \in A(E)$ implies that $F \in C^\infty(I)$ and

$$\sup_{n, x \in I} \mid M_n^{-1} F^{(n)}(x) \mid^{1/n} < \infty.$$

The main tool in order to study these extensions of Katznelson's theorem is the theory of tensor algebras, introduced by N. Varopoulos in 1965. The proofs of the above mentioned results can be found in Kahane 1970.

There are several variations on the theme of the dichotomy conjecture. One of them suggested that, given a homogeneous algebra $B(G)$ between $A(G)$ and $C(G)$, either $B(G) = C(G)$ or only analytic functions operate on $B(G)$. Here homogeneous means that the norm is translation-invariant and the translation norm-continuous. This was first disproved by Zafran in 1978. A most interesting counter example is the Pisier algebra $P(G)$, which consists of continuous functions on G whose Fourier series $\sum a_\gamma \gamma$ have the following property : for almost all changes of signs, $\sum \pm a_\gamma \gamma$ is the Fourier series of a continuous function. With the notations of Chapter 9,

$$P(G) = C(G) \cap C_{a.s.}(G).$$

(G. Pisier 1979 ; or Kahane 1985).

5. A problem on spectral synthesis with a negative solution.

Originally spectral analysis consists in finding the harmonics contained in a given function, together with their respective weights or coefficients, and spectral synthesis consists in reconstructing the function from these harmonics and coefficients. In this respect Fejér's theorem is a good answer for the problem of spectral synthesis of continuous functions on the circle $\mathbf{T}$.

Generally speaking the problems of spectral analysis and spectral synthesis occur when we consider translation invariant spaces, either on G or on Γ. In the simplest case spectral analysis consists in finding the one-dimensional translation-invariant subspaces, spectral synthesis consists in reconstructing the space from these one-dimensional subspaces.

Let us consider $\ell^\infty(\Gamma)$. For each given $g \in G$, the sequence $(\gamma(g))_{\gamma\in\Gamma}$ generates a one-dimensional translation-invariant subspace. Is it possible to generate $\ell^\infty(\Gamma)$ as the closure of the finite dimensional spaces generated by the $(\gamma(g))_{\gamma\in\Gamma}$ when g belongs to F, F running over all finite subsets of G ? Of course, this depends on the topology that we choose. There is no hope with the strong topology (= norm topology) of $\ell^\infty(\Gamma)$, for the uniform limits of linear combinations $\sum a_n\gamma(g_n)$ are the almost periodic sequences on Γ. However it works when $\ell^\infty(\Gamma)$ is endowed with the weak topology, as the dual space of $\ell^1(\Gamma)$.

In order to see this and to go further, we may represent elements (a_γ) of $\ell^1(\Gamma)$ by the corresponding functions in $A(G)$, $\sum a_\gamma\gamma$, and elements (b_γ) of $\ell^\infty(\Gamma)$ by formal series $\sum b_\gamma\gamma$, called pseudomeasures. When $\Gamma = \mathbb{Z}^\nu$, pseudomeasures can be viewed as Schwartz distributions with Fourier coefficients b_γ. Let us denote the space of pseudomeasures by $PM(G)$, and define duality between $T = \sum b_\gamma\gamma$ and $f = \sum a_\gamma\gamma$ by

$$< T, f >= \sum a_\gamma b_{-\gamma}.$$

Then $PM(G)$ is a modul on the algebra $A(G)$:

$$< Tf, g >=< T, fg > .$$

The support of T is defined as for distributions : T vanishes on an open set if $< T, f >= 0$ for all $f \in A(G)$ carried by this set, and the support of T is the smallest closed subset of G such that T vanishes outside. The sequence $(\gamma(g))_{\gamma\in\Gamma}$ in $\ell^\infty(\Gamma)$ corresponds to the Dirac measure δ_{-g} in $PM(G)$. Since a function in $A(G)$ orthogonal to all Dirac measures vanishes, the Dirac measures generate $PM(G)$ as the dual space of $A(G)$. In other words, $\ell^\infty(\Gamma)$ is generated by the $(\gamma(g))_{\gamma\in\Gamma}$ $(g \in G)$ in the weak topology.

Now, is it true that, given a weakly closed and translation invariant subspace of $\ell^\infty(\Gamma)$, it is generated by the $(\gamma(g))_{\gamma\in\Gamma}$ which it contains ? In other words, given $T \in PM(G)$ carried by a closed set E (in this case we write $T \in PM(E)$), is it true that T belongs to the weakly closed subspace generated by the δ_g, $g \in E$? Or again, using duality between $PM(G)$ and $A(G)$, is it true that every closed ideal I in $A(G)$ is the intersection of the maximal ideals containing I ? This is the way we described the problem after Wiener's theorem.

The answer is negative.

For $\nu \geq 3$ here is a counter example in $\mathbf{T}^\nu$. The area measure on a small ball E has Fourier coefficients $O(|\gamma|^{-1/2(\nu-1)})$; this is an easy exercise in Bessel functions.

Since $\nu \geq 3$ the radial derivative T has bounded Fourier coefficients. However there is a $f \in A(G)$ vanishing on E (therefore, orthogonal to all δ_g, $g \in E$) and not orthogonal to T ($\in PM(E)$). This is Schwartz's example.

The general case (including the case of T) needs other ideas. Here is a brief account of the three methods in use.

First, instead of considering E first, we can first define f as a rather wild function, so that

$$\int_{\mathbb{R}} 2\pi i u f \, e^{2\pi i u f} du$$

has a meaning as a vectorial integral in $PM(G)$. This defines $T(=\delta'(f)) \in PM(E)$, E being the zero-set of f. Now, adding a constant to f if necessary, we may assume $< T, f > \neq 0$. Then E, T, f provide the example. This is Malliavin's method.

Further, it is possible to use the tensor algebra

$$V(\mathbb{D}) = C(\mathbb{D}) \hat{\otimes} C(\mathbb{D})$$

(projective tensor algebra according to Grothendieck) which consists of continuous functions on $\mathbb{D} \times \mathbb{D}$ of the form

$$f(x_1, x_2) = \sum_{n=1}^{\infty} g_n(x_1) h_n(x_2)$$

with

$$\sum_{1}^{\infty} \| g_n \|_{C(\mathbb{D})} \| h_n \|_{C(\mathbb{D})} < \infty.$$

$V(\mathbb{D})$ is a Banach algebra. The failure of spectral synthesis in *one* algebra $A(G)$ (for example, $A(\mathbb{D})$, or $A(\mathbf{T}^3)$) implies the same for $V(\mathbb{D})$ (that is, the existence of a closed ideal strictly smaller than the intersection of the maximal ideals in which it is contained). And this in turn implies the same for *all* algebras $A(G)$. This is Varopoulos's method.

Finally one can consider a Helson set E ($A(E) = C(E)$). A basic theorem of Helson says that E carries no measure $\neq 0$ with Fourier coefficients tending to 0 at infinity ($M_0(E) = \{0\}$). If spectral synthesis holds for E, $PM(E)$ is orthogonal to all functions in $A(G)$ vanishing on E, therefore $PM(E)$ consists of bounded linear forms on $A(E)$, therefore $PM(E) = M(E)$, the space of measures carried by E. As a consequence $PM(E)$ contains no $T \neq 0$ whose Fourier coefficients tend to zero (what we call a pseudofunction). From the beginning of the theory of Helson sets it was tempting to construct a Helson set E and a pseudofunction $T \neq 0$ carried by E ; for such an E spectral synthesis fails. This was performed by T. Körner in 1972 and a brief account of the subsequent work has already been given in Chapter 6.

6. Another negative result on spectral synthesis.

Here is a problem of spectral synthesis of a different kind : given a Fourier series, is it possible to reconstruct the function through trigonometric polynomials obtained by operating on coefficients ? That is, given $f \sim \sum a_\gamma \gamma$ (a_γ are defined as $\int f\,\bar{\gamma}$ and integration is performed with the Haar measure), is it possible to find a sequence of functions $\varphi_m : \mathbf{C} \to \mathbf{C}$, vanishing near 0, such that

$$f = \lim_{m\to\infty} \sum \varphi_m(a_\gamma)\gamma \quad ?$$

Of course the answer depends on the function space we consider. It is positive for $A(G)(= \mathcal{F}\ell^1(\Gamma))$, $L^2(G)(= \mathcal{F}\ell^2(\Gamma))$ and generally for $\mathcal{F}\ell^p(\Gamma)$ $(1 \le p \le 2)$.

The answer is negative for $C(G)$ and $L^p(G)$ $(1 \le p < \infty,\ p \ne 2)$ when G is infinite. The proof consists in constructing a partition of Γ into classes E_j, all finite but one, and a function whose Fourier coefficients are constant on each E_j, such that trigonometric polynomials with the same property cannot approximate f.

The question was raised by W. Rudin in his book (1962) for L^1 ; in this case it was solved by Kahane (1966). For L^p $(1 < p < 2)$ the solution is due to D. Rider (1969). For L^p $(2 < p < \infty)$ it was obtained independently by G. Bachelis and J. Gilbert (1979) and D. Oberlin (1982). A construction for C and L^1 with finite classes E_j consisting of one or two elements only was given by Y. Katznelson and J.-P. Kahane (1978). All these constructions are made when $G = \mathbf{T}$; the general case follows the same lines. It seems likely that a construction with cardinal $E_j \le 2$ works also for L^p $(1 \le p < \infty, p \ne 2)$.

The question was motivated by the structure of closed subalgebras in the convolution algebras L^p : are they generated by the idempotents they contain ? The answer is negative except for $p = 2$.

A positive solution would have had a computational interest. From a numerical point of view small coefficients are nothing but zero and it is tempting to approximate a function in replacing coefficients with moduli $< \epsilon$ by 0. This is not possible in L^p $(p \ne 2)$ with ordinary Fourier series. It is possible with wavelets.

7. Homomorphisms of algebras A(G).

Here is a theorem of P. J. Cohen (1960) whose proof can be found in chapters 3 and 4 of Rudin's book (1962).

THEOREM. - *Let G_0 and G_1 be two compact abelian groups and φ a continuous map from G_0 to G_1 such that it carries $A(G_1)$ into $A(G_0)$: $f \in A(G_1) \Rightarrow f(\varphi) \in A(G_0)$. Then φ is locally affine, meaning that*

$$\varphi(t+s') - \varphi(s') = \varphi(t+s) - \varphi(s)$$

for all s and s' in G_0 when t belongs to a convenient neighbourhood of 0 in G_0.

Here is a short proof. For simplicity we restrict ourselves to the case when Γ_1, the dual group of G_1, is countable.

Using the closed graph theorem the assumption on φ reads

$$\forall \gamma \in \Gamma_1 \qquad \| \gamma(\varphi) \|_{A(G_0)} \leq C$$

where C is a constant. Moreover $\| \gamma(\varphi) \|_{L^2(G_0)} = 1$. Since $A(G_0) = \mathcal{F}\ell^1$ and $L^2(G_0) = \mathcal{F}\ell^2$ we can use Hölder's inequality

$$\| \quad \|_2^2 \leq \| \quad \|_1^{2/3} \| \quad \|_4^{4/3}$$

and we obtain

$$\| \gamma(\varphi) \|_{\mathcal{F}\ell^4}^4 \geq \| \gamma(\varphi) \|_{\mathcal{F}\ell^1}^{-2} .$$

Therefore

$$\forall \gamma \in \Gamma_1 \qquad \| \gamma(\varphi) \|_{\mathcal{F}\ell^4}^4 \geq C^{-2}.$$

Now, using Parseval's formula, and writing d for Haar measure,

$$\| \gamma(\varphi) \|_{\mathcal{F}\ell^4}^4 = \int\int\int_{G_0^3} \gamma(\Phi) ds\, ds'\, dt$$

where

$$\Phi = \varphi(t + s') - \varphi(s') - \varphi(t + s) + \varphi(s).$$

Let K be the set t-set where $\Phi = 0$ whatever s and s'. It is a closed subgroup of G_0. We simply have to prove that the Haar measure of K is > 0.

Let us enumerate Γ_1 in the form $(\gamma_m)_{m\in\mathbf{N}}$, take a strictly positive sequence $(\lambda_m)_{m\in\mathbf{N}}$ such that $\sum \lambda_m = 1$, and consider

$$b = \sum \lambda_m \gamma_m.$$

Then b^n has a similar form for $n = 1, 2, \cdots$ Therefore

$$\int\int\int_{G_0^3} b^n(\Phi) ds\, ds'\, dt \geq C^{-2}.$$

Moreover $b(0) = 1$ and $| b(g) | < 1$ when $g \neq 0$. Therefore $b^n(\Phi)$ converges to $1_{\Phi=0}$ pointwise, and the set $(\Phi = 0)$ contains $G_0 \times G_0 \times K$. By the Lebesgue convergence theorem

$$\text{Haar measure}(G_0 \times G_0 \times K) \geq C^{-2}$$

therefore the Haar measure of K is $\geq C^{-2}$. This ends the proof.

As a corollary, $A(G_0)$ and $A(G_1)$ are algebraically isomorphic if and only if G_0 and G_1 are isomorphic as locally abelian groups. For example, all $A(\mathrm{T}^n)$ and $A(\mathbb{D})$ are different algebras.

Moreover, considering the case $G_1 = G_2 = \mathrm{T}$, the only changes of variables φ which carry $A(\mathrm{T})$ into $A(\mathrm{T})$ are of the form $\varphi(t) = kt + \ell$ $(k \in \mathbb{Z}, \ell \in \mathrm{T})$.

The prototype of these results is a theorem of A. Beurling and H. Helson on $A(\mathbb{R}) = \mathcal{F}L^1(\mathbb{R})$ (1953) : if φ carries $A(\mathbb{R})$ into $A(\mathbb{R})$, then $\varphi(t) = at+b$ $(a \in \mathbb{R}\backslash\{0\}$, $b \in \mathbb{R})$. The theorem of P. J. Cohen was stated for all locally compact abelian groups (see Rudin 1962).

The theorems of Beurling-Helson and Katznelson (expressed for $A(\mathbb{R})$) answer both questions asked by Paul Lévy in 1934 : to characterize all functions φ and F such that $f(\varphi)$ and $F(f)$ belong to $A(\mathbb{R})$ whenever f belongs to $A(\mathbb{R})$.

Chapter 11

MARTINGALES AND H^p SPACES

1. Taylor series, Walsh series, and martingales.

It may seem paradoxical, not to devote a chapter to complex methods in Fourier series. We know that Fourier series and Taylor series inside the unit disc have an intimate relation. The use of complex methods goes back to Cauchy, and we have seen how Riemann corrected Cauchy's mistakes. In chapter 8 we discussed the Fatou theorem on bounded Taylor series and its generalization to H^p (Hardy classes) through the Nevanlinna class. Complex methods have many more aspects, described in three chapters of Zygmund's book ; in particular they play an essential role in the classical Littlewood-Paley theory, first published under the title "Theorems on Fourier series and power series" (1931). Several books are devoted entirely or partially to H^p spaces (Duren 1970, Petersen 1977, Koosis 1980, Garnett 1981) and their uses in analysis have been popularized by Coifman and Weiss (1977).

Instead of explaining Taylor series and complex methods we chose Walsh series and martingales.

Walsh series were introduced in chapter 10, and we have already seen that they may be more manageable than ordinary Fourier series. As a hors d'œuvre for the present chapter we view their use for a best possible Khintchin inequality, a nice and recent result of Rafal Latala and Krzysztof Oleszkiewicz (1994). Then we explain why their partial sums of order 2^j are so easy to deal with : they constitute a dyadic martingale. The Littlewood-Paley theory, which needed hard complex analysis when it appeared, has a more manageable martingale version, due to Paley, that we try to explain in a complete way. Then we mention the main points in the theory of dyadic H^p spaces (the analogues of classical H^p spaces when Taylor series are replaced by Walsh series). Complements can be found in the book of Long Rui-lin (1993).

Finally we evoke another kind of martingales, provided by plane Brownian motion. Because of its isotropic property Brownian motion is well adapted to the study of harmonic functions, conformal mappings, analytic functions. We sketch its use for boundary properties of analytic functions in the unit disc and a new interpretation of classical H^p spaces through Brownian motion. The topic was initiated by D. Burkholder ; the book of R. Dürrett (1984) is a complete exposition of the subject.

2. A typical use of Walsh expansions : a best possible Khintchin inequality.

Khintchin inequalities were met in chapter 9. They show that the L^p-norms of sums of Rademacher series are equivalent, that is

$$0 < A_{p,q} \leq \left(\int |f|^p\right)^{-1/p} \left(\int |f|\right)^{+1/q} \leq B_{p,q} < \infty$$

when f is the sum of a Rademacher series or, equivalently, when f is a finite linear combination of Rademacher functions r_j :

$$f = \sum r_j u_j \qquad (u_j \in \mathbb{C}).$$

This holds for some $A_{p,q}$ and $B_{p,q}$ as soon as $0 < p, q < \infty$. If $p < q$ one can choose $A_{p,q} = 1$. The example $f = r_1 + r_2$ shows that $B_{1,2} \geq \sqrt{2}$.

Khintchin inequalities exist also for Rademacher series in a Banach space. Here the u_j are vectors and the inequalities read

$$0 < A^*_{p,q} \leq \left(\int \| f \|^p\right)^{-\frac{1}{p}} \left(\int \| f \|^q\right)^{\frac{1}{q}} \leq B^*_{p,q} < \infty,$$

where $A^*_{p,q}$ and $B^*_{p,q}$ do not depend on the Banach space. Of course, $A^*_{p,q} = 1$ holds when $p < q$, and $B^*_{1,2} \geq \sqrt{2}$.

We shall prove that $B^*_{1,2} = B_{1,2} = \sqrt{2}$. The second inequality is due to S. J. Szarek (1976), the first to R. Latala and K. Oleszkiewicz (1994) and the proof is theirs in a version communicated to us by Stanislaw Kwapien.

Let $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_N) \in \{-1,1\}^N = G$, with $N \geq 2$. The characters on G are the 2^N Walsh functions $w_n = \prod_{j \in J} \epsilon_j$ when J runs over all subsets of $\{1, 2, \cdots, N\}$ and $n = \sum_{j \in J}(\epsilon_j + 1)2^{j-2}$. Let us write $| n |= \text{card}\, J$. Given ϵ, we denote by η each of the N neighbours of ϵ, obtained by changing one ϵ_j into $-\epsilon_j$. Given a function $g : G \to \mathbb{C}$, let

$$Lg(\epsilon) = \frac{1}{2} \sum_\eta (g(\epsilon) - g(\eta)).$$

L is a kind of differentiation and we shall verify that it has a simple expression when we develop g in the form $\sum a_n w_n$, namely

$$L\left(\sum a_n w_n\right) = \sum | n | a_n w_n.$$

Since L is linear we simply have to prove $Lw_n =| n | w_n$ and that is clear because, given ϵ, $w_n(\epsilon) - w_n(\eta) = 2w_n(\epsilon)$ for exactly $| n |$ neighbours of ϵ, and $w_n(\epsilon) - w_n(\eta) = 0$ for the remaining $N- | n |$ neighbours.

Now let us imbed $\{-1,1\}^N$ in $\mathbb{R}^N$ and suppose that $g : \mathbb{R}^N \to \mathbb{R}^+$ is even, convex and homogeneous of degree 1. Since it is convex

$$\frac{1}{N} \sum_\eta g(\eta) \geq g\left(\frac{1}{N} \sum \eta\right)$$

and a simple inspection gives $\sum \eta = (N-2)\epsilon$. Since it is homogeneous of degree 1

$$g\left(\frac{N-2}{N}\epsilon\right) = \frac{N-2}{N} g(\epsilon).$$

Therefore, considering g restricted to $G = \{-1,1\}^N$,

$$Lg(\epsilon) \le \frac{N}{2} g(\epsilon) - \frac{N}{2}\frac{N-2}{N} g(\epsilon) = g(\epsilon).$$

Comparing with the previous expression for Lg, we have

$$\sum (\mid n \mid -1) a_n w_n \le 0.$$

Multiplying by g (positive function) and integrating on G we obtain

$$\sum (\mid n \mid -1) a_n^2 \le 0.$$

Hence

$$a_0^2 \ge \sum_{|n|\ge 2} a_n^2.$$

Now, since g is even, all a_n with $\mid n \mid = 1$ vanish, therefore

$$a_0^2 \ge \sum_{|n|\ge 1} a_n^2 = \int_G g^2 - a_0^2, \qquad a_0 = \int_G g.$$

We can choose $g(x_1, \cdots, x_N) = \| \sum_1^N x_j u_j \|$. Then $g(\epsilon) = \| f \|$ and the former inequality reads

$$\int \| f \|^2 \le 2 \left(\int \| f \| \right)^2$$

what we had to prove.

3. Walsh series and dyadic martingales.

Many problems on ordinary Fourier series (that is, Fourier series on the group **T**) can be attacked for Walsh series (that is, Fourier series on the group $\mathbb{D}$) in a much simpler way. This is the case for the Littlewood-Paley theory, stated in 1931 without any proof for ordinary Fourier series (the proof waited until 1937, four years after Paley's death, before being published). The Walsh version was expounded by Paley in 1932 and we shall explain the main features of the different proofs now existing.

Beforehand let us explain why Walsh series are easier to handle (as we have already seen in the preceding chapter). First, we have two simple positive kernels :

1) $$K_j = \prod_{i=1}^{j}(1+r_i) = \sum_{n=0}^{2^j-1} w_n \qquad (j = 1, 2, \cdots)$$

2) $$P_\rho = \prod_{i=1}^{\infty}(1+\rho r_i) = \sum_{n=0}^{\infty} \rho^{|n|} w_n \qquad (0 < \rho < 1)$$

where $| n |$ has the same meaning as in the previous section. K_j is a kind of Dirichlet kernel of order 2^j, but it is positive, like the Fejér kernel, so that the partial sums of order 2^j of a Walsh series converge in the same manner as the Fejér sums of an ordinary Fourier series. P_ρ is a kind of Poisson kernel and it also looks like a Riesz product ; both aspects are interesting to consider. Let us remark that L, the operator introduced in the previous section, can be defined as $\left(\frac{d}{d\rho}P_\rho\right)_{\rho=1}$: it is actually a derivative.

Secondly, the partial sums of order 2^j of a series $\sum_0^\infty a_n w_n$,

$$f_j = \sum_{0 \le n < 2^j} a_n w_n,$$

have the structure of a dyadic martingale. That is, considering that $r_1, r_2, r_3, \cdots$ represent what we obtain in playing heads and tails at time $1, 2, 3, \cdots$ (for example $r_j = 1$ for head and -1 for tail), the conditional expectation of $f_j - f_{j-1}$ at time $j-1$ when $r_1, r_2, \cdots, r_{j-1}$ are known is zero. We may write

$$\Delta_j = f_j - f_{j-1} = r_j B_j(r_1, r_2, \cdots, r_{j-1})$$

and consider B_j as a bet at time $j-1$; then Δ_j is the instant earning and f_j the total earning of the player at time j. The theory of martingales applies. Roughly speaking, the series

$$\sum \Delta_j$$

has much in common with a sum of independent random variables - it is nothing but the original Walsh series to which we apply a summation process by dyadic blocks.

Let us start with $f \in L^1(\mathbb{D})$. We can consider f either as a random variable defined on the probability space $\mathbb{D}$, or as a function defined on the interval $[0, 1]$ on which $\mathbb{D}$ is mapped through $\epsilon \to \sum_1^\infty \epsilon_j 2^{-j}$. A theorem of Kaczmarz (1929) says that f_j converges to f almost everywhere. This can be seen easily using either a martingale argument or the kernel K_j.

If, moreover, $f \in L^p(\mathbb{D})$ $(1 \le p < \infty)$, f_j converges to f in $L^p(\mathbb{D})$; this can be seen in the same way.

The situation on T is not exactly the same. The Kolmogorov example shows that the partial sums of order 2^j of a Fourier-Lebesgue series can diverge everywhere. On the other hand, it is true that the partial sums of order 2^j of a function $f \in L^p(\mathrm{T})$ ($p > 1$) converge to f almost everywhere. Before Carleson 1966, this was considered a very deep result, depending on the Littlewood-Paley theory.

4. The Paley theorem on Walsh series.

The Littlewood-Paley theory has two aspects. Firstly, it exhibits a special kind of multipliers of Fourier series : starting with a Fourier series of a L^p-function ($1 < p < \infty$), if we multiply the n-th terms by ϵ_k ($\epsilon_k = \pm 1$) when $2^{k-1} \leq n < 2^k$, we obtain the Fourier series of a L^p-function again. Secondly, it involves the so called square function,

$$Sf = \left(\sum |\Delta_j|^2 \right)^{1/2}$$

where Δ_j is the sum of terms of order $n \in [2^{j-1}, 2^j[$ of the Fourier series of f, and states that $Sf \in L^p$ if and only if $f \in L^p$ ($1 < p < \infty$).

Here is the version for Walsh series. Now Δ_j is the j-th dyadic block of the Walsh series of $f : \Delta_0 = a_0$, $\Delta_1 = a_1 w_1$, $\Delta_2 = a_2 w_2 + a_3 w_3 \cdots$

THEOREM (Paley 1932). - *Let* $1 < p < \infty$. *For some* $B_p < \infty$

1)
$$B_p^{-1} \int |Sf|^p \leq \int |f|^p \leq B_p \int |Sf|^p$$

and

2)
$$B_p^{-1} \int |\sum \epsilon_j \Delta_j|^p \leq \int |f|^p \leq B_p \int |\sum \epsilon_j \Delta_j|^p$$

whenever $\epsilon_j = \pm 1$ *and* $f \in L^p(\mathbb{D})$.

Since Sf is not modified when we change the Δ_j into $\epsilon_j \Delta_j$, condition 1) implies condition 2) with B_p^2 instead of B_p. Conversely, if condition 2) holds, integrating with respect to the ϵ_j's and using Khintchin's inequality for $(p, 2)$ gives 1), with a different B_p. Therefore it is enough to prove either 1) or 2).

Moreover, using duality between L^p and $L^{p'}$ when p and p' are conjugate exponents (meaning $(p-1)(p'-1) = 1$), it is enough to provide a proof either for $p > 2$ or for $1 < p < 2$. Anyway, the case $p = 2$ is obvious.

A common idea of all proofs is to use an interpolation theorem between function spaces. There are three interpolation theories available now, and three proofs of Paley's theorem. Let us explain the three of them, and give one full proof (the second).

At the time of Paley there was only one interpolation theorem, due to Marcel Riesz : if T is a linear and bounded operator from L^{p_0} to L^{q_0} and from L^{p_1} to L^{q_1}, it is also a bounded operator from L^p to L^q whenever the point $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the line segment joining $\left(\frac{1}{p_0}, \frac{1}{q_0}\right)$ and $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$. Paley used it with $p_0 = q_0 = 2$ and $p_1 = q_1 = 2k$, where k is a positive integer. He had to apply 2) in order to have a linear operator ($Tf = \sum \epsilon_j \Delta_j$), and he used 1) in order to obtain the result when $p = 2k$. Interpolation gives 2) when $2 < p < 2k$. It is a beautiful proof, needing Paley's conceptual and computational power.

In 1939 Marcinkiewicz gave another interpolation theorem. The conclusion is the same as above, but the assumption is much weaker : T (replacing $| T |$) is supposed to be positive, sublinear, and of weak type (p_0, q_0) and (p_1, q_1). Weak type (p, q) means that

$$| \{Tf > \lambda\} | \leq \lambda^{-q} \| f \|_p \qquad (\lambda > 0),$$

the first member denoting the measure of the set where $Tf > \lambda$. The mapping $f \to Sf$ is positive, sublinear, and $\int (Sf)^2 = \int | f |^2$ implies that S has weak type $(2,2)$. In order to prove Paley's theorem in the form 1) for $1 < p < 2$, it suffices to show that S has weak type $(1,1)$. Let us complete the proof.

Given $\lambda > 0$, we introduce the random variable

$$\tau = \inf\{j : | f_j | > \lambda\}$$

($0 \leq \tau \leq \infty$), called a "stopping time". It is classical, and easy to check, that $f_{j\wedge\tau}$ is a martingale which converges to f_τ ($j \to \infty$) whenever $\tau < \infty$ and that

$$\int | f_{j\wedge\tau} | \leq \int | f | \qquad (j = 0, 1, \cdots).$$

Therefore

$$| \{\tau < \infty\} | \leq \lambda^{-1} \| f \|_1 .$$

In order to study S it is convenient to suppose that f is a finite Walsh series, that is, f_j is a stationary sequence when j is large enough ; to prove weak type $(1,1)$ it is enough to consider such f's. Let us write

$$(Sf)^2 = \sum \Delta_j^2 = \sum f_j(\Delta_j - \Delta_{j-1}) ;$$

$\sum$ is a finite sum starting from $j = 0$, $\Delta_{-1} = 0$, $\Delta_0 = f_0 = \int f$, $\Delta_j = f_j - f_{j-1}$ ($j \geq 1$). Let us integrate on the set $\{\tau = \infty\}$ (where $| f_j | \leq \lambda$ for every j) and use Schwarz's inequality :

$$\begin{aligned}
\int_{\tau=\infty} (Sf)^2 &= \sum \int_{\tau=\infty} f_j(\Delta_j - \Delta_{j-1}) \\
&\leq \sum \left(\int_{\tau=\infty} f_j^2 \int_{\tau=\infty} (\Delta_j - \Delta_{j-1})^2 \right)^{1/2} \\
&\leq \lambda \sum \left(\int | f_j | \int_{\tau=\infty} 2(\Delta_j^2 + \Delta_{j-1}^2) \right)^{1/2} \\
&\leq 2\lambda^{1/2} \left(\int | f | \right)^{1/2} \left(\int_{\tau=\infty} (Sf)^2 \right)^{1/2} .
\end{aligned}$$

Hence

$$\int_{\tau=\infty} (Sf)^2 \leq 4\lambda \parallel f \parallel_1 .$$

Now

$$\begin{aligned} \mid \{Sf > \lambda\} \mid &\leq \mid \{\tau < \infty\} \mid + \lambda^{-2} \int_{\tau=\infty} (Sf)^2 \\ &\leq \lambda^{-1} \parallel f \parallel_1 + 4\lambda^{-1} \parallel f \parallel_1 \end{aligned}$$

and the proof is complete.

This proof goes back to Shigeki Yano (1959). The extension to general martingales is due to D. Burkholder (1966). A different version was given by R. Gundy (1980). Our version is adapted from Long Rui-lin (1993).

The third proof goes back to Paley's point of view : to prove 2) when $p > 2$. But it makes use of a space, BMO (bounded mean oscillation), whose theory was developed only in the 60's and 70's, and on an interpolation theory between BMO and L^2. We refer to Long Rui-lin (1993) again for references and a detailed account of the proof. We shall say more on BMO in the next section.

5. The $\mathrm{H^p}$ spaces of dyadic martingales.

Let us go on with dyadic martingales (f_j) and with their differences $\Delta_j = f_j - f_{j-1}$ $(j \geq 1)$, $\Delta_0 = f_0$. The maximal function is defined as

$$Mf = \sup_j \mid f_j \mid .$$

Considering again the stopping time $\tau = \inf\{j : \mid f_j \mid > \lambda\}$ associated with a $\lambda > 0$. The sets $\{Mf > \lambda\}$ and $\{\tau < \infty\}$ are the same, therefore

$$\mid Mf > \lambda \mid \leq \lambda^{-1} \parallel f \parallel_1$$

when the f_j arise from $f \in L^1(\mathbb{D})$. It is not difficult to establish that

$$\parallel f \parallel_p \leq \parallel Mf \parallel_p \leq \frac{p}{p-1} \parallel f \parallel_p$$

when $f \in L^p$, $1 < p < \infty$ (see Long Rui Lin (1993) p. 35). We see that the square operator Sf and the maximal operator Mf have a similar behaviour when $f \in L^p$, $1 < p < \infty$. Strikingly they have also a similar behaviour for $p = 1$ (though now $f \in L^1$ does not insure $Mf \in L^1$ nor $Sf \in L^1$) and also for $0 < p < 1$ (though now f is just a notation for the martingale (f_j) and not a function from which the martingale derives). Here is the result : whenever $0 < p < \infty$,

$$\int (Mf)^p < \infty \Leftrightarrow \int (Sf)^p < \infty.$$

DEFINITION. - When $\int (Mf)^p < \infty$ (or, alternatively, $\int (Sf)^p < \infty$) we write $f \in H^p$. This defines the H^p spaces of dyadic martingales.

Dyadic martingales are nothing but formal Walsh series $\sum a_n w_n$. We can write

$$f_j = \sum_{0 \le n < 2^j} a_n w_n$$

and

$$f = \sum_0^\infty a_n w_n$$

the last equality being purely formal.

A duality theory can be formulated in the following way : $f\,(= \sum a_n w_n)$ and $g\,(= \sum b_n w_n)$ are in duality whenever the series $\sum a_n b_n$ converges by dyadic blocks; then we write

$$(f,g) = \sum_0^\infty a_n b_n \underset{\text{def}}{=} \lim_{j\to\infty} \sum_{0\le n<2^j} a_n b_n.$$

Equivalently

$$\begin{aligned}(f,g) &= \sum_0^\infty \int \Delta_j(f)\Delta_j(g)\\ &= \sum_0^\infty f_j(g_j - g_{j-1}) = \sum_0^\infty (f_j - f_{j-1})g_j\end{aligned}$$

with the usual convention $f_{-1} = g_{-1} = 0$. The dual space of a collection of f is the space of g which are in duality with those f.

Let us describe the dual space of H^p $(0 < p < \infty)$.

For $1 < p < \infty$, H^p is nothing but L^p, therefore the dual space is $L^{p'}(= H^{p'})$ where $(p'-1)(p-1) = 1$.

For $0 < p \le 1$ the dual space $(H^p)'$ consists of those g for which

$$\left(\frac{1}{|I|}\int_I |g - g_I|^2\right)^{1/2} \le C\,|I|^\alpha, \qquad \alpha = \frac{1}{p} - 1$$

for some C and all dyadic intervals I, g_I being the mean value of g on I. Let us remember that a dyadic interval of order j is defined by the values of $r_1, r_2, \cdots r_j$; the measure of such an interval, I, is $|I| = 2^{-j}$, and $g_I = g_j$ on I. This definition can be modified in several ways.

When $p < 1$, that is, $\alpha > 0$, it can be written

$$\left(\frac{1}{|I|}\int_I |g_{j+1} - g_j|^2\right)^{1/2} \le C\,|I|^\alpha \qquad (|I| = 2^{-j})$$

with a different C, hence

$$|\Delta_j(g)| = |g_j - g_{j-1}| \le C2^{-j\alpha}.$$

Then we say that g belongs to Λ_α, the Lipschitz (or Hölder) class of order α.

When $p = 1$, this is no longer true, but we may write the definition in the form

$$\frac{1}{|I|}\int_I |g - g_I|^r \leq C_r$$

for any given $r \geq 1$, some C_r, and all dyadic intervals I. The space of such g is by definition the dyadic BMO.

Therefore

$$(H^p)' = H^{p'} \quad (p' = \frac{p}{p-1}) \quad \text{when} \quad 1 < p < \infty$$

$$(H^p)' = \Lambda_\alpha \quad (\alpha = \frac{1-p}{p}) \quad \text{when} \quad 0 < p < 1$$

$$(H^1)' = \text{BMO}.$$

Let us remark that BMO is defined by the condition

$$\frac{1}{|I|}\int_I \sum_{j+1}^{\infty} \Delta_k^2(g) \leq C$$

for all dyadic intervals I, and $|I| = 2^{-j}$. That proves that

$$\sum \Delta_k \in \text{BMO} \Rightarrow \sum \epsilon_k \Delta_k \in \text{BMO}$$

whenever the ϵ_k have values ± 1. This is the starting point of the third proof in the last section.

6. The classical $\mathbf{H^p}$ spaces and Brownian motion.

From a historical point of view, Hardy classes H^p came first (Hardy 1914). The theory originated from Taylor series, or analytic functions in the unit disc, as noted in chapter 8. Then it moved to harmonic functions of two or more variables. The functions of bounded mean oscillation (in $\mathbb{R}^n$) were introduced by F. John and L. Nirenberg in 1961. However the role of BMO as a dual space and the completion of the theory of H^p spaces in $\mathbb{R}^n$ are due to Ch. Fefferman (1971) and Ch. Fefferman and E. Stein (1972). The H^p theory of martingales was developed by D. Burkholder (1966), D. Burkholder, R. Gundy and M. Silverstein (1971), and BMO for martingales was introduced by A. Garsia (1973) and continued by C. Herz (1974) and many others. Historical complements can be found in R. Gundy (1980) and Long Rui-lin (1993).

Until now, following Paley, we have considered only dyadic martingales attached to Walsh series in a very natural way. However, the most important martingales are provided by Brownian motion. The book of Richard Dürrett (1984) is a good account of their use in analysis, initiated by Burkholder (1977) and B. Davis (1979). Here we shall rely on an intuitive notion of plane Brownian motion as the continuous version of a random walk in the plane. We denote by $B(t,\omega)$ the position at time t of the ω-trajectory of the plane Brownian motion starting from 0 at time $t = 0$. The essential connection with analytic functions is that, $F(z)$ being an analytic function ($\not\equiv 0$) defined in a disc centered at 0 and vanishing at 0, $F(B(t,\omega))$ can be considered as another Brownian motion modulo a change of time ; this is intuitive because the mapping $z \to F(z)$ is conformal - when z moves randomly in an isotropic way, $F(z)$ does the same - and is expressed by the formula

$$F(B(t,\omega)) = B^*(T,\omega)$$

$$T = \int_0^t \mid F'(B(s,\omega)) \mid^2 ds = T(t,\omega)$$

due to Paul Lévy. A nice application is that, given a point $z_0 \neq 0$ in the plane, the probability that the Brownian motion hits z_0 is zero (proof : $F(z) = z_0(1-e^z)$ never takes the value z_0).

Let us suppose that F is analytic and bounded in the unit disc. When $B(t,\omega)$ moves inside the disc from 0 to the boundary, $B^*(T,\omega)$ stays bounded, therefore it has a limit. Suppose now that $B(t,\omega)$ and $B(t,\omega')$ hit the boundary at the same point ς (the hitting time being $\theta(\omega)$ for the first and $\theta(\omega')$ for the second), the trajectories will cross each other infinitely often near ς. Therefore the trajectories of $B^*(T,\omega)$ and $B^*(T,\omega')$ will cross infinitely often near their respective end points. Hence these end points are the same :

$$B^*(T(\theta(\omega),\omega),\omega) = B^*(T(\theta(\omega'),\omega'),\omega').$$

This defines the boundary values of $F(z)$ along Brownian trajectories : when $\varsigma = B(\theta,\omega)$, then, writing $\tau = T(\theta(\omega),\omega)$,

$$F(\varsigma) = B^*(\tau,\omega).$$

Since the Brownian trajectories ending at ς sweep any small triangle with vertex ς contained in the open unit disc, we conclude that $F(\varsigma)$ is also the non-tangential limit of $F(z)$ when z tends to ς. Now all that we said was valid except on a ω-set of probability zero. That means that $F(\varsigma)$ exists except on a ς-set of Lebesgue measure 0. It is the probabilistic proof of the first part of Fatou's theorem (the existence of non tangential limits almost everywhere). The second part (the limit is $\neq 0$ a.e.) comes from the fact that the probability that $B^*(T,\omega)$ hits 0 is zero.

Of course this is not the simplest proof of Fatou's theorem. However, when we relax the assumption that $F(z)$ is bounded, most of it stays valid provided that we have a convenient control on τ, the stopping time for $B^*(T,\omega)$. For the H^p spaces the main tool is a theorem of Burkholder, expressing the equivalence

$$F \in H^p \Leftrightarrow \tau^{p/2} \in L^1(\Omega).$$

This is a way to avoid the Nevanlinna theory in the study of H^p spaces. The interested reader can find more information in Dürrett's book.

Chapter 12

A FEW CLASSICAL APPLICATIONS

1. Back to Fourier.

Fourier considered the propagation of heat, a problem of physics. He expressed the heat equation, a functional equation, and reduced the problem to finding a solution under prescribed boundary conditions. Trigonometric series, and also trigonometrical integrals, known as Fourier integrals now, were a tool for this purpose, and Fourier insisted on their computational value : let us remember his observation that the series which gives the temperature *inside* an infinite rectangular body is *extremely convergent.*

This chapter will begin with a recollection of the partial differential equations of physics which can be treated with the help of Fourier series. Then other classical problems will be considered (isoperimetric surfaces, curves with constant width), where Fourier series give optimal solutions. After the Poisson summation formula and its application to the Riemann functional equation, the chapter will end with practical and computational questions : sampling, fast Fourier transforms.

2. The three typical PDE's.

The simplest forms of elliptic, hyperbolic and parabolic equations are

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad (u=u(x,y))$$

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0 \qquad (u=u(t,x))$$

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (u=u(t,x)).$$

The first can be written as $\Delta u=0$, where Δ is the Laplacian operator; it shows that u is a harmonic function. Since potentials in physics are usually harmonic functions, at least in a large part of their domain, it is called the potential equation. We have already encountered its use and treatment by Fourier in a question of heat equilibrium.

The second and the third are evolution equations : t is the time and x a position. We recognize the equation of vibrating strings and the heat equation for a homogeneous bar.

The Dirichlet problem consists in solving $\Delta u=0$ in a domain with prescribed values of u at the boundary. Let us consider the case of the disc $x^2+y^2<1$, or $x=r\cos t$, $y=r\sin t$ with $0<r<1$, with continuous data at the boundary, expressed by a given real-valued function $f(t)$. We are looking for a harmonic function in the open disc, continuous in the closed disc, taking the given values at the boundary. A real-valued harmonic function is the real part of an analytic function in $z = x+iy = re^{it}$, therefore it can be written as

$$u(x,y) = \Re\mathrm{e} \sum_0^\infty c_n z^n = \sum_0^\infty r^n (a_n \cos nt + b_n \sin nt).$$

Given a continuous 2π-periodic function $f(t)$, we can compute the Fourier coefficients a_n and b_n through Fourier formulas and we know that

$$f(t) = \lim_{r\uparrow 1} \sum_0^\infty r^n (a_n \cos nt + b_n \sin nt)$$

with uniform convergence ; this is the Abel-Poisson method of summation for Fourier series. Its role in the Dirichlet problem was recognized by Schwarz. Defining $u(x,y)$ as above we obtain the solution and see that it is unique. There are variations on this theme : it is possible to relax the continuity condition on f, and also to consider other domains, such as annuli, or domains obtained from discs or annuli by conformal mappings.

We have already discussed the problem of vibrating strings ; let us remember that the Bernoulli treatment of the problem was the first use of trigonometric series in this type of question. Here the boundary conditions are $u(t,0) = u(t,\ell) = 0$ (the string is fixed at $x = 0$ and $x = \ell$), $u(0,x) = \varphi(x)$ and $\frac{\partial u}{\partial t}(0,x) = \psi(x)$ (the initial position and velocity are given, $0 \leq x \leq \ell$). The solution is

$$u(t,x) = \sum_0^\infty \sin \frac{n\pi x}{h} \left(a_n \cos \frac{n\pi t}{\ell} + b_n \sin \frac{n\pi t}{\ell} \right)$$

the coefficients a_n and b_n being computed through the Fourier expansions of $\varphi(x)$ and $\psi(x)$.

For the heat equation a physical assumption is that the solution should tend to a limit as $t \to \infty$. Elementary solutions are given by $\exp(-\omega^2 t + i\omega x)$. When we integrate with respect to $d\omega$ we obtain the Gaussian distributions

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2} \frac{x^2}{t} \right)$$

which express the law at time t of the linear Brownian motion starting from $x = 0$ at time $t = 0$, or, equivalently, the distribution of temperatures at time t when there is a unit point source of heat at $t = 0$, $x = 0$. This solution is called the heat kernel. When we integrate the elementary solutions $\exp(-\omega^2 t + i\omega x)$ with respect to $\varphi(\omega) d\omega$ instead of $d\omega$ we obtain other solutions, the most interesting being convolutions of the heat kernel with positive measures $d\mu(x)$ which express the distribution of heat at time $t = 0$ (then $\varphi(\omega)$ is the Fourier transform of $d\mu(x)$).

If we consider the case of an annulus (*"une armille"*, as Fourier said), x has to be considered modulo the length of the annulus, say 2π, and ω should be restricted to integral values. A general solution is

$$\sum_{n\in\mathbb{Z}} c_n e^{-n^2 t} e^{inx}.$$

The initial distribution is given by

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}$$

(it may be the Fourier series of a positive measure). For $t > 0$ the series converge very fast ; it is called the Weierstrass transform of the initial distribution, and it can be expressed as a convolution of this initial distribution with the Weierstrass kernel

$$W_t(x) = \sum_{n\in\mathbb{Z}} e^{-n^2 t} e^{inx}.$$

Since the Weierstrass kernel is positive (it is a sum of translates of the heat kernel) we have uniform convergence of the solution as $t \to 0$ to the initial distribution when it is given by a continuous function of x.

The discretization of the heat equation for the annulus has an interesting feature : instability may appear even with very tame initial values. The discretization consists of considering values of (t, x) in a grid : $t = m\alpha$, $x = n\beta$, where $\beta = \frac{2\pi}{N}$. Let us write $v(m, n) = u(m\alpha, n\beta)$ and replace the heat equation by

$$\frac{v(m+1,n) - v(m,n)}{\alpha} = \frac{v(m,n+1) + v(m,n-1) - 2v(m,n)}{\beta^2}.$$

Then, assuming $v(m,n) = e^{ipn\beta}$, we have

$$v(m+1,n) = (1 - 4\frac{\alpha}{\beta^2}\sin^2\frac{p\beta}{2})v(m,n),$$

therefore, starting with $v(0,n) = e^{ipn\beta}$, we obtain

$$v(m+1,n) = (1 - 4\frac{\alpha}{\beta^2}\sin^2\frac{p\beta}{2})^{m+1}e^{ipn\beta},$$

an explosion when

$$\frac{\alpha}{\beta^2}\sin^2\frac{p\beta}{2} > \frac{1}{2}.$$

The process is stable when $2\alpha < \beta^2$; when $2\alpha = \beta^2$, and N is even, its instability is sensitive to frequencies which are odd multiples of $N/2$.

3. Two extremal problems on curves.

Fourier series were applied to the isoperimetric problem by A. Hurwitz in 1901, as noted in chapter 7. They were applied by Albert Pfluger in 1980 to curves of constant width. In both cases we have to deal with closed rectifiable curves in the plane, in particular with their diameters, their lengths, and the areas that they define.

Let Γ be the curve, L its length, A the area of the domain limitated by Γ, with the usual sign conditions. The isoperimetric problem consists in finding the supremum of A when L is given and the curves Γ for which this supremum is attained. Let us use $t \in \mathbf{T}$ as a parameter and write the trajectory in the form

$$z = x + iy = f(t) \sim \sum_{-\infty}^{\infty} c_n e^{2\pi i n t},$$

$\sim$ meaning as usual that the c_n are the Fourier coefficients of f. The signed area A is

$$A = \frac{1}{2}\int_\Gamma (x\,dy - y\,dx) = \frac{1}{4i}\int_\Gamma (\bar{z}dz - z\,d\bar{z})$$

and the length L is

$$L = \int_\Gamma |\, dz \,| .$$

Let us assume $L = 1$ and choose t as the arc length. Then

$$\left|\frac{dz}{dt}\right| = |\, f'(t) \,| = 1$$

and

$$\frac{dz}{dt} = f'(t) \sim \sum_{-\infty}^{\infty} 2\pi i n c_n e^{2\pi i n t}.$$

Parseval's formula gives

$$4\pi^2 \sum_{-\infty}^{\infty} n^2 \,|\, c_n \,|^2 = \int |\, f' \,|^2 = 1$$

and

$$A = \pi \sum_{-\infty}^{\infty} n \,|\, c_n \,|^2,$$

hence

$$A \leq \pi \sum_{-\infty}^{\infty} n^2 \,|\, c_n \,|^2 = \frac{1}{4\pi}$$

and equality holds if and only if $c_n = 0$ when $n \neq 0, 1$ and $|\, c_1 \,| = \frac{1}{2\pi}$, that is, when Γ is a circle of length 1 with the direct orientation.

Therefore the solution of the isoperimetric problem is $A \leq \frac{1}{4\pi}L^2$, and equality holds if and only if Γ is a circle with the direct orientation.

The second problem is about diameter D and length L, in case Γ is convex. Clearly $L \geq 2D$. What is the supremum of L when D is given, and for which curves Γ is it attained ? The circles are natural candidates, so that we can guess that $L \leq \pi D$, with equality in case of circles. This is true in a way, but it is not a complete answer. Let us proceed.

Now we choose $2\pi t$ ($t \in \mathrm{T}$) as the oriented tangent angle with a fixed direction, and write $z = f(t)$ again. If Γ has an angular point, $f(t)$ is constant on an interval of values of t. In any case

$$(*) \qquad f'(t) = | f'(t) | e^{2\pi it} = | f'(t) | e_1(t) \quad \text{a.e.}$$

The length is

$$L = \int_\Gamma | dz | = \int_{\dot{\mathrm{T}}} | f'(t) | \, dt = \int_{\dot{\mathrm{T}}} f'\bar{e}_1 = 2\pi i c_1$$

and the Fourier coefficient c_1 can be expressed as

$$c_1 = \frac{1}{2} \int_{\dot{\mathrm{T}}} (f(t) - f(t + \frac{1}{2}))\bar{e}_1(t)\, dt,$$

hence

$$(**) \qquad | c_1 | \leq \frac{1}{2} \int_{\dot{\mathrm{T}}} | f(t) - f(t + \frac{1}{2}) | \, dt \leq \frac{1}{2} D.$$

Therefore, as we guessed,

$$L \leq \pi D.$$

In order to have equality it is necessary and sufficient to have $(*)$ and twice equality in $(**)$. Twice equality in $(**)$ means

$$f(t) - f(t + \frac{1}{2}) = 2Ce_1, \qquad | C | = D,$$

that is

$$f'(t) - f'(t + \frac{1}{2}) = 4\pi i C e_1 \ \text{ a.e.}, \qquad | C | = D.$$

That can be written as

$$\begin{cases} f'(t) = 2\pi i C e_1 (1 + r(t)), & | C | = D \\ r(t) + r(t + \frac{1}{2}) = 0 \ \text{ a.e..} \end{cases}$$

Now $(*)$ means $iC = D$ and $1 + r(t) \geq 0$ (hence $1 - r(t + \frac{1}{2}) \geq 0$). The function $r(t)$ should satisfy

$$\begin{cases} -1 \leq r(t) \leq 1 \\ \int r\bar{e}_1 = 0, \quad \int r\bar{e}_{2k} = 0 \quad (k \in \mathbb{Z}). \end{cases}$$

Conversely, given such a function $r(t)$, the choice

$$f'(t) = \pi D e_1(t)(1 + r(t))$$

guaranties that Γ satisfies $L = \pi D$. Taking $r = 0$ gives the circle, other choices will give different figures ; for example, $r(t) = s(3t)$ with $s(t) = \frac{t}{|t|}$ on $[-\frac{1}{2}, \frac{1}{2}[$ gives the so-called Reuleaux triangle. In any case we have

$$\begin{cases} f(t) - f(t + \dfrac{1}{2}) = -iDe_1 \\ f' = \mid f' \mid e_1 \ \text{a.e.} \end{cases}$$

and that expresses that Γ has constant width (its orthogonal projections have length D in any direction). This gives the general construction of all convex curves of constant width.

4. The Poisson formula and the Shannon sampling.

From now on we are interested in Fourier transforms. Given $f \in L^1(\mathbb{R})$, we consider

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x) dx.$$

$\hat{f}$ is continuous and tends to zero at infinity. If, moreover, $\hat{f} \in L^1(\mathbb{R})$, then

$$f(x) = \int_{\mathbb{R}} e^{2\pi i \xi x} \hat{f}(\xi) d\xi \quad \text{a.e.} ;$$

equality holds everywhere when f is continuous.

Fourier transforms and Fourier series are related in many ways. We shall restrict ourselves to one question : what can we say about f when we known $\hat{f}$ on $\mathbb{Z}$?

Firstly we may write

$$\hat{f}(n) = \int_{[0,1]} \sum_{m \in \mathbb{Z}} f(x + m) e^{-2\pi i n x} dx.$$

The 1-periodic function

$$Pf(x) = \sum_{m \in \mathbb{Z}} f(x + m)$$

exists a.e., it is locally integrable, and its Fourier coefficients are the $\hat{f}(n)$:

$$Pf \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

Whenever the Fourier series converges to $Pf(x)$ and $Pf(x)$ is given by the formula above, the equality

$$\sum_{m \in \mathbb{Z}} f(x + m) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

holds. The case $x = 0$,

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

is known as the Poisson formula. For example, the Poisson formula holds when f and $\hat{f}$ are integrable and continuous on $\mathbb{R}$, and moreover

$$\begin{cases} \sum_{n\in\mathbb{Z}} \mid \hat{f}(n) \mid < \infty \\ \sum_{m\in\mathbb{Z}} \sup_{0\leq x<1} \mid f(x+m) \mid < \infty. \end{cases}$$

For a discussion one may consult Katznelson 1967 and Kahane-Lemarié 1994.

The Poisson formula has several forms and many applications. If we assume that f (therefore $\hat{f}$) is even, subtracting $\hat{f}(0) + f(0)$ from both members we obtain

$$\int_0^\infty f(x)d(Ex - x) = \int_0^\infty \hat{f}(\xi)d(E\xi - \xi)$$

where Ex denotes the integral part of x. Let us choose

$$f(x) = \frac{1}{a}\gamma(\frac{x}{a}), \hat{f}(\xi) = \hat{\gamma}(a\xi) \qquad (a > 0)$$
$$\gamma(x) = e^{\pi x^2}, \hat{\gamma}(\xi) = e^{\pi\xi^2}$$

and integrate both members with respect to $a^{-\sigma}da$ $(0 < \sigma < 1)$. Formal integration is justified through an integration by parts

$$\int_0^\infty f(x)d(Ex - x) = \int_0^\infty f'(x)(Ex - x)dx$$

and the use of Fubini's theorem for positive functions. Since

$$\int_0^\infty \frac{1}{a}\gamma(\frac{x}{a})\frac{da}{a^\sigma} = x^{-\sigma}\int_0^\infty t^{\sigma-1}\gamma(t)dt = x^{-\sigma}\frac{1}{2}\pi^{-\frac{\sigma}{2}}\Gamma(\frac{\sigma}{2})$$
$$\int_0^\infty \hat{\gamma}(a\xi)\frac{da}{a^\sigma} = \xi^{\sigma-1}\frac{1}{2}\pi^{-\frac{1-\sigma}{2}}\Gamma\left(\frac{1-\sigma}{2}\right)$$

we obtain

$$\pi^{-\frac{\sigma}{2}}\Gamma(\frac{\sigma}{2})\int_0^\infty x^{-\sigma}d(Ex - x) = \pi^{-\frac{1-\sigma}{2}}\Gamma\left(\frac{1-\sigma}{2}\right)\int_0^\infty \xi^{\sigma-1}d(E\xi - \xi).$$

Now

$$\int_1^\infty x^{-\sigma}d(Ex - x) = \varsigma(\sigma) - \frac{1}{\sigma - 1} \qquad (\sigma > 0)$$

so that

$$\int_0^\infty x^{-\sigma} d(Ex - x) = \varsigma(\sigma) \qquad (0 < \sigma < 1).$$

where $\varsigma(s)$ denotes the Riemann zeta function. We obtain the functional equation of Riemann

$$\xi(\sigma) = \xi(1-\sigma) \qquad (0 < \sigma < 1)$$

where

$$\xi(\sigma) = \pi^{-\frac{\sigma}{2}} \Gamma(\frac{\sigma}{2}) \varsigma(\sigma).$$

Analytic continuation allows to replace σ by any complex s. This use of the Poisson formula provides the quickest proof of the Riemann functional equation.

The sampling problem of Shannon has a similar nature. It deals with a signal $f(t)$, knowing that it has the form

$$f(t) = \int_{[-\lambda,\lambda]} e^{2\pi i \nu t} d\mu(\nu).$$

Given λ, but ignoring the measure $d\mu$, we want to define f through its restriction to $a\mathbb{Z}$ (the sampling). How to choose a ? The example $f(t) = \sin\frac{\pi}{a}t$ shows that the condition

$$q < \frac{1}{2\lambda}$$

is necessary. It is sufficient as well because, assuming for simplicity $a = 1$ and $\lambda < \frac{1}{2}$, the measure $d\mu$ is well defined by its Fourier coefficients $f(n)$; moreover, writing $d\mu = \psi\, d\mu$, where ψ is a window function, equal to 1 on $[-\lambda, \lambda]$ and vanishing outside $[-\frac{1}{2}, \frac{1}{2}]$, we can reconstitute f in the form

$$f(t) = \sum_{n \in \mathbb{Z}} f(n) \hat{\psi}(t-n).$$

A smooth window ψ gives a rapidly decreasing $\hat{\psi}$, allowing a good reconstruction of f.

5. Fast Fourier transform.

In the 1960's computers made the use of fast programs very important. On the other hand Fourier transforms had a number of applications in physics and engineering ; in particular, all problems of convolutions $(f, g) \to f * g)$ or deconvolutions (g given, $f * g \to f$) can be treated via Fourier transforms or their analogues (Fourier series, Laplace transforms). Fast programs for the computation of Fourier transforms (FFT) were introduced at this time and they were used immediately on a very wide scale. Moreover they were considered as a paradigm in the newborn theory of complexity of algorithms. The names of their initiators, Cooley and Tukey, became quite popular.

Let us explain the principle.

First, we have to reduce the computation of a continuous family of integrals to a finite collection of finite sums. This needs a sampling both in x and ξ, made on either a theoretical or empirical basis. Modulo a change of scale we may assume that we compute $\hat{f}$ on $\mathbb{Z}$ and we have f defined on T. We may think of f as a trigonometric polynomial of a high degree, N, that is

$$f(t) = \sum_{-N}^{N} a_n e_n(t) = \sum_{-N}^{N} a_n e^{2\pi i n t},$$

and we are interested in computing the Fourier coefficients a_n without using too many elementary operations.

Let us choose an integer $M \geq 2N + 1$. The a_n are well defined by the M values $f(\frac{m}{M})$ ($m \in \mathbb{Z}_M = \mathbb{Z}/M\mathbb{Z}$) by means of the discrete Fourier formulas

$$a_n = \frac{1}{M} \sum_m f(\frac{m}{M}) e_{-n}(\frac{m}{M}).$$

There are $2N + 1$ such formulas, and $2M$ operations (additions or multiplications) in each formula. The total number of operations appears as $O(N^2)$.

The FFT consists in reducing this order of magnitude to $O(N \log N)$. We shall ignore the Fourier formulas and try to extract the best possible information on the a_n by considering $f(0)$, $f(\frac{1}{2})$, $f(\frac{1}{4})$, $f(\frac{3}{4})$, $f(\frac{1}{8}) \cdots$ in succession. To begin with $f(0)$ gives the total sum $\sum a_n$. Then, using $f(\frac{1}{2}) = \sum (-1)^n a_n$, we obtain the sums of a_n restricted to even n and the same for odd n. Going further we obtain more and more restricted sums of a_n, until finally each sums contains one term only.

To be more specific we need some notations. First let us assume $M = 2^\mu$, a power of 2. Given $0 \leq j \leq \mu$ and $0 \leq k < 2^j$ we write

$$A_j(k) = \sum_{n=k \pmod{2^j}} a_n$$

$$A_j(k, \varsigma) = \sum_{n=k \pmod{2^j}} a_n \varsigma^n$$

We increase j from 0 to μ and compute the $A_j(k)$ by steps, using the $f(\frac{m}{2^j})$. The initial step is

$$A_0(0) = f(0).$$

Assume that we can derive the $A_j(k)$ $(0 \leq k < 2^j)$ from the $f(\frac{m}{2^j})$ $(0 \leq m < 2^j)$ by means of K_j operations. Then, using the same procedure, we derive the $A_j(k, \varsigma)$ from the $f\left(\frac{m+\frac{1}{2}}{2^j}\right)$, with $\varsigma = e(2^{-j-1})$, and we have

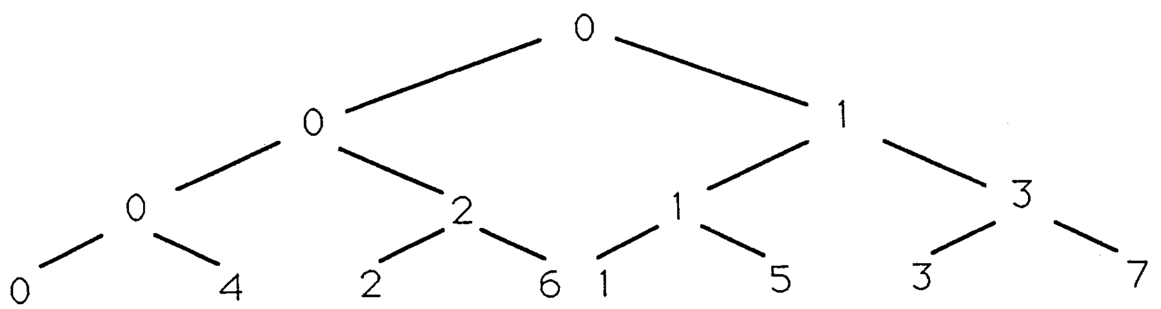
$$A_j(k) = A_{j+1}(k) + A_{j+1}(k + 2^j)$$
$$A_j(k, \varsigma) = \varsigma^k (A_{j+1}(k) - A_{j+1}(k + 2^j)).$$

Therefore $A_{j+1}(k)$ and $A_{j+1}(k+2^j)$ are derived from $A_j(k)$ and $A_j(k,\varsigma)$ by means of four operations : multiplying $A_j(k,\varsigma)$ by $\frac{1}{2}\bar{\varsigma}^k$ and $A_j(k)$ by $\frac{1}{2}$, adding and subtracting. On the whole the $A_{j+1}(k)$ $(0 \leq k < 2^{j+1})$ are defined from the $f\left(\frac{m}{2^{j+1}}\right)$ $(0 \leq m < 2^{j+1})$ by means of $K_{j+1} = 2K_j + 4 \cdot 2^j$ operations. Since $K_0 = 0$, we have $K_j = 4j2^{j-1}$. At the μ-th step the $A_\mu(k)$ contain only one term a_n, and

$$K_\mu = 4\mu 2^{\mu-1} = O(M \log M),$$

what we had to prove.

The procedure can be viewed as a binary tree. Let us figure how the values of k occur at times $j = 0, 1, 2, 3$:



We see that either a final reordering or a convenient parsing of the tree is necessary in order to obtain the a_n in the natural order ; the order provided by the tree is based on the binary expansion of n considered from right to left, while the usual order involves binary expansions read from left to right.

There are many variations on this principle. Instead of taking M a power of 2, and using binary expansions for m and k, it is possible to choose $M = c^\mu$ and a c-adic expansion. One can also choose $M = c_1 c_2 \cdots c_\mu$ and adapted expansions. Moreover, one can take advantage of peculiarities of the data, such as reality, positivity, symmetries. This is very important when dealing with Fourier transforms in three dimensions, as is the case in cristallography.

In the mid 60's, after the seminal paper of Cooley and Tukey (1965), the FFT was immediately applied to electrical engineering and radioastronomy. Applications to cristallography came later, in the mid 70's. Now the best exposition, from a mathematical as well as applied point of view, can be found in the last issue of *International Tables in Cristallography*, in an extensive study on *Fourier transforms in Cristallography*, by Gérard Bricogne (1993). The use of FFT expanded in such a way that, at the time when Bricogne wrote his paper, 50 % of all supercomputer CPU time was alleged to be spent calculating FFT. By the way, the modern way to designate the foundation of FFT is DFT, discrete Fourier transforms. Both FFT and DFT are important and popular in industry and applied mathematics (see the recent books by Loan, 1993, and Briggs and Henson, 1995).

FFT forced a number of users to learn and understand the relations between Fourier transforms on different groups : $\mathbb{R}^n$, $\mathbb{R}$, T, $\mathbb{Z}$, $\mathbb{Z}_N$, in particular to see the role of discrete Fourier transforms (on finite cyclic groups). A great deal of analysis, geometry and algebra is needed by now, as Bricogne's paper shows. FFTs prove to be a good way to look at many parts of mathematics and to go back in their history. Historical investigations resulted in attributing FFT to Gauss as early as 1805 (Bricogne, 1.3.2.0). Anyway DFT, including discrete Fourier formulas, was expressed very clearly by Clairaut as we have seen.

Let us end with a few data. In 1970, the FFT programs used by astrophysicists at CIRCE (computing centre at Orsay) computed 10^6 values of a Fourier transform in less than ten minutes. Now the same is done in four seconds on standard machines (twenty megaflops). Here the Fourier transform is bidimensional and its values are considered on a square grid 1000×1000.

In crystallography the Fourier transform is tridimensional, and the usual grid is $640 \times 640 \times 640$. Here we can appreciate the importance of $O(N \log N)$ compared to $O(N^2)$; the computations are impossible without FFT, and improvements using crystal *and* molecular symmetries are needed too. The first complete description of the atomic structure of a virus (tomato bushy stunt virus) was given in 1978 using these tools (see G. Bricogne, loc. cit.).

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