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ON THE STATIONARY STATISTICAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS AND TURBULENCE

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Abstract.

This paper constitutes a continuation and an improvement of the study [11] on the stationary statistical solutions of the Navier-Stokes equations in bounded domains. Also it contains some new results pertaining to the asymptotic behaviour of the non stationary individual solutions or to the global behaviour of the stationary individual solutions. A discussion on the possible meaning in the theory of the turbulence of the results we establish here, is also given.

Acknowledgement.

This research was initiated (§§.2-4) by the first author (C.F.), while he enjoyed a Fulbright lecturer award at Courant Institute of Mathematical Sciences (New York University). Its main results (given in §§.5-6) represent the joint effort of the authors, made while C.F. enjoyed a visinting professorship (at Collège de France and Université Paris-Sud).

§.1. Introduction.

1.1. Among the most tantalizing problems of the nowadays mathematical physics is that of the occurence of turbulent phenomena (quoted by P.D. Lax [21] as one of the three typical pattern of the physical meaningful nonlinear problems) in the evolution of the solutions of equations of the Navier-Stokes type. Therefore it seems whorth to look for a better understanding of the global and the asymptotic behaviour of the solutions of the initial value problem for the Navier-Stokes equations

$$(1.1.1)\qquad \begin{cases} \partial_t u_i - \nu\Delta u_i + \sum_1^n u_j\, \partial_{x_j} u_i = -\partial_{x_i} p + g_i \quad (i = 1,2,\ldots,n)\ , \\ \sum_1^n \partial_{x_j} u_j = 0 \ \text{ in } \ \Omega \times [0,\infty)\ , \text{ and} \\ u_i = 0 \quad (i = 1,2,\ldots,n) \ \text{ on } \ \partial\Omega \times (0,\infty)\ , \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2,3)$, $\nu>0$ and $g = \{g_1,\ldots,g_n\}$ represent the kinematic viscosity, resp. the external body forces and p the pressure. We will assume that

$$g \in (L^2(\Omega))^n$$

Our belief is that an efficient way of answering to the above expressed axis is to study the statistical solutions of the Navier-Stokes equations, especially the stationary ones.

Heuristically these solutions can be defined as follows (see for instance [23], Ch.I, §.6) : The equations (1.1.1) are viewed as an evolution equation

$$(1.1.2)\qquad \frac{du}{dt} + A(u) = 0$$

in a suitable real infinite dimensional Hilbert space H, where $A(u)$ is a specific continuous (non linear) map from a subspace $H^1 \subset H$ (dense in H and endowed with a supplementary stronger norm) into its dual $H^{-1} \supset H$ (for details, see n° 2.3 below). Roughly speaking, a stationary statistical solution of (1.1.1-2) is a Borel probability measure μ in H, carried by H^1 which is invariant under the infinitezimal translation

$$T_{\delta t} : u \longmapsto T_{\delta t}\, u = u - A(u)\, \delta t$$

along the vector field $A(u)$ $(u \in H^1)$ in H^{-1}. A simple non rigorous computation leads easily to the equation

$$(1.1.3) \qquad \int \langle A(u),\phi'(u)\rangle \, d\mu(u) = 0$$

for an enough large class of adequate functionals ϕ on H, where $\phi'(u)$ denotes the Fréchet derivatives of ϕ, while $\langle .,. \rangle$ denotes the duality between H^{-1} and $\underset{\smile}{H}^{1}$. (For the rigourous definitions see n°2-4 below). Though probability measures on H which actually are stationary statistical solutions of (1.1.1-2) explicitely occur in [17], [31], and [32], their first rigourous and systematic study is (as far as we are aware) contained in [11]. The present paper continues this study, substantially improving parts of [11] and exhibiting new properties of these solutions, concerning their generation as ergodic means (§.3) and the structure of their supports (§§.4-5).

1.2. The main new results (announced in [15]) of the paper are contained in §§.5-6, where we prove that for two-dimensional fluids (i.e. for n=2) all stationary statistical solutions are carried by the stationary individual solutions (settling thus a question raised, fourteen years ago, by G. Prodi [31]) and that "in general" the number of the latter ones is finite.

We also give some three-dimensional versions of these results which seem to redeem Leray's point of view on the occurence of turbulence.

1.3. The justification of the preceding statement as well as a discussion of the possible meaning and for the corresponding consequences, of our results for the theory of turbulence, will be given in §.7 ; this paragraph should also be considered as a postface to this paper.

§ 2. The framework.

2.1. Let Ω be a bounded domain in R^n (n=2 or 3) with a C^2-boundary $\partial\Omega$ and let H, rep. H^1, be the closure in $(L^2(\Omega))^n$, resp $(H^1(\Omega))^n$, of

$$H_0 = \{ v : v \in (C_0^\infty(\Omega))^n, \ \mathrm{div}\, v = \sum_{j=1}^{n} \partial_j v_j = 0 \}.$$

The spaces H, resp. H^1 are endowed with the scalar products

$$(u,v) = \int_\Omega \left(\sum_{j=1}^{n} u_j v_j \right) dx, \text{ resp. } ((u,v)) = \int_\Omega \left(\sum_{j,k=1}^{n} \partial_k u_j \cdot \partial_k v_j \right) dx,$$

and the corresponding norms $|u| = (u,u)^{\frac{1}{2}}$, resp. $\|u\| = ((u,u))^{\frac{1}{2}}$. Let A be the Friedrichs' extension of $-\Delta | H_0$. Then

(2.1.1) $\quad D_{A^{\frac{1}{2}}} = H^1$ and $\|u\| = |A^{\frac{1}{2}} u|$ $\quad (u \in D_{A^{\frac{1}{2}}})$

(2.1.2) $\quad D_A = H^1 \cap (H^2(\Omega))^n$ and $c_1^{-1} \|u\|_{(H^2(\Omega))^n} \leq$
$$\leq |Au| \leq c_1 \|u\|_{(H^2(\Omega))^n} \quad (u \in D_A),$$

where $D_\square$ represents the domain of the operator $\square$ and c_1 (as well as the different constants c_j, c_j', c_j'', etc., occuring in the sequel) is a constant depending only on Ω. The relation (2.1.1) is a direct consequence of the definition of H^1 and A, while (2.1.2) results easily from the Cattabriga-Solonnikov-Vorovich-Yudovich theorem ([6], [40]; see [42], § 2.3).

By Rellich's lemma, A^{-1} is compact, hence there exists an orthonormal basis $\{w_m\}_{m=1}^\infty$ in H such that

(2.1.3) $\quad A w_m = \lambda_m w_m \quad (m=1,2,..)$ and $0 < \lambda_1 \leq \lambda_2 \leq \ \ldots$.

We have

(2.1.4) $\quad \lambda_m \geq c_2 m^{2/n} \quad (m = 1,2,\ldots)$

where, as already stated above, c_2 is a constant depending only on Ω. This can be easily obtained by a method of Agmon (see [1],), using (2.1.2); as sample, see [11], § 7, where the case n=2 is proved.

In the sequel P_m (m=1,2,..) will always denote the orthogonal projection of H onto the space spanned by $w_1, w_2, \ldots, w_m$.

2.2. For $u, v, w \in H^1$ we set

(2.2.1) $$b(u,v,w) = \int_\Omega \Big(\sum_{j,k=1}^{n} u_j \cdot \partial_j v_k \; w_k \Big) dx$$

which is a bilinear continuous functional on H^1 such that

(2.2.2) $$b(u,v,w) = -b(u,w,v);$$

this is readily verified, first on H_0.

Moreover we have the following useful inequalities

(2.2.3) $$\begin{cases} |b(u,v,w)| \le c_3 \; |A^\alpha u| \cdot |A^\beta v| \cdot |A^\gamma w| \\ \qquad (\text{for all } u \in D_{A^\alpha},\ v \in D_{A^\beta},\ w \in D_{A^\gamma}) \\ \text{where} \quad 0 \le \alpha, \gamma \le \tfrac{1}{2} \le \beta \le 1 \quad \text{and} \\ \qquad \alpha + \beta + \gamma = \tfrac{5}{4} \quad \text{in case} \quad n=3, \\ \qquad \alpha + \beta + \gamma = 1 + \varepsilon \quad \text{with} \quad 0 < \varepsilon \le \tfrac{1}{4}, \ \text{in case} \quad n=2 \\ (\text{and in this case the constant } c_3 \text{ depends also on } \varepsilon). \end{cases}$$

Let us sketch the proof of (2.2.3) in case n=2. (For the analogous proof of (2.2.3) in case n=3, see [12], Chap.I) To this aim we notice that by (2.1.1-2) and the Sobolev's embedding theorem $H^1(\Omega) \subset L^q(\Omega)$ (for all $1 \le q < \infty$), the maps $u = (u_j)_{j=1}^{2} \longmapsto u_j$, $u \longmapsto \partial_k u_j$ are continuous from $D_{A^{1/2}}$ (resp. D_A) into $L^q(\Omega)$; plainly they are also continuous from $H = D_{A^0}$ (resp. $D_{A^{1/2}}$) into $L^2(\Omega)$. Thus for

$q \geq 2$, we infer by interpolation (see [24] ,)
that these maps are also continuous from $D_A^{\theta/2}$ (resp. $D_A^{(1+\theta)/2}$)
into $L^r(\Omega)$ (for any $0 \leq \theta \leq 1$) with $r = r_\theta = 2q \cdot (2\theta + q - q\theta)^{-1}$. Therefore for α, β, γ and ε as in (2.2.3), taking $q = 2 + \varepsilon^{-1}$ we will have

$$|b(u,v,w)| \leq \sum_{j,k} \|u_k\|_{L^{r_{2\alpha}}(\Omega)} \|\partial_k v_j\|_{L^{r_{2\beta-1}}(\Omega)} \|w_j\|_{L^{r_{2\gamma}}(\Omega)} \leq$$
$$\leq c_3 |A^\alpha u| \cdot |A^\beta v| \cdot |A^\gamma w|$$

for any $u \in D_{A^\alpha}$, $v \in D_{A^\beta}$, $w \in D_{A^\gamma}$.

In the case $n = 2$ we can supplement (2.2.3) with the following

(2.2.4)
$$\begin{cases} |b(u,v,w)| \leq c_4 |A^{\alpha_1} u|^{\frac{1}{2}} \cdot |A^{\alpha_2} u|^{\frac{1}{2}} |A^{\beta_1} v|^{\frac{1}{2}} \cdot |A^{\beta_2} v|^{\frac{1}{2}} \\ \quad \cdot |A^{\gamma_1} w|^{\frac{1}{2}} \cdot |A^{\gamma_2} w|^{\frac{1}{2}} \quad \text{(for all } u \in D_{A^{\alpha_1}} \cap D_{A^{\alpha_2}}, \\ \quad v \in D_{A^{\beta_1}} \cap D_{A^{\beta_2}}, \ w \in D_{A^{\gamma_1}} \cap D_{A^{\gamma_2}}) \\ \text{Where } \{\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2\} = \{0, \tfrac{1}{2}, 1\},\ 1 \leq \beta_1, \beta_2 \leq \tfrac{3}{2} \\ \alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 2 \quad \text{and} \\ \max\{\alpha_1, \alpha_2, \gamma_1, \gamma_2\} = 1 \quad \text{in case} \quad \beta_1 + \beta_2 = 1 \end{cases}$$

These inequalities result directly from the following well-known inequalities:

$$\|u_j\|_{L^4(\Omega)} \leq 2^{1/4} |u|^{1/2} \|u\|^{1/2} \quad \text{for } u = (u_j)_{j=1}^2 \in H^1,$$
$$\|\partial_k u_j\|_{L^4(\Omega)} \leq c_5 \|u\|^{1/2} |Au|^{1/2} \quad \text{for } u = (u_j)_{j=1}^2 \in D_A,$$
$$\|u_j\|_{L^\infty(\Omega)} \leq c_6 |u|^{1/2} |Au|^{1/2} \quad \text{for } u = (u_j)_{j=1}^2 \in D_A;$$

for the first two, see [19] , and [13] , respectively, while for the last one see (2.1.2) and [1], §.13,

Moreover in case n=3, we can use instead of the last inequality the following

$$\|u_j\| \le c_8 |u|^{1/4} \cdot |Au|^{3/4} \text{ for } u=(u_j)_{j=1}^3 \in D_A$$

(see [1] , §.13), obtaining readily

$$(2.2.5)\quad |b(u,v,w)| \le c_4 \cdot \begin{cases} |u|^{1/4}|Au|^{3/4}\|v\|\cdot|w| & (u\in D_A, v\in H^1, w\in H) \\ |u|^{1/4}|Au|^{3/4}|v|\cdot\|w\| & (u\in D_A, v\in H, w\in H^1) \\ |u|\cdot\|v\|\cdot|w|^{1/4}|Aw|^{3/4} & (u\in H, v\in H^1, w\in D_A) \\ |u|\,|v|^{1/4}|Av|^{3/4}\|w\| & (u\in H, v\in D_A, w\in H^1). \end{cases}$$

Let H^{-1} be the dual of H^1 and for $\varphi\in H^{-1}$, $u\in H^1$ let denote $\varphi(u)$ by $\langle\varphi,u\rangle$. We embed H into H^{-1} through the identification $\langle h,u\rangle = (h,u)$ (for $h\in H$, $u\in H^{-1}$). Plainly, the operator A can be extended by continuity to a continuous operator from H^1 into H^{-1} (actually, onto). For $u,v \in H^1$, let $B(u,v) \in H^{-1}$ be defined by

$$(2.2.6)\qquad \langle B(u,v),w\rangle = b(u,v,w) \quad \text{for all } w\in H^1.$$

We remark that (2.2.3) (with the choice $\alpha=\frac{1}{2}$, $\beta=\frac{3}{4}$, $\gamma=0$) yields

$$(2.2.7)\qquad B(H^1,\ D_A) \subset H.$$

2.3. The precise form of the operator A(u) in (1.1.2) is

$$(2.3.1)\qquad A(u) = \nu Au + B(u,u) - f \qquad (u\in D_A)$$

where f is the orthogonal projection on H of the $\mathbb{R}^n$- valued function g occuring in (1.1.1). Plainly A(u) applies D_A into H and H^1 into H^{-1}. In case we want to emphasize the dependence of A(u) on $\nu>0$ and $f\in H$, we will write $A(u;f,\nu)$ instead of A(u).

We shall be concerned with the initial value problem

(2.3.2) $\frac{du}{dt} + A(u) = 0$ for $t > 0$, $u(0)=u_0 \in H$.

By a solution (or an <u>individual solution</u>) of (2.3.2) we mean a function $u(t)$ from $[0, \infty)$ to H which is weakly continuous at any $t_0 \in [0, \infty)$ and strongly continuous from the right at any to $t_0 \in [0, \infty) \setminus \vartheta$ (where $\vartheta \subset (0, \infty)$ depends on the solution and meas $\vartheta = 0$), and which satisfies

(2.3.3) $u(t) \in H^1$ a.e. on $(0, \infty)$ and $\int_0^t \|u(\tau)\|^2 d\tau < \infty$ $(t \in (0,\infty))$,

(2.3.4) $\frac{1}{2} |u(t)|^2 + \nu \int_{t_0}^t \|u(\tau)\|^2 d\tau \leq \frac{1}{2} |u(t_0)|^2 + \int_{t_0}^t (f, u(\tau)) d\tau$ (for all $t \in (0,\infty)$ and $t_0 \in [0,t] \setminus \vartheta$),

(2.3.5) $u(t) = u_0 + \int_0^t u'(\tau) d\tau$, $u'(\tau) \in L^1(0,t; H^{-1})$ $(t \in (0,\infty))$,

(2.3.6) $u'(t) + A(u(t)) = 0$ a. e. on $(0, \infty)$,

where (2.3.5-6) are considered relations in H^{-1} (see [23], Ch. I, § 6 and especially [12], §.II.1). It can be proved (as in [23], Chap.I, §.6 or [12], §.II.1) that for any $u_0 \in H$ there exists at least one solution of (2.3.2). Moreover if the n=2, one easily verifies that this solution is unique (see for instance [23], Chap.I, §.6).

A time-independent solution $u(t)=u_0$ (or rather u_0) is called a stationary (individual) solution; that is, u_0 H is a stationary solution of (2.3.2) if $u_0 \in H^1$ and $A(u_0) = 0$ (in H^{-1}). Plainly (2.2.2-3) yield

(2.3.7) $\|u_0\| \leq \lambda_1^{-1/2} \nu^{-1} |f|$.

2.4. In order to define the stationary statistical solutions of (2.3.2) let us agree to call <u>test functional</u> any real functional $\phi(u)$ on H enjoying the following properties:

1° For any $u \in H^1$, the Fréchet derivative $\phi'(u)$ of ϕ, taken in H along H^1 exists (that is

$$\frac{1}{|v|}\,|\phi(u+v)-\phi(u)-(\phi'(u),v)| \to 0 \text{ for } v\in H^1,\ |v|\to 0).$$

2° $\phi'(u) \in H^1$ for all $u \in H^1$ and, as function from H^1 into H^1, ϕ' is continuous and bounded.

Let now μ denote a Borel probability measure on H such that

$$\int \|u\|^2 d\mu(u) < \infty \tag{2.4.1}$$

(where $\|u\|$ is taken $=\infty$ for $u \in H \setminus H^1$). Then, for any test functional ϕ, the function

$$\langle A(u), \phi'(u)\rangle = \nu((u,\phi'(u))) + \langle B(u,u),\phi'(u)\rangle - (f,\phi'(u)) \tag{2.4.2}$$

is continuous on H^1 and satisfies (by (2.2.3))

$$|\langle A(u),\phi'(u)\rangle| \le \nu\|u\|\cdot\|\phi'(u)\| + c_3 |u|^{1/2}|A^{1/2}u|^{1/2}\|u\|\cdot \|\phi'(u)\| + |f|\cdot|\phi'(u)| \le (\nu\|u\| + c_3\lambda_1^{-1/4}\|u\|^2 + \lambda_1^{-1/2}|f|)\sup_{v\in H^1}\|\phi'(v)\| \quad (\text{for all } u\in H^1).$$

Thus by (2.4.1), the integral

$$\int \langle A(u), \phi'(u)\rangle\, d\mu(u)$$

makes sense. By definition a <u>stationary statistical solution</u> of (2.3.2) is any Borel probability measure μ on H satisfying (2.4.1.) such that

$$\int \langle A(u),\phi'(u)\rangle\, d\mu(u) = 0 \tag{2.4.3}$$

for any test functional ϕ, and

$$\int_{\{E_1\le |u|^2 < E_2\}} [\nu\|u\|^2 - (f,u)]\, d\mu(u) \le 0 \tag{2.4.4}$$

for any $0 \le E_1 < E_2 \le \infty$ (see [11], § 6). The equation (2.4.3) is the rigorous form of the equation (1.1.3) introduced in an heuristic manner in the § 1, while (2.6.4) is a strengtened energy inequality, which has the following direct consequences (see [11], §.6):

(2.4.5) $$\int \|u\|^2 \, d\mu(u) \leq \nu^{-2} \lambda_1^{-1} |f|^2$$

(2.4.6) $$\operatorname{supp} \mu \subset \{ |u| \leq \nu^{-1} \lambda_1^{-1} |f| \},$$

where supp μ denotes the support of μ (i.e. the smallest closed set F in H such that $\mu(F)=1$).

2.5. In case n=2 , then, as already said above, the individual solution u(t) of (2.3.2) is uniquely determined by its initial data $u(0) = u_0$; moreover it is easy to verify that, if $S(t_0)\, u_0$ denotes the value $u(t_0)$ at the time $t_0 > 0$ of u(t), then $S(t_0)$ is a continuous map from H into H^1 (see for instance; [12], §.III.2). We showed in [11], §.6, that a Borel probability measure μ on H is a stationary statistical solution of (2.3.2) if and only if it is invariant with respect to the functional flow $\{ S(t) \}_{t \geq 0}$,

that is, if

(2.5.1) $$\begin{cases} \mu(S(t)^{-1}\omega) = \mu(\omega) \text{ for all } t \geq 0 \text{ and all} \\ \text{Borel subsets } \omega \text{ of } H. \end{cases}$$

Moreover any such μ has its support compact in H^1 (see [11] §.6,); in particular one can prove that

(2.5.2) $$\operatorname{supp} \mu \subset \{ \|u\| \leq c_7 \nu^{-1} |f| \cdot \exp (c_8 \nu^{-8} |f|^4) \}$$

where (as already stated in Sec. 2.1), c_{7-8} are some constants depending only on Ω .

2.6. In case n=3, a similar functional flow might not exist; therefore in this case a more laborious discussion is necessary in order to clarify our definition of the stationary statistical solutions.

Let us begin this discussion by agreeing to call <u>regular</u> on $[0,T]$ $(T > 0)$ any individual solution $u(t)$ such that $u(t) \in C([0,T]; H^1)$.

One can verify that $u(t)$ is regular on $[0,T]$ if and only if $u(t) \in L^\infty(0,T; H^1)$ and that if $u(t)$ is regular on $[0,T]$ then any individual solution of (2.3.2) with the same initial date as $u(t)$ coincides with $u(t)$ on $[0,T]$ (see for instance [12] §.III.1). Thus we can define the map $RS(t_0)$, on the set $D_{RS(t_0)}$ of those $u_0 \in H^1$ for which there exists an individual solution $u(t)$, regular on $[0,t_0]$ such that $u(0)=u_0$, by $RS(t_0)u_0 = u(t_0)$. One can also verify that $D_{RS(t_0)}$ is an open subset of H^1 such that

(2.6.1) $$\begin{cases} u_0 \in D_{RS(t_0)} \text{ for } u_0 \in H^1 \text{ and} \\ t_0 = c_{10}\,\nu^3 \min\{(\nu^{-1}\lambda_1^{-1/2}|f|)^{-4}, \|u_0\|^{-4}\} \end{cases}$$

and that $RS(t_0)$, as a map from $D_{RS(t_0)}$, endowed with the topology of H^1, into H^1, is continuous (see [32], or [12], §.III.1). Therefore for any Borel subset ω of H the set

$$RS(t_0)^{-1}\omega = \{u \in D_{RS(t_0)} ; RS(t_0)u \in \omega\}$$

is also a Borel set in H. In [11], §.6 it was proved that <u>if</u> μ <u>is a Borel probability measure in</u> H, <u>the support of which is bounded in</u> H^1, <u>then</u> μ <u>is a stationary statistical solution of</u> (2.3.2) <u>if and only if</u>

(2.6.2) $$\begin{cases} \mu(RS(t)^{-1}\omega) = \mu(\omega) \text{ for all } t \geq 0 \text{ and} \\ \qquad \text{all Borel subsets } \omega \text{ of } H. \end{cases}$$

In this case $\{RS(t)\}_{t \geq 0}$ defines on $\operatorname{supp}\mu$ a functional flow with respect to which μ is invariant.

2.7. Let us denote by $S(f;\nu)$, resp. $\mathcal{S}(f;\nu)$ the set of all stationary individual, resp. statistical, solutions of (2.3.2). In virtue of (2.3.7) it is plain that any Borel probability mesure carried by the set $S(f;\nu)$ belongs to $\mathcal{S}(f;\nu)$. Also in virtue of the relation

(2.7.1) $$\gamma^2 A(u;f,\nu) = A(\gamma u;\gamma^2 f,\gamma\nu) \qquad (\gamma>0)$$

it is plain that

(2.7.2) $$\gamma S(f;\nu) = S(\gamma^2 f,\gamma\nu) \qquad (\gamma>0).$$

Moreover, if for a Borel measure $\bar{\mu}$ on H and $\gamma > 0$ we denote by $\bar{\mu}\circ\gamma^{-1}$ the measure : $\omega \mapsto \bar{\mu}(\gamma^{-1}\omega)$ (for all Borel sets $\omega \subset H$), then it follows readily that

(2.7.3) $$\mathcal{S}(f;\nu)\circ\gamma^{-1} = \mathcal{S}(\gamma^2 f,\gamma\nu) \qquad (\gamma>0).$$

In particular, we have

(2.7.4) $$\mathcal{S}(f;\nu) = \mathcal{S}(\nu^{-2} f;1)\circ\nu^{-1}.$$

§3. Generation of stationary statistical solutions

3.1. It is a classical fact that $S(f;\nu) \neq \emptyset$ (see for instance [23], Ch. I, §6,), so that $\mathcal{S}(f;\nu) \neq \emptyset$ too. Therefore the problem concerning the existence of stationary statistical solutions has to be replaced by that of their occurence; namely we shall show that stationary statistical solutions naturally occur in studying the asymptotic behaviour of any (non-stationary) individual solution. To this aim we shall agree on a precise meaning for a time average of an individual solution u(t) of (2.3.2).

Let $\mathcal{C}$ denote the space of all (real) functionals on H, weakly continuous on any (bounded) ball in H. Since (2.3.4) implies

$$|u(t)|^2 \leq e^{-\nu\lambda_1 t}|u_0|^2 + \nu^{-2}\lambda_1^{-2}|f|^2 \quad (t \in (0,\infty)) \tag{3.1.1}$$

it results that

$$\mathcal{M}_t(\phi) = \frac{1}{t}\int_0^t \phi(u(\tau))d\tau \qquad (t \in (0,\infty)) \tag{3.1.2}$$

makes sense for any $\phi \in \mathcal{C}$. Using the fact that

$$K_0 = \{u \in H : |u|^2 \leq |u_0|^2 + \nu^{-2}\lambda_1^{-2}|f|^2\}$$

endowed with the weak topology of H, is a separable compact space, we can infer that for any sequence of t's tending to ∞ there exists a subsequence $\{t_j\}_{j=1}^{\infty}$ such that

$$\lim \mathcal{M}_{t_j}(\phi) = \mathcal{M}(\phi) \text{ exists} \tag{3.1.3}$$

for any $\phi \in \mathcal{C}$, or what is the same, for every $\phi \in C(K_0)$

(= the space of all continuous real functions on K_0).
Therefore, by the Riesz-Kakutani representation theorem we obtain a Borel probability measure μ on K_0 such that

(3.1.4) $$\mathfrak{M}(\phi) = \int_{K_0} \phi \, d\mu \qquad (\phi \in \mathcal{C}).$$

But a subset $K \subset K_0$ is a Borel set in K_0 if and only if it is a Borel set in H (endowed with its norm topology). Thus we can consider that μ is a Borel probability measure on H with supp $\mu \subset K_0$. Any such measure will be called a time average of the individual solution u(t).

3.2. Theorem. Any time average of an individual solution is a stationary statistical solution.

Proof. Ad (2.4.1). Set $\phi(u) = \|P_m u\|^2$ (where m = 1,2,... ; see Sec. 2.1). Then $\phi \in \mathcal{C}$, thus

$$\int \|P_m u\|^2 d\mu(u) = \lim_{t_j \to \infty} \frac{1}{t_j} \int_0^{t_j} \|P_m u(\tau)\|^2 d\tau$$

for a suitable sequence $\{t_j\}_{j=1}^{\infty}$, tending to . From (2.3.4) we infer easily

(3.2.1) $$\int_0^{t_j} \|P_m u(\tau)\|^2 d\tau \le \int_0^{t_j} \|u(\tau)\|^2 d\tau \le$$

$$\le \nu^{-1} |u_0|^2 + \nu^{-2} \lambda_1^{-1} |f|^2 t_j,$$

whence

$$\int \|P_m u\|^2 d\mu(u) \le \nu^{-2} \lambda_1^{-1} |f|^2 \qquad (m = 1, 2, \ldots).$$

Letting $m \to \infty$, we can apply B. Levi's convergence theorem, since $\|P_m u\|^2 \nearrow \|u\|^2$, obtaining

(3.2.2) $$\int \|u\|^2 d\mu(u) \le \nu^{-2} \lambda_1^{-1} |f|^2.$$

Ad (2.4.3). Let ϕ be a test functional and for some $k, m (=1,2,\dots)$, set $\psi(u) = \phi(P_m u)$ and

$$b(u) = b(P_k u, P_k u, \psi'(u)) \qquad (u \in H).$$

Then the functional

$$\Theta(u) = \nu((u, \psi'(u))) + b(u) - (f, \psi'(u))$$

belongs to $\mathcal{C}$, thus

$$\int \Theta(u)\,d\mu(u) = \lim_{t_j \to \infty} \frac{1}{t_j} \int_0^{t_j} \Theta(u(\tau))\,d\tau. \tag{3.2.3}$$

It is easy to verify that, since P_m is a linear continuous map from H^{-1} into H^1, the function $\psi(u(t))$ is absolutely continuous on any compact interval of $[0,\infty)$ and

$$\frac{d}{dt}\psi(u(t)) + \langle A(u(t)), \psi'(u(t))\rangle = 0 \quad \text{a.e. on } (0,\infty),$$

so that

$$\frac{1}{t_j}\int_0^{t_j} \langle A(u(\tau)), \psi'(u(\tau))\rangle\,d\tau = \frac{\psi(u_0) - \psi(u(t_j))}{t_j} \to 0. \tag{3.2.4}$$

Or, by (2.2.2-3), we obtain

$$|\langle A(u), \psi'(u)\rangle - \Theta(u)| = |b(u,u,\psi'(u)) - b(u)| \leq \tag{3.2.5}$$

$$\leq |b(P_k u, \psi'(u), u - P_k u)| + |b(u - P_k u, \psi'(u), u)| \leq$$

$$\leq 2\left(\sup_{v \in H^1} |A^{3/4}\psi'(v)|\right) c_3 \|u\|\,|u - P_k u| \leq 2c_3 \lambda_m^{1/4}\left(\sup_{v\in H^1}\|\phi'(v)\|\right)\cdot$$

$$\cdot \|u\|\,|(I-P_k)u| \leq 2c_3' \lambda_m^{1/4} \lambda_{k+1}^{-1/2} \|u\| \cdot \|(I-P_k)u\| \leq$$

$$\leq 2c_3' \lambda_m^{1/4} \lambda_{k+1}^{-1/2} \|u\|^2 \qquad (u \in H^1)$$

where c_3' is a constant depending on Ω and also on ϕ.
The relations (3.2.3-5) yield

$$\left|\int \langle A(u), \varphi'(u)\rangle d\mu(u)\right| \leq 2c_3' \lambda_m^{1/4} \lambda_{k+1}^{-1/2} \int \|u\|^2 d\mu(u) + $$

$$+ \left|\lim_{t_j\to\infty} \frac{1}{t_j}\int_0^{t_j} \Theta(u(\tau))d\tau\right| \leq$$

$$\leq 2c_3' \lambda_m^{1/4} \lambda_{k+1}^{-1/2}\left[\int \|u\|^2 d\mu(u) + \lim_{t_j\to\infty} \frac{1}{t_j}\int_0^{t_j} \|u(\tau)\|^2 d\tau\right],$$

so that, by (3.2.1-2) we obtain

$$\left|\int \langle A(u), \varphi'(u)\rangle d\mu(u)\right| \leq \frac{4c_3' \lambda_m^{1/4} |f|^2}{\nu^2 \lambda_1} \cdot \frac{1}{\lambda_{k+1}^{1/2}} \to 0$$

for $k \to \infty$. It results

$$\int \langle A(u), P_m \Phi_\square' (\boxed{P_m u}) \rangle d\mu(u) = 0.$$

Letting $m \to \infty$ and using (2.4.1-2), the property 2^0, in Sec. 2.4, of Φ and Lebesgue's dominated convergence theorem we obtain (2.4.3).

<u>Ad</u> (2.4.4). Let $\varphi(\sigma)$ be a non-decreasing twice continuously differentiable bounded function on $[0, \infty)$. Let

$t_{j0} = 0 < t_{j1} < \dots < t_{jk_j} = t_j$ be such that the t_{jk}'s do not belong to the exceptional set in (2.3.4).

We have

$$\varphi(|u(t_j)|^2) - \varphi(|u_0|^2) = \sum_{k=0}^{k_j-1} \left[\varphi(|u(t_{j,k+1})|^2) - \varphi(|u(t_{j,k})|^2)\right] =$$

$$= \sum_{k=0}^{k_j-1} \varphi'((1-\theta_{jk})|u(t_{jk})|^2 + \theta_{jk}|u(t_{j,k+1})|^2)\left[|u(t_{j,k+1})|^2 - |u(t_{jk})|^2\right] \leq$$

$$\leq \sum_{k=0}^{k_j-1} \varphi'(\dots) \int_{t_{jk}}^{t_{j,k+1}} 2\left[(f, u(\tau)) - \nu\|u(\tau)\|^2\right] d\tau \longrightarrow$$

$$\longrightarrow 2\int_0^{t_j} \varphi'(|u(\tau)|^2)[(f,u(\tau)) - \nu\|u(\tau)\|^2]d\tau$$

as $\max\limits_k (t_{j,k+1} - t_{jk}) \to 0$, because $|u(t)|^2$ is continuous from the right in any point outsidethe exceptional set in (2.3.4). (Here above $0 < \theta_{jk} < 1$, $k = 0,1,\ldots,t_j - 1$, are suitably chosen.) We can thus infer that

$$(3.2.6) \quad \lim_{t_j\to\infty} \frac{1}{t_j}\int_0^{t_j} \varphi'(|u(\tau)|^2)[(f,u(\tau)) - \nu\|P_k u(\tau)\|^2]d\tau \geq$$

$$\geq \lim_{t_j\to\infty} \frac{1}{t_j}\int_0^{t_j} \varphi'(|u(\tau)|^2)[(f,u(\tau)) - \nu\|u(\tau)\|^2]d\tau \geq 0$$

for any $k = 1,2,\ldots$. Let us suppose moreover that $\varphi''(\sigma)$ is bounded on $[0,\infty)$. Then for a fixed $\ell (=1,2,\ldots)$ we will have

$$(3.2.7) \quad \left| \frac{1}{t_j}\int_0^{t_j} [\varphi'(|u(\tau)|^2) - \varphi'(|P_\ell u(\tau)|^2)]\cdot[(f,u(\tau)) - \right.$$

$$\left. - \nu\|P_k u(\tau)\|^2]d\tau \right| \leq c(1+\lambda_k)\frac{1}{t_j}\int_0^{t_j} |(I-P_\ell)u(\tau)|^2 d\tau =$$

$$= c(1+\lambda_k)\, T_{j,\ell}$$

where (see (3. .1)) the constant c depends on Ω, f, ν and φ'', but not on ℓ, k or j. For p = 1,2,.. let denote by $\alpha_{j,p}$ the set of those $t \in [0, t_j]$ for which $u(t) \in \mathcal{B}_p = \{u \in H^1 : \|u\| \leq p\}$ and let

$$\varepsilon_{p,\ell} = \sup\{|(I-P_\ell)u|^2 : u \in \mathcal{B}_p\}$$

Then $\varepsilon_{p,\ell} \longrightarrow 0$ for $\ell \longrightarrow \infty$ and, by (3.1.1) and (3.2.1) we get

$$(3.2.8)\quad T_{j,\ell} \leq \varepsilon_{p,\ell} + c' \frac{1}{t_j}\, meas([0,t_j] \setminus \alpha_{j,p}) \leq$$

$$\leq \varepsilon_{p,\ell} + c' \frac{1}{p^2} \frac{1}{t_j} \int_{[0,t_j]\setminus \alpha_{j,p}} \|u(\tau)\|^2 d\tau \leq$$

$$\leq \varepsilon_{p,\ell} + \frac{c''}{p^2}$$

where c'-c" are some constants independent of j, k, ℓ, p. In virtue of (3.2.6-8) we have

$$(3.2.9)\quad \varliminf_{t_j \to \infty} \int_0^{t_j} \varphi'(|P_\ell u(\tau)|^2)[(f, u(\tau)) - \nu \| P_k u(\tau)\|^2] d\tau \geq$$

$$\geq -c(1+\lambda_k) \varliminf_{j\to\infty} T_{j,\ell} \geq -c(1+\lambda_k)(\varepsilon_{p,\ell} + \frac{c''}{p^2}).$$

But $\varphi'(|P_\ell u|^2)\, [(f,u) - \nu \| P_k u \|^2]$ *belongs to* $\mathcal{C}$, thus (3.2.9) yields

$$\int \varphi'(|P_\ell u|^2)[(f,u) - \nu \| P_k u\|^2]\, d\mu(u) \geq$$

$$\geq -c(1+\lambda_k)(\varepsilon_{p,\ell} + c'' p^{-2}).$$

Letting first $\ell \to \infty$, secondly $p \to \infty$ and finally $k \to \infty$ we obtain

$$(3.2.10)\quad \int \varphi'(|u|^2)[(f,u) - \nu \|u\|^2]\, d\mu(u) \geq 0.$$

In (3.2.10), φ is any non-decreasing twice continuously differentiable function on $[o, \infty)$ such that

$$\sup_{[o,\infty)} |\varphi(\sigma)| + \sup_{[o,\infty)} |\varphi''(\sigma)| < \infty .$$

Consequently (3.2.10) will remain valid also in case φ' is any non-negative continuous function with compact support in $[o, \infty)$; we can therefore easily infer (2.4.4) from (3.2.10).

This achives the proof of the theorem.

3.3. For any set $\omega \subset H$ let denote by $\omega(t_o)$ $(t_o \geqslant 0)$ the set of all values $u(t_o)$ of all solutions $u(t)$ which have their initial date u_o in ω . It can be shown that whenever ω is a Borel subset of H, the set $\omega(t_o)$ is measurable with respect to any Borel measure on H (see [12] , § II,2). This allows us to make the following definition: A Borel probability measure μ on H will be called <u>accretive</u> if

(3.3.1) $$\mu(\omega(t)) \geqslant \mu(\omega) \quad \text{(for all } t \geqslant 0 \text{ and Borel sets } \omega \subset H.)$$

Since for any set $\omega \subset H$ we have

(3.3.2) $$\omega(t_1)(t_2) \subset \omega(t_1+t_2) \qquad (t_1, t_2 \geqslant 0)$$

it results easily that if μ is accretive then actually $\mu(\omega(t))$ is a nondecreasing function of $t \in [o, \infty)$, for any Borel subset ω of H.

In case n = 2 (see Sec. 2.5), then any stationary statistical solution is accretive, since

$$\mu(\omega(t)) = \mu(S(t)\omega) = \mu(S(t)^{-1}(S(t)\omega)) \geqslant \mu(\omega)$$

for any $t \geqslant 0$ and Borel set $\omega \subset H$.

3.4. Theorem. Any time average of an individual solution is accretive.

Proof. Let $u(t)$ be an individual solution, $t_0 > 0$ and K a bounded subset of H, closed in H^{-1}. For $u \in H$ and $t \geq 0$ let $d_t(u)$ denote the distance in H^{-1} from u to $K(t)$; for fixed t, $d_t(u)$ is continuous in the topology induced on H by H^{-1}, thus weakly continuous on the bounded sets of H. Moreover for every $p = 1,2,\ldots$, there exists a q $(= 1,2,\ldots)$ such that

(3.4.1) $$d_{t_0}(u(t+t_0)) \leq \frac{1}{p^2} \quad \text{whenever } d_0(u(t)) \leq \frac{1}{q} \text{ and } t \geq 0.$$

(The proof of this fact will be given in the next section.) Thus if we set

$$\varphi(u) = \begin{cases} 1 & \text{if } d_0(u) \geq \frac{1}{q} \\ q d_0(u) & \text{if } d_0(u) < \frac{1}{q} \end{cases}, \quad \psi(u) = \begin{cases} 1 & \text{if } d_{t_0}(u) \geq \frac{1}{p} \\ p d_{t_0}(u) & \text{if } d_{t_0}(u) < \frac{1}{p}, \end{cases}$$

then

(3.4.2) $$\psi(u(t+t_0)) > \frac{1}{p} \quad \text{implies} \quad \varphi(u(t)) > 1 \quad (\text{for any } t \geq 0)$$

But $\varphi, \psi \in \mathcal{C}$; therefore, for a suitable sequence $\{t_j\}_{j=1}^{\infty}$ converging to ∞, we have

$$\int \psi(u)\,d\mu(u) = \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} \psi(u(t))\,dt = \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} \psi(u(t+t_0))\,dt$$

and

$$\int \varphi(u)\,d\mu(u) = \lim_{j\to\infty} \frac{1}{t_j} \int_0^{t_j} \varphi(u(t))\,dt,$$

where, in virtue of (3.4.2)

$$\frac{1}{t_j}\int_0^{t_j}\psi(u(t+t_0))dt \leq \frac{\text{meas}\{t\in[0,t_j]:\ \psi(u(t+t_0))>1/p\}}{t_j}+$$

$$+\frac{1}{p} \leq \frac{1}{p} + \frac{1}{t_j}\cdot \text{meas}\{t\in[0,t_j]:\ \varphi(u(t))>1\} \leq$$

$$\leq \frac{1}{p} + \frac{1}{t_j}\int_0^{t_j}\varphi(u(t))dt.$$

We can conclude that

$$\int\psi(u)d\mu(u) \leq \frac{1}{p} + \int\varphi(u)d\mu(u) \leq \frac{1}{p} + \mu(H\setminus K).$$

Letting $p\to\infty$ we finally obtain

(3.4.3) $\quad \mu(H\setminus K(t_0)) \leq \mu(H\setminus K)$, i.e. $\mu(K)\leq\mu$

Let ω be any Borel subset of H. Then there exists a sequence $K_1, K_2, \ldots$ of bounded subsets of H, closed in H^{-1} such that $\mu(\omega\setminus\bigcup_{j=1}^{\infty}K_j)=0$, because μ regarded, as Borel measure on H^{-1}, is regular (see [8] , Chap.III, §.9); consequently

$$\mu(\omega) = \lim_{j\to\infty}\mu(K_j) \leq \overline{\lim_{j\to\infty}}\ \mu(K_j(t_0)) \leq \mu(\omega(t_0)),$$

where we used the obvious property $K_j(t_0)\leq\omega(t_0)$ $(j=1,2,\ldots)$.

This achives the proof of the theorem.

3.5. Proof of (3.4.1). Suppose that the property (3.4.1) does not hold. Then there exist $t_k\in[0,\infty)$ $(k=1,2..)$ such that

(3.5.1) $\quad d_{t_0}(u(t_k+t_0))>\frac{1}{p}$ and $d_0(u(t_k))\to 0$ for $k\to\infty$;

since $d_{t_0}(u(t+t_0))$ and $d_0(u(t))$ are continuous (because $u(t)$ is weakly continuous in H), we can suppose that $u(t)$ is

strongly continuous from the right at each $t_k (k=1,2,\ldots)$ (and that $\sup_{k=1,2,\ldots} \|u(t_k)\| < \infty$).
Therefore if $v_k(t) = u(t+t_k)$ for $t \in [0, \infty)$ then $v_k(t)$ is an individual solution of (2.3.2) for any $k=1,2,\ldots$.
Therefore, by (3.1.1) and (3.2.1), we have

$$(3.5.2)\begin{cases} |v_k(t)| \le \rho = (|u_0|^2 + \nu^{-2}\lambda_1^{-2}|f|^2)^{1/2} & (t \ge 0), \\ \int_0^t \|v_k(\tau)\|^2 d\tau \le \nu^{-1}\rho + \nu^{-2}\lambda_1^{-1}|f|^2 t & (t \ge 0), \end{cases}$$

for any $k=1,2,\ldots$. Moreover for $0 \le t_1 \le t_2 < \infty$ we have

$$\|v_k(t_2) - v_k(t_1)\|_{H^{-1}} \le \int_{t_1}^{t_2} \|v_k'(t)\|_{H^{-1}} dt = \int_{t_1}^{t_2} \|A(v_k(t))\|_{H^{-1}} dt \le$$

$$\le \nu \int_{t_1}^{t_2} \|v_k(t)\| dt + c_3 \int_{t_1}^{t_2} |v_k(t)|^{1/2} \|v_k(t)\|^{3/2} dt +$$

$$+ \lambda_1^{-1/2} |f| (t_2 - t_1) \le \nu (t_2 - t_1)^{1/2} \left(\int_{t_1}^{t_2} \|v_k(t)\|^2 dt \right)^{1/2} +$$

$$+ c_3 \rho^{1/2} (t_2 - t_1)^{1/4} \left(\int_{t_1}^{t_2} \|v_k(t)\|^2 dt \right)^{3/4} + \lambda_1^{-1/4} |f| (t_2 - t_1),$$

from which we can infer that $v_k(t) (k=1,2,\ldots)$ are equicontinuous (on any compact interval $\subseteq [0, \infty)$) as H^{-1}-valued functions. Since $\{u \in H : |u| \le \rho\}$ is compact in H^{-1}, the Arzela-Ascoli theorem allows us to select a subsequence $\{v_{k_j}(t)\}_{k_1 < k_2 < \cdots}$ converging in H^{-1}, uniformly on every compact interval $\subset [0, \infty)$. Let $v(t) = \lim_{j \to \infty} v_{k_j}(t)$ for $t \ge 0$, where the limit is taken in H^{-1}.
Since on bounded subsets of H, the strong topology of H^{-1} and

the weak topology of H coincide, $v(t)$ is an H-valued weakly continuous function on $[0, \infty)$. Moreover in virtue of (3.5.2) we can also suppose that $\{v_{k_j}(t)\}_{k_j}$ is weakly convergent in $L^2(0,t;H^1)$ for all $t > 0$. Using these properties we can infer now, as in the proofs of the existence of individual solutions (see [23], Ch. I, § 6 and [12], §.II.2), that $v(t)$ is an individual solution of (2.3.2).

Since $v_k(0) = u(t_k)$ and $d_0(u(t_k)) \longrightarrow 0$, it results $v(0) \in K$. Therefore $v(t_0) \in K(t_0)$. Consequently, because $v_k(t_0) \longrightarrow v(t_0)$ in H^{-1}, we obtain (see (3.5.1))

$$\frac{1}{p} \leq d_{t_0}(u(t_{k_j}+t_0)) = d_{t_0}(v_{k_j}(t_0)) \leq \|v_{k_j}(t_0) - v(t_0)\|_{H^{-1}} \longrightarrow 0,$$

that is, a contradiction.

This conclude our proof.

3.6. Let $n = 3$ and define $t(u_0)$ by

$$(3.6.1) \quad t(u_0) = \begin{cases} \sup\{t_0 : u_0 \in D_{RS(t_0)}\} & \text{if } u_0 \in H^1, \\ 0 & \text{if } u_0 \in H \setminus H^1. \end{cases}$$

Then (see Sec. 2.6), $t(u)$ is a Borel function on H, since $\{u: t(u) > t_0\} = D_{RS(t_0)}$ is open in H^1 (for any $t_0 > 0$) and thus a Borel set in H.

3.7. _Proposition._ _Let_ μ _be an accretive Borel probability on_ H. _Then_

(3.7.1) _either_ $\int \frac{1}{t(u)} d\mu(u) = \infty$ _or_ $\int \frac{1}{t(u)} d\mu(u) = 0.$

In the last case, μ _is invariant with respect to_ $\{RS(t)\}_{t \geq 0}$.

Proof. Let us suppose that

(3.7.2) $$I = \int \frac{1}{t(u)}\, d\mu(u) < \infty$$

and let t_0 be fixed, $t_0 > 0$. It is clear that

$$I = \lim_{k\to\infty} \sum_{j=k+1}^{\infty} \frac{k}{(j-k)t_0}\, \mu\left(\left\{u \in H : \frac{j-k-1}{k} t_0 < t(u) \le \frac{j-k}{k} t_0\right\}\right)$$

and that, for $j \geq k+1$,

$$\left\{u \in H : \frac{j-1}{k} t_0 < t(u) \le \frac{j}{k} t_0\right\}(t_0) \subset \left\{u \in H : \frac{j-k-1}{k} t_0 < t(u) \le \frac{j-k}{k} t_0\right\}.$$

Therefore, by the fact that μ is accretive, we obtain

$$I_k = \sum_{j=k+1}^{\infty} \frac{k}{(j-k)t_0}\, \mu\left(\left\{u \in H : \frac{j-1}{k} t_0 < t(u) \le \frac{j}{k} t_0\right\}\right) \le$$

$$\le \sum_{j=k+1}^{\infty} \frac{k}{(j-k)t_0}\, \mu\left(\left\{u \in H : \frac{j-k-1}{k} t_0 < t(u) \le \frac{j-k}{k} t_0\right\}\right),$$

whence

(3.7.3) $$\int_{\{u \in H : t(u) > t_0\}} [t(u) - t_0]^{-1}\, d\mu(u) = \lim_{k\to\infty} I_k \le I .$$

Integrating (3.7.3) with respect to $t_0 \in [0, T]$, we obtain, by the Fubini theorem,

$$TI \geq \int \left\{ \int_0^T [t(u) - t_0]^{-1} \chi_{\{v \in H : t(v) > t_0\}}(u)\, dt_0 \right\} d\mu(u) =$$

$$= \int \left\{ \int_0^{\min\{T, t(u)\}} [t(u) - t_0]^{-1}\, dt_0 \right\} d\mu(u) =$$

$$= \int_{\{u \in H : t(u) > T\}} \left[\log \frac{t(u)}{t(u) - T}\right] d\mu(u) + \int_{\{u \in H : t(u) \le T\}} \infty \, d\mu(u)$$

which plainly implies that $\mu(\{u \in H : t(u) \le T\}) = 0$. It results readily that $t(u) = \infty$, μ-a.e.; plainly, the last conclusion is equivalent to the second alternative in (3.7.1).

In order to prove the second statement, let ω be a Borel subset of H and let $\omega_0 = RS(t_0)^{-1}\omega$, where $t_0 > 0$. Then

(3.7.4) $\mu(\omega) \geq \mu(RS(t_0)\omega_0) = \mu(\omega_0(t)) \geq \mu(\omega_0) = \mu(RS(t_0)^{-1}\omega)$

and analogously

(3.7.5) $\mu(H\setminus\omega) \geq \mu(RS(t_0)^{-1}(H\setminus\omega))$.

Since

$$H\setminus\{[RS(t_0)^{-1}\omega]\cup[RS(t_0)^{-1}(H\setminus\omega)]\} = H\setminus D_{RS(t_0)}$$

is of null μ-measure, (3.7.4-5) imply that

(3.7.6) $\mu(\omega) = \mu(RS(t_0)^{-1}\omega)$.

Because in (3.7.6), $t_0 > 0$ and the Borel set $\omega \subset H$ are arbitrary, we have thus proved that μ is invariant with respect to $\{RS(t)\}_{t\geq 0}$.

Remark. In virtue of (2.6.1), the second alternative in (3.7.1) will occur whenever

(3.7.7) $\int \|u\|^4 d\mu(u) < \infty$.

This fact was essentially already known by G. Prodi long time ago (see [33]).

3.8. Let $X \subset H$ (or $X \subset H^1$) be given. An individual solution $u(t)$ is weakly, resp. strongly, asymtotically convergent in H (or in H^1) to X if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_G (u(\tau))\, d\tau = 1 \qquad (3.8.1)$$

for every neighbourhood G of X, in the weak, resp. strong topology of H (resp. H^1).

The set $X \subset H$ will be called an asymptotic attractor if

(α) X is weakly closed in H;

($\alpha\alpha$) there exists a weak neighbourhood V_X of X in H such that any individual solution $u(t)$ starting from V_X (i.e. such that $u(0) \in V_X$) weakly asymptotically converges in H to X;

($\alpha\alpha\alpha$) if $Y \subset X$ satisfies the properties (α) and ($\alpha\alpha$), then $Y = X$.

3.9. Theorem. Every asymptotic attractor is the weak closure of the support of an accretive stationary statistical solution.

Proof. Since any individual solution $u(t)$ satisfies

$$\overline{\lim_{t\to\infty}} |u(t)| \leq \frac{|f|}{\nu\lambda_1}, \qquad (3.9.1)$$

the set $Y = X \cap \{u \in H : |u| \leq \nu^{-1} \lambda_1^{-1}|f|\}$ will also enjoy (α), ($\alpha\alpha$). Thus by ($\alpha\alpha\alpha$) we must have $Y = X$, i.e.

$$X \subset \{u \in H : |u| \leq \nu^{-1} \lambda_1^{-1}|f|\}. \qquad (3.9.2)$$

Let $K = \{u \in H : |u| \leq \nu^{-1} \lambda_1^{-1}|f|+1\}$. K is a compact metrizable space in H endowed with its weak topology, thus there exists a countable system $\{G_1, G_2, \ldots, G_p, \ldots\}$ of weakly open sets in H such that $G_1 \cap K$, $G_2 \cap K$, ..., $G_p \cap K$, ... form a basis for the (weak) topology of K. Let us assume that for a fixed G_j, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_{G_j \cap V_X} (u(\tau))\, d\tau = 0 \qquad (3.9.3)$$

for any individual solution $u(t)$. Then it is plain that $X \setminus G_j$ will enjoy (α) and ($\alpha\alpha$; with $V_{X\setminus G_j} = V_X$), thus, by ($\alpha\alpha\alpha$), $X \setminus G_j = X$, i.e. $G_j \cap X = \emptyset$. Therefore for every G_j such that $G_j \cap X \neq \emptyset$ there exists an individual solution $u_j(t)$ satisfying

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int_o^t \chi_{G_j \cap V_X}(u_j(\tau))\, d\tau > 0 \; ; \tag{3.9.4}$$

in virtue of (3.9.1) the $\overline{\lim}$ in (3.9.4) will remain unchange if we replace G_j by $G_j \cap K$. Thus we can infer that

$$\lim_{p\to\infty} \frac{1}{t_{jp}} \int_o^{t_{jp}} \chi_{G_j \cap V_X \cap K}(u_j(\tau))\, d\tau = \chi_j > 0 \tag{3.9.5}$$

for an adequate sequence $t_{j} < t_{j_2} < \ldots \longrightarrow \infty$. By n°3.1-5 we can even assume that

$$\lim_{p\to\infty} \frac{1}{t_{jp}} \int_o^{t_{jp}} \phi(u_j(\tau))\, d\tau = \int \phi(u)\, d\mu_j(u) \tag{3.9.6}$$

for every real (functional ϕ (weakly) continuous on K , where μ_j is an accretive stationary statistical solution. Since individual solutions are (weakly) continuous, it is plain that (replacing, if necessary, $u_j(t)$ by an adequate time translation $u_j(t + t_{oj})$) we can assume $u_j(0) \in V_X$. Let now ϕ be any (weakly) continuous functional on K such that $0 \leqslant \phi \leqslant 1$, $\phi(u) = 1$ in a (weak) neighbourhood $G \cap K$ of X in K . Then by (3.8.1), (3.9.1) and (3.9.6) we have

$$\int \phi\, d\mu_j = \lim_{p\to\infty} \frac{1}{t_{jp}} \int_o^{t_{jp}} \phi(u_j(\tau))\, d\tau \geqslant \lim_{p\to\infty} \frac{1}{t_{jp}} \int_o^{t_{jp}} \chi_G(u_j(\tau))\, d\tau = 1 \quad .$$

In this manner

$$\int \phi\, d\mu_j \geqslant 1 \tag{3.9.7}$$

for any (weakly) continuous functional ϕ on K , $0 \leqslant \phi \leqslant 1$, $\phi = 1$ in a (weak) neighbourhood of X in K . This implies obviously that $\mu_j(X) = 1$. Since X is weakly closed, it is also strongly closed, thus we can infer that

$$\operatorname{supp} \mu_j \subset X \; . \tag{3.9.8}$$

We will prove now that

$$(\operatorname{supp} \mu_j) \cap H_j \neq \emptyset \tag{3.9.9}$$

where H_j denotes the (weak) closure of $G_j \cap K$. (Recall that we are considering the case $G_j \cap X \neq \emptyset$!)

Indeed let ψ_j be a functional as is ϕ in (3.9.7) but with X replaced by H_j in its definition. Then as above we will infer from (3.9.5) (instead of (3.8.1)), (3.9.1) and (3.9.6) that

$$\int \psi_j \, d\mu_j \geqslant x_j \, . \tag{3.9.10}$$

Again, since ψ_j , up to its designed properties, is arbitrary, from (3.9.10) it follows readily that

$$\mu_j(H_j) \geqslant \chi_j \, ,$$

whence also (3.9.9). Let us set

$$\mu = \sum_{G_j \cap X \neq \emptyset} \varepsilon_j \mu_j \quad \text{with} \quad \sum_{G_j \cap X \neq \emptyset} \varepsilon_j = 1 \, , \quad \varepsilon_j > 0 \ (\text{for } G_j \cap X \neq \emptyset) \tag{3.9.11}$$

It is plain that μ is an accretive stationary statistical solution, which by (3.9.8), satisfies $\operatorname{supp} \mu \subset X$. On the other hand (3.9.9) shows that

$$(\operatorname{supp} \mu) \cap H_j \neq \emptyset \quad \text{for any } j \text{ such that } G_j \cap X \neq \emptyset \, .$$

Since $\{H_j \cap X : G_j \cap X \neq \emptyset\}$ is a basis for the (weak) topology of X it results that $\operatorname{supp} \mu$ is weakly dense in X . This achieves the proof.

3.10. Corollary. Any asymptotic attractor bounded in H^1 is the support of an invariant stationary statistical solution ; in case $n=2$, any asymptotic attractor is bounded in H^1 .

Proof. If X is an asymptotic attractor, bounded in H^1 , then (with the notations of n° 3.9) $\operatorname{supp} \mu$ will be bounded in H^1 . In case $n=2$, even without any supplementary assumption on X , $\operatorname{supp} \mu$ will be bounded in H^1 (see n° 2.5). Thus in both cases, $\operatorname{supp} \mu$ is compact in H , thus also a compact subset of K endowed with the weak topology of H . In particular this implies that $\operatorname{supp} \mu$ is weakly cloded thus, by the preceding theorem, $\operatorname{supp} \mu = X$.

§4. Analytic properties

4.1. In order to unify the notation we shall set, in case $n = 2$, $t(u_0) = \infty$ and $RS(t)u_0 = S(t)u_0$ for all $u_0 \in H^1$ and $t \geqslant 0$; this is justified by the fact that in this case $S(t)u_0 \in \in C([0,\infty); H^1)$.

Thus we can state our next results in an unified manner, for both cases $n = 2$ and $n = 3$.

4.2. Lemma. For $u_0 \in H^1$ let

(4.2.1) $t_0(u_0) = \nu^3 \min\left\{ (\nu^{-1}\lambda_1^{-1/2}|f|\quad)^{-4}, \|u_0\|^{-4}\right\}$ (see (2.6.1))

There exists a constant c_{11} such that $t(u_0) \geqslant c_{11}\ t_0(u_0)$. $RS(t)u_0$ is analytic on $(0, c_{11} t_0(u_0)]$ as a D_A - valued function (D_A being normed by $|Au|$) and

(4.2.2)
$$|ARS(t)u_0| \leq c_{12}\nu^{-1}\Big\{\lambda_1^{-1/2}t^{-1} + \nu^{-1}\cdot \Big[\max\{\lambda_1^{-1/2}\nu^{-1}|f|, \|u_0\|\}\Big]^2\Big\}\cdot \max\{\lambda_1^{-1/2}\nu^{-1}|f|, \|u_0\|\} + \nu^{-1}|f| \quad \text{for all } 0 < t \leq c_{11}\, t_0(u_0)$$

(where c_{11-12} depend only on Ω).

Proof. Let $H_{\mathbb{C}}$, resp. $H_{\mathbb{C}}^{\pm 1}$, denote the complexified space of H, resp. $H^{\pm 1}$, that is the space of those $\mathbb{C}^n$-valued distributions $\{u_j + iv_j\}_{j=1}^{n}$ on Ω such that $u_j, v_j \in H$, resp. $u_j, v_j \in H^{\pm 1}$ $(1 \leq j \leq n)$. By linearity, A, P_m and B extend to a selfadjoint operator in H, resp. to the orthonormal projection of H on $\mathbb{C}w_1 + .. + \mathbb{C}w_m$ and respectively to a bilinear operator from $H_{\mathbb{C}}^1 \times H_{\mathbb{C}}^1$ into $H_{\mathbb{C}}^{-1}$.
Consider now in $P_m H_{\mathbb{C}}$ the following differential system

(4.2.3) $$\frac{d}{ds}u_m(s) + \nu A u_m(s) + P_m B(u_m(s), u_m(s)) = P_m f$$

which for small $|\zeta|$ has an analytic solution with values in $P_m H_{\mathbb{C}}$ and such that $u_m(0)=P_m u_0$. For $|\theta| < \pi/2$ and $\zeta = s e^{i\theta}$ we can easily infer from (4.2.3) that

$$\frac{1}{2}\frac{d}{ds}\|u_m(se^{i\theta})\|^2 = \mathrm{Re}\,\left(\left(\frac{du_m(\zeta)}{d\zeta}, u_m(\zeta)\right)\right) =$$

$$= \mathrm{Re}\, e^{i\theta}\left(\frac{du_m(\zeta)}{d\zeta}, Au_m(\zeta)\right) =$$

$$= -\mathrm{Re}\, e^{i\theta}\nu\,|Au_m(\zeta)|^2 - \mathrm{Re}\, e^{i\theta}\left(B(u_m(\zeta),u_m(\zeta)), Au_m(\zeta)\right)$$

$$+ \mathrm{Re}\, e^{i\theta}(f, Au_m(\zeta)) \leq -(\nu\cos\theta)\,|Au_m(\zeta)|^2 +$$

$$+ c_3\,\|u_m(\zeta)\|^{3/2}\,|Au_m(\zeta)|^{3/2} + |f|\cdot|Au_m(\zeta)| \leq$$

$$\leq -\frac{\nu\cos\theta}{2}\,|Au_m(\zeta)|^2 + \frac{|f|^2}{\nu\cos\theta} + \frac{27\,c_3^4}{4\nu^3\cos^3\theta}\,\|u_m(\zeta)\|^6,$$

that is

$$(4.2.4)\quad \frac{d}{ds}\|u_m(\zeta)\|^2 + (\nu\cos\theta)|Au_m(\zeta)|^2 \leq \frac{2|f|^2}{\nu\cos\theta} + \frac{27\,c_3^4}{2\nu^3\cos^3\theta}\cdot\|u_m(\zeta)\|^6,$$

whence

$$\frac{d}{ds}\|u_m(se^{i\theta})\|^2 + (\lambda_1\nu\cos\theta)\|u_m(se^{i\theta})\|^2 \leq \frac{2|f|^2}{\nu\cos\theta} + \frac{27\,c_3^4}{2\nu^3\cos^3\theta}\cdot\|u_m(se^{i\theta})\|^6.$$

Integrating this differential inequality we obtain easily that

$$(4.2.5)\quad \|u_m(se^{i\theta})\|^2 \leq 4\max\left\{\lambda_1^{-1}\nu^{-2}\frac{|f|^2}{\cos^2\theta},\ \|u_0\|^2\right\}$$

as long as

$$(4.2.6)\quad 0 \leq s \leq 8c_{11}(\nu^3\cos^3\theta)\min\left\{\left(\lambda_1^{-1}\nu^{-2}\frac{|f|^2}{\cos^2\theta}\right)^{-2},\ \|u_0\|^{-4}\right\},$$

where c_{11} is a suitable constant depending only on c_3, i.e. only on Ω ; plainly we can suppose that $8c_{11} < c_{10}$ (see (2.6.1)).The preceding conclusion (4.2.5-6) shows that $u_m(\zeta)$ which was defined and analytic in a neighbourhood of $\zeta = 0$, actually extends to an analytic solution of (4.2.3) in a neighbourhood of

(4.2.7) $$\Delta(u_0) = \{ se^{i} \; : \; 0 \leq s \leq s(u_0;\theta) = $$

$$=8c_{11} \; (\nu^3 \cos^3\theta) \; \min \left[\left(\lambda_1^{-1}\nu^{-2}\frac{|f|^2}{\cos^2\theta}\right)^{-2}, \|u_0\|^{-4} \right] ; |\theta| < \frac{\pi}{2} \}.$$

Also by (4.2.5-7) we can extract a subsequence $\{u_{m_j}(\zeta)\}_{j=1}^{\infty}$ such that it converges in $H_{\mathbb{C}}$ (by the vector version of the classical Vitali's theorem, which can be applied since $\{u \in H^1_{\mathbb{C}} \; : \; \|u\| \leq \rho\}$ is compact in $H_{\mathbb{C}}$ for any $0 \leq \rho < \infty$), uniformly on every compact set included in the interior $\Delta(u_0)^0$ of $\Delta(u_0)$, to an analytic $H_{\mathbb{C}}$-valued function $u_0(\zeta)$. But we can prove, as in [10], § 4, Sec. 2,c), that the sequence $\{u_m(t)\}$ converges in H, uniformly on $[0, s(u_0;0)]$ to $RS(t)u_0$. We can therefore infer that $u_0(\zeta)$ is the analytic $H_{\mathbb{C}}$-valued extension to $\Delta(u_0)^0$ of $RS(t)u_0$. This uniqueness of $u_0(\zeta)$ implies now that actually $\{u_m(\zeta)\}_{m=1}^{\infty}$ is convergent in $H_{\mathbb{C}}$, uniformly on every compact subset of $\Delta(u_0)^0$, to $u_0(\zeta)$. Therefore the same is true for the following convergence in $H_{\mathbb{C}}$

$$AP_k u_m(\zeta) \longrightarrow AP_k u_0(\zeta) \text{ for } m \to \infty,$$

where $k = 1,2,\ldots$ is fixed. It results that for any compact set $K \subset (u_0)^0$ we have, by (4.2.4-7),

$$\int_K |AP_k u_0(\zeta)|^2 |\zeta| d|\zeta| d\theta = \lim_{m\to\infty} \int_K |AP_k u_m(\zeta)|^2 |\zeta| d|\zeta| d\theta \leq$$

$$\leq \lim_{m\to\infty} \int_K |Au_m(\zeta)|^2 |\zeta| d|\zeta| d\theta \leq$$

$$\leq c(K) \cdot \sup\left\{ \int_0^{s(\theta)} |Au_m(se^{i\theta})|^2 ds : m=1,2,\ldots, \text{ the ray } se^{i\theta}\ (s>0) \text{ intersects } K \right\} \leq$$

$$\leq c(K;\nu,f)(1+\|u_0\|^2)^3$$

where $c(K)$, resp. $c(K;\nu,f)$ are some adequate constants depending only on K, resp. K, ν and f. Thus we infer that

(4.2.8) $$\sup_{k=1,2,\ldots} \int_K |AP_k u_0(\zeta)|^2 |\zeta| d|\zeta| d\theta < \infty$$

The sequence $\{|AP_k u_0(\zeta)|^2\}_{k=1}^{\infty}$ being increasing it results that

(4.2.9) $$\lim_{k\to\infty} |AP_k u_0(\zeta)|^2 < \infty \quad \text{a.e. on } K.$$

It is easy to check that if for a $v_0 \in H_{\mathbb{C}}$ we have $\lim_{k\to\infty} |AP_k v_0|^2 < \infty$ then $v_0 \in D_A$ and

(4.2.10) $$\lim_{k\to\infty} |Av_0 - AP_k v_0|^2 = \lim_{k\to\infty} (|Av_0|^2 - |AP_k v_0|^2) = 0.$$

In this manner (4.2.9) implies that $u_0(\zeta) \in D_A$ a.e. in $\Delta(u_0)^0$ (since K is an arbitrary compact $\subset \Delta(u_0)^0$). Moreover (4.2.8) and (4.2.10) imply that

$$\int_K |Au_0(\zeta)|^2 |\zeta| d|\zeta| d\theta < \infty$$

and consequently also that

(4.2.11)
$$\int_K |Au_o(\zeta) - AP_k u_o(\zeta)|^2 \, |\zeta| \, d|\zeta| \, d\theta =$$
$$= \int_K \left[|Au_o(\zeta)|^2 - |AP_k u_o(\zeta)|^2 \right] |\zeta| \, d|\zeta| \, d\theta \xrightarrow[k\to\infty]{} 0$$

for any compact $K \subset \Delta(u_o)^o$.

Since $AP_k u_o(\zeta)$ is analytic on $\Delta(u^o)^o$ for any $k=1,2,\ldots$, the convergence (4.2.11) implies that the function $Au_o(\zeta)$ is also an analytic $H_{\mathbb{C}}$ -valued function on the whole $\Delta(u_o)^o$. Thus, in particular $RS(t)u_o$ is a D_A-analytic function on $(0, c_{11} t_o(u_o)] \subset (0, s(u_o;0)) \subset \Delta(u_o)^o$.

In order to obtain the relation (4.2.2), let us first remark that the disk $\{\zeta ; |\zeta - t| \le \frac{1}{2}t\}$ is, for $0 < t \le c_{11} t_o(u_o)$, contained in $\Delta(u_o)^o$, so that for such a t, by the Cauchy formula and (4.1.5),

(4.2.12)
$$\left|\frac{du_o(t)}{dt}\right| = \left|\frac{1}{2\pi i} \int_{|\zeta - t| = \frac{1}{2}t} u_o(\zeta)\zeta^{-2} d\zeta\right| \le \frac{5}{t\lambda_1^{1/2}} \cdot$$
$$\cdot \max\{\lambda_1^{-1/2} \nu^{-1}|f|, \|u_o\|\}.$$

Using now the equation (2.3.6) (which is satisfied everywhere on $(0, s(u_o;o))$ because of the analyticity of $u_o(t), Au_o(t)$ there) and the relations (2.2.3) and (4.1.12) we obtain (again for $0 < t \le c_{11} t_o(u_o)$)

$$\nu|Au_o(t)| \le \left|\frac{du_o(t)}{dt}\right| + |B(u_o(t), u_o(t))| + |f| \le$$
$$\le \left|\frac{du_o(t)}{dt}\right| + c_3 \|u_o(t)\| \cdot |A^{3/4} u_o(t)| + |f| \le$$

$$\leq \left|\frac{du_0(t)}{dt}\right| + c_3 \|u_0(t)\|^{3/2} |Au_0(t)|^{1/2} + |f| \leq$$

$$\leq \left|\frac{du_0(t)}{dt}\right| + \frac{\nu}{2}|Au_0(t)| +$$

$$+ \frac{c_3^2}{2\nu}\|u_0(t)\|^3 + |f|,$$

whence (4.2.2) results in virtue of (4.2.5).

4.3. Corollary. For $u \in H^1$, $RS(t)u$ is a D_A-valued analytic function on $(0, t(u))$. In case $n=2$ and $u \in H$, then $S(t)u$ is also a D_A- valued analytic function on $(0, \infty)$.

The first statement is a direct consequence of the preceding lemma, while for the second one we use the fact that in case $n=2$ we have (see Sec. 2.5 and 4.1) $S(t+t_0)u = RS(t)S(t_0)u$ for any $u \in H$ and $t, t_0 > 0$.

4.4. Theorem. If for a stationary statistical solution μ we have

(4.4.1) $$\operatorname{supp}\mu \subset \{u \in H^1 : \|u\| \leq b_1\}$$

then

(4.4.2) $$\operatorname{supp}\mu \subset \{u \in D_A : |Au| \leq b_2\}$$

where

(4.4.3) $$b_2 = c_{13}\, \nu \lambda^{1/2} (\max\{\nu^{-2}\lambda_1^{-1/2}|f|, \nu^{-1}b_1\}) \left[\lambda_1^{-1/2}(\max\{\nu^{-2}\lambda_1^{-1/2}|f|, \nu^{-1}b_1\}^2+1\right]^2$$

(here c_{13} is a suitable constant depending only on Ω).

Proof. By Lemma 4.2, if $\|u_0\| \leq b_1$ and

(4.4.4) $$t_1 = c_{11}\,\nu^3 \min\{(\nu^{-1}\lambda_1^{-1/2}|f|)^{-4}, b_1^{-4}\},$$

then

(4.4.5) $$|ARS(t_1)u_0| \leq c_{12}\,\nu^{-2}\{\lambda_1^{-1/2}\nu^{-2}c_{11}^{-1}[\max\{\lambda_1^{-1/2}\nu^{-1}|f|, b_1\}]^2+1\}\cdot$$

$$\cdot[\max\{\lambda_1^{-1/2}\nu^{-1}|f|, b_1\}]^3 + \nu^{-1}|f| = b_2$$

with b_2 as in (4.4.3), where c_{13} is chosen a suitable large constant, depending only c_{12}, thus only on Ω.
Since supp μ is bounded in H^1, μ is invariant with respect to $\{RS(t)\}_{t\geq 0}$ (see Sec. 2.6), hence, by (4.4.5),

$$1 \geq \mu(\{u \in D_A : |Au| \leq b_2\}) = \mu(RS(t_1)^{-1}\{u \in D_A : |Au| \leq b_2\}) \geq$$
$$\geq \mu(\{u \in H^1 : \|u\| \leq b_1\}) = 1 .$$

We can now infer (4.4.3) because $\{u \in D_A : |Au| \leq b_2\}$ is compact in H.

4.5. As a direct consequence of the preceding theorem we obtain readily from (2.5.2) the following

Corollary. Let $n = 2$. Then any stationary statistical solution μ has the support bounded in D_A; more precisely

(4.5.) $$\text{supp}\,\mu \subset \{u \in D_A : |Au| \leq c_{14} \nu^{-1} |f| \exp(c_{15} \nu^{-8} |f|^4)$$

(where c_{14-15} are some suitable constants depending only Ω).

4.6. Another consequence (however less obvious) of Theorem 4.4 is the following

Corollary. Let μ be a stationary statistical solution with a bounded support in H^1. Then for any $t \geq 0$, we have

(4.6.1) $$RS(t)\,\text{supp}\,\mu = \text{supp}\,\mu .$$

Proof. Let b_1 and t_1 be as in (4.4.1), resp. in (4.4.4); then it is sufficient to prove (4.6.1) for $0 \leq t \leq t_1$.
Since, supp μ is closed in H^1 and, by Theorem 4.4, bounded in D_A, supp μ is compact in H^1. But $RS(t)$ is for $0 \leq t \leq t_1$ continuous from $D_{RS(t)}$ (endowed by the H^1-topology) into H^1. But, again in virtue of Theorem 4.4,

$D_{RS(t)} \supset \operatorname{supp}\mu$ for $0 \leq t \leq t_1$. We can therefore infer that $RS(t)\operatorname{supp}\mu$ is compact in H^1, and so much the more closed in H, for any $0 \leq t \leq t_1$. Fix such a t. By the invariance of μ we have $\mu(RS(t)\operatorname{supp}\ ) = 1$, thus $RS(t)\operatorname{supp}\mu \supset \supset \operatorname{supp}\mu$. On the other hand if $G = RS(t)^{-1}(H^1 \setminus \operatorname{supp}\mu)$ then G is open in $D_{RS(t)}$, henceforth in H^1. By the invariance of μ we have $\mu(G)=1$, hence for $K = (H \setminus G) \cap \operatorname{supp}\mu = (H^1 \setminus G) \cap \operatorname{supp}\mu$, which is closed in H^1, we have $\mu(K)=1$. But K, being closed in a compact subset of H^1 (namely $\operatorname{supp}\mu$) is compact in H^1, and so much the more it is closed in H. Therefore $K \supset \operatorname{supp}\mu$ (since $\mu(K)=1$). Consequently $K=\operatorname{supp}\mu$ which obviously implies that $RS(t)\operatorname{supp}\mu \subset \operatorname{supp}\mu$.

4.7. Theorem. Let μ be a stationary statistical solution with a bounded support in H^1. Then on $\operatorname{supp}\mu$ the topologies induced by H, H^1 and D_A coincide. (Thus $\operatorname{supp}\mu$ is also compact in D_A!)

Proof. By Corollary 4.6 and Lemma 4.2 there exists an $\eta > 0$ such that for any $u \in \operatorname{supp}\mu$, $RS(t)u$ can be extended to an $H_{\mathbb{C}}$-valued analytic function $u(\zeta)$ defined on a whole strip $\Sigma = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < 2\eta\}$ and $|Au(\zeta)| \leq \alpha$ for any $\zeta \in \Sigma$ and some constant $\alpha < \infty$ depending on μ, but independent of $u \in \operatorname{supp}\mu$ and $\zeta \in \Sigma$; plainly, we will have also that

$$RS(t_0)u(t) = u(t+t_0) \qquad (t_0 \geq 0, -\infty < t < \infty).$$

Therefore, if $u, v \in \operatorname{supp}\mu$, we can infer, for $w(t) = u(t) - v(t)$ and any $t \in (-\infty, \infty)$, that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2 + \nu|Aw(t)|^2 = -b(u(t), w(t), Aw(t)) -$$

$$- b\,(w(t),v(t),\, Aw(t)) \leq c_3\, \|u(t)\| \cdot |A^{3/4}\, w(t)| \cdot |Aw(t)| +$$

$$+ c_3 \cdot \|w(t)\| \cdot |A^{3/4}\, w(t)| \cdot |Aw(t)| \leq c_3\, \lambda_1^{-1/2} \alpha\, \|w(t)\|^{\frac{1}{2}} \cdot$$

$$\cdot |Aw(t)|^{3/2} + c_3\, \lambda_1^{-1/4} \alpha\, \|w(t)\| \cdot |Aw(t)| \leq$$

$$\leq \gamma \|w(t)\|^2 + \frac{\nu}{2} |Aw(t)|^2$$

where γ is a constant (with respect to $u,v \in \operatorname{supp}\mu$ and $-\infty < t < \infty$). Thus we can easily conclude, first

$$\|w(t)\|^2 \leq e^{2\gamma(t-t_0)}\, \|w(t_0)\|^2 \qquad (-\infty < t_0 \leq t < \infty)$$

and then, secondly,

$$\int_{t_0}^{t} |Aw(\tau)|^2 d\tau \leq \nu^{-1} e^{2\gamma(t-t_0)}\, \|w(t_0)\|^2 \quad (-\infty < t_0 \leq t < \infty);$$

this last relation obviously implies

$$(4.7.1) \qquad \int_0^{\eta} |Aw(\tau)|^2 d\tau \leq \gamma_1\, |w(o)|$$

where $\gamma_1 = 2\,\nu^{-1} \alpha\, e^{2\gamma\eta}$. Consider, for $|\zeta| \leq \eta$,

$$F(\zeta) = \int_0^{\zeta} Aw(z)\,dz.$$

Then $F(\zeta)$ is analytic in $\{\zeta : |\zeta| < \eta\}$, $|F(\zeta)| \leq 2\alpha$ for $|\zeta| \leq \eta$ and

$$(4.7.2) \qquad |F(t)| \leq \gamma_1^{1/2}\, \eta^{1/2}\, |w(o)|^{1/2} \quad \text{for} \quad 0 \leq t \leq \eta.$$

Let ω denote the maximum on the circle $\{\zeta : |\zeta| = \frac{1}{2}\eta\}$ of the harmonic function $\omega(\zeta)$ in $\{\zeta : |\zeta| < \eta, \zeta \notin [0,1]\}$ such that $\omega(\zeta) = 0$ on $\{\zeta : |\zeta| = \eta\}$ and $\omega(\zeta) = 1$ for $\zeta \in [0,1)$. Then the Nevanlinna's classical maximum principle yields

$$|F(\zeta)| \leq [\max\{1, (2\alpha)^{1-\omega}\}] \cdot [\min\{1, (\gamma_1^{1/2}\, \eta^{1/2}\, |w(0)|^{1/2})^{\omega}\}]$$

for all $\zeta, |\zeta| = \frac{1}{2}\eta$. Thus (with suitable constants γ_2, γ_3)

$$|Aw(o)| = |F'(o)| \leq \eta^{-1}\gamma_2 |w(o)|^{\omega/2} = \gamma_3 |w(o)|^{\omega/2}$$

whenever $|w(o)| < \gamma_1^{-1}\eta^{-1}$. We can finally conclude that for some positive constants γ_4 (depending on Ω, ν, f; and on μ in case n = 3!) and $\omega > 0$, we have

(4.7.3) $|Au - Av| \leq \gamma_4 |u-v|^{\omega/2}$ for all $u, v \in \operatorname{supp}\mu$.

It is plain that (4.7.3) is sufficient for the validity of the theorem.

4.8. As a direct consequence of Corollary 3.10 and Theorem 4.7 we have the following

Corollary. If X is an asymptotic attractor bounded in H^2, then it is already bounded in $\mathcal{D}_A$ and the topologies induced by H, H^1 and $\mathcal{D}_A$ on X coincide.

4.9. In the case n=2, Theorem 4.7 and Corollary 4.8 concern any stationary statistical solution, resp. any asymptotic attractor. In the case n=3, these proposition concern only those stationary statistical solutions, resp. asymptotic attractors, which have bounded supports in H^1, resp. are bounded in H^1. Moreover as far as we are concerned the existence of nontrivial such entities is in this case (n=3) not known. However we have the following

Proposition. Let n = 3 and let μ be a time average of an individual solution (see sec. 3.1-3). Then

(4.9.1) $$\int |Au|^{2/5}\, d\mu(u) < \infty$$

(Thus, so much the more,

(4.9.2) $$\mu(\mathcal{D}_A) = 1 \text{ .)}$$

Proof. Let u(t) be a fixed individual solution. For every $t_o \in (0,\infty)$ such that $\|u(t_o)\| < \infty$ (i.e. $u(t_o) \in H^1$) we have, by virtue of Lemma 4.2,

$$|Au(t)| \leq c'_{12}\left[((t-t_o)^{-1}+1+\|u(t_o)\|^2)(1+\|u(t_o)\|)+1\right]$$

for all t such that

$$t_o < t \leq t_o + c'_{11}(1+\|u(t_o)\|^4)^{-1}$$

where c'_{11-12} are constants independent of t_o and t (but depending on Ω, ν and f). We can easily infer that if for some $p = 1,2,\ldots$ we have

$$\|u(t_o)\|^2 < p \tag{4.9.3}$$

then

$$|Au(t_o+\tau_p)|^{2/5} \leq c''_{12}(1+p) \ , \tag{4.9.4}$$

where

$$\tau_p = c'_{11}(1+p^2)^{-1} \tag{4.9.5}$$

and c''_{12} is a constant independent of t_o and p. We set

$$\alpha_p(t) = \{0 \leq t_o \leq t : p - 7 \leq \|u(t_o)\|^2 < p\}$$

and

$$\beta_p(t) = [\alpha_p(t)+\tau_p] \cap [0,t] \ .$$

Then

$$\text{meas}\ \alpha_p(t) \geq \text{meas}\ \beta_p(t) \geq \text{meas}\ \alpha_p(t) - \tau_p \ . \tag{4.9.6}$$

Taking into account (4.9.3-6) we obtain (for any $m = 1,2,\ldots$)

$$\frac{1}{t}\int_0^t |AP_m u(\tau)|^{2/5}\,d\tau = \frac{1}{t}\int_{\bigcup_{p=1}^{\infty}\beta_p(t)} |AP_m u(\tau)|^{2/5}\,d\tau + \frac{1}{t}\int_{[0,t]\setminus\bigcup_{p=1}^{\infty}\beta_p(t)} |AP_m u(\tau)|^{2/5}\,d\tau$$

$$\leq \frac{1}{t}\sum_{p=1}^{\infty}\int_{\beta_p(t)} |Au(\tau)|^{2/5}\,d\tau$$

$$+ \frac{\lambda_m^{2/5}}{t}\sup_{0\leq\tau\leq t}|u(\tau)|^{2/5}\,\text{meas}([0,t]\setminus\bigcup_{p=1}^{\infty}\beta_p(t))$$

$$\frac{1}{t}\int_0^t |AP_m u(\tau)|^{2/5} d\tau \leq \frac{1}{t}\sum_{p=1}^{\infty} c''_{12}(1+p) \text{ meas } \beta_p(t) + \frac{\lambda_m^{2/5}}{t}\left(|u(0)|^2 + \frac{|f|^2}{\nu^2\lambda_1^2}\right)^{1/5}$$

$$\cdot \text{ meas}\left([0,t]\setminus \bigcup_{p=1}^{\infty} \beta_p(t)\right)$$

$$= \frac{c''_{12}}{t}\sum_{p=1}^{\infty} (1+p) \text{ meas } \beta_p(t) + \frac{c'''_{12}}{t}\left(t - \sum_{p=1}^{\infty} \text{meas } \beta_p(t)\right)$$

$$\leq \frac{c''_{12}}{t}\sum_{p=1}^{\infty} (1+p) \text{ meas } \alpha_p(t) + \frac{c'''_{12}}{t}\left[t - \sum_{p=1}^{\infty} (\text{meas } \alpha_p(t) - \tau_p)\right]$$

$$= \frac{c''_{12}}{t}\left[2t + \sum_{p=1}^{\infty} (p-1) \text{ meas } \alpha_p(t)\right] + \frac{c'''_{12}}{t}\sum_{p=1}^{\infty} \tau_p$$

$$\leq c''_{12}\left(2 + \frac{1}{t}\int_0^t \|u(\tau)\|^2 d\tau\right) + \frac{c''''_{12}}{t}$$

where $c'''_{12}-''''$ are two suitable constants independent of t (depending however on Ω, ν, f, $u(0)$ and m). Using (3.2.1) we finally obtain

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t |AP_m u(t)|^{2/5} d\tau \leq c''_{12}\left(2 + \frac{|f|^2}{\lambda_1\nu^2}\right) \tag{4.9.7}$$

But $\phi(u) = |AP_m u|^{2/5}$ is weakly continuous on any bounded set of H, so that from (4.9.7) we can readily infer that

$$\int |AP_m u|^{2/5} d\mu(u) \leq c''_{12}\left(2 + \frac{|f|^2}{\lambda_1\nu^2}\right) \tag{4.9.8}$$

for every time average μ of $u(t)$ and every $m = 1,2,\ldots$ (4.9.1) follows obviously from (4.9.8), by letting $m \to \infty$.

Remark. The preceding proof shows that the integral in (4.9.1) is bounded by a constant (for instance $c''_{12}(2+\lambda^{-1/2}\nu^{-2}|f|^{-2}$; see (4.9.8)) which depends only on Ω, ν and f but neither on the individual solution $u(t)$ nor on its time average μ

4.10. Finally let us also remark a useful proprety of $S(f;\nu)$ which is a direct consequence of Lemma 4.2, the estimation (2.3.7) and the fact that $RS(t)u_o = u_o$, $(t\geq 0)$, for any $u_o \in S(f;\nu)$, namely :

$$\sup\{|Au| : u \in S(t;\nu)\} < \infty . \tag{4.10.1}$$

§.5. Structure of stationary statistical solutions.

5.1. We shall denote by $\mathcal{S}_a(f;\nu)$ the set formed by all accretive stationary solutions ; obviously

$$\mathcal{S}_a(f;\nu) \subset \mathcal{S}(f;\nu) \tag{5.1.1}$$

and $\mathcal{S}_a(f;\nu) = \mathcal{S}(f;\nu)$ if $n=2$. Let moreover $K_o = \{u \in H : |u| \leq \nu^{-1} \lambda_1^{-1}|f|\}$ be endowed with the weak topology of H and let $C(K_o)$ denote the space of all real continuous functions on K_o . Plainly we can identify in a natural way (as we did for instance in sec. 3.1), the sets $\mathcal{S}(f;\nu)$ and $\mathcal{S}_a(f;\nu)$ with subsets of the dual $C(K_o)'$ of $C(K_o)$. Moreover we endow $C(K_o)'$, with its $\sigma(C(K_o)',C(K_o))$ topology.

5.2. <u>Proposition</u>. <u>$\mathcal{S}(f;\nu)$ and $\mathcal{S}_a(f;\nu)$ are convex compact subsets of</u> $C(K_o)'$.

<u>Proof</u>. That $\mathcal{S}(f;\nu)$ and $\mathcal{S}_a(f;\nu)$ are convex is obvious. Since $\mathcal{S}(f;\nu) \subset M_1 = \{\mu \in C(K_o)' : \|\mu\| \leq 1\}$ and the latter is compact in $C(K_o)'$, it remains to prove that $\mathcal{S}(f;\nu)$ and $\mathcal{S}_a(f;\nu)$ are closed. Using the fact that M_1 is metrizable, we can consider the cases when $\mu_j \longrightarrow \mu$ in M_1 and $\{\mu_j\}_{j=1}^{\infty} \subset \mathcal{S}(f;\nu)$ or $\mathcal{S}_a(f;\nu)$. In the first case, since for all fixed $k, m = 1,2\ldots$ the restriction to K_o of the functionals $\phi_{km}(u) = \langle A(P_k u), P_m \phi'(P_m u)\rangle$ belong also to $C(K_o)$ for any test functional ϕ , we can infer that

$$\int \langle A(P_k u),P_m\phi'(P_m u)\rangle \, d\mu(u) = \lim_{j\to\infty} \int \langle A(P_k u),P_m\phi'(P_m u)\rangle \, d\mu_j(u) \; . \tag{5.2.1}$$

On the other hand for $k \geq m$ (see sec. 3.2; formula (3.2.5))

$$|\langle A(P_k u),P_m\phi'(P_m u)\rangle - \langle A(u),P_m\phi'(P_m u)\rangle| \leq 2\, c_3' \, \lambda_m^{1/4} \, \lambda_{k+1}^{-1/2} \|u\|^2 \quad (u \in H^1) \tag{5.2.2}$$

where c_3' is (as in sec. 3.2) a constant depending only on Ω and ϕ . But, since (the restriction to K_o of) $\phi_k(u) = \|P_k u\|^2$ also belongs to $C(K_o)$ we have (see (2.4.5))

$$\int \|P_k u\|^2 \, d\mu(u) = \lim_{j\to\infty} \int \|P_k u\|^2 \, d\mu_j(u) \leq \nu^{-2} \lambda_1^{-1} |f|^2$$

whence (letting $k \longrightarrow \infty$)

$$\int \|u\|^2 \, d\mu(u) \leq \nu^{-2} \lambda_1^{-1} |f|^2 \; . \tag{5.2.3}$$

Taking into account (5.2.1-3), (2.4.5), (2.4.3) (the last two for μ_j) and the fact that

$$\phi'_{\Box}(P_m \Box) = P_m \, \phi'_{\Box}(\boxed{P_m u}) \qquad (u \in H^1)$$

we obtain at once

$$\left| \int \langle A(P_k u), P_m \phi'(P_m u) \rangle d\mu(u) \right| \leqslant \frac{4c_3' \lambda_m^{1/4}}{\lambda_{k+1}^{1/2}} \; \frac{|f|^2}{\nu^2 \lambda_1}$$

whenever $k \geqslant m$. Letting $k \longrightarrow \infty$ and then $m \longrightarrow \infty$ we obtain the relation (2.4.3) for μ . The relation (2.4.4) for μ can be obtained easily using the functionals

$$\phi_{\ell,k}(u) = \phi'(|P_\ell u|^2)\left[(f,u) - \|P_k u\|^2\right] \quad (u \in K) ,$$

which belong to $C(K_o)$ $(k,\ell = 1,2,\ldots)$ in a manner similar to that in sec. 3.2.

It remains to prove that if $\{\mu_j\}_{j=1}^{\infty} \subset \mathcal{S}_a(f;\nu)$ then μ is accretive. By an argument similar to that at the end of sec. 3.4 we can infer that it is sufficient to prove

(5.2.4) $$\mu(K(t)) \geqslant \mu(K) \qquad (t \geqslant 0)$$

for any bounded subset of H , closed in H^{-1} . For p,q = 1,2,... , let us set

$$K_{p,q} = \{u \in H^1 : \|u\| \leqslant p \ , \ d_K(u) \leqslant \frac{1}{q}\}$$

where for a subset $X \subset H$ and $u \in H$ we denote by $d_X(u)$ the distance in H^{-1} from u to X . Obviously $K_{p,q}$ is compact in H (thus also closed in H^{-1}) ; consequently by [12], Lemma II.2.5, $K_{p,q}(t)$ is closed in H^{-1} . Let ψ_r denote the function (from $[0,\infty)$ to $[0,\infty)$) defined by $\xi \longmapsto (1-k\xi)^+$ (k=1,2...) , and let

$$K_q = \bigcup_{p=1}^{\infty} K_{p,q} .$$

Then $\psi_r(d_{K_{p,q}(t)}(u))$ and $\psi_r(d_{K_q}(u))$ (as function of $u \in K_o$) belong to $\mathcal{C}(K_o)$ for every $k = 1,2,\ldots)$ so that we have

$$\int \psi_r(d_{K_{p,q}(t)}(u))d\mu(u) = \lim_{j\to\infty} \int \psi_r(d_{K_{p,q}(t)}(u))d\mu_j(u)$$

$$\geq \limsup_{j\to\infty} \mu_j(K_{p,q}(t)) \geq \limsup_{j\to\infty} [\mu_j(K_q) - \mu(\{u \in H^1 : \|u\| \geq p\})]$$

$$\geq \limsup_{j\to\infty} \mu_j(K_q) - \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2 \geq \limsup_{j\to\infty} \int \psi_q(d_K(u))d\mu_j(u) - \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2$$

$$= \int \psi_q(d_K(u))d\mu(u) - \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2 \geq \mu(K) - \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2$$

that is

$$\mu(K) \leq \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2 + \int \psi_r(d_{K_{p,q}(t)}(u))\, d\mu(u) .$$

Letting $r \longrightarrow \infty$ and taking into account that $K_{p,q}(t)$ is closed in H^{-1} we obtain

$$\mu(K) \leq \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2 + \mu(K_{p,q}(t)) . \tag{5.2.5}$$

It is obvious that

$$K_{p,1}(t) \geq K_{p,2}(t) \geq \dots . \tag{5.2.6}$$

In virtue of [12], Lemma 2.5, we have also

$$\bigcap_{q=1}^{\infty} K_{p,q}(t) \subset K(t) . \tag{5.2.7}$$

Making $q \longrightarrow \infty$ in (5.2.5) and taking into account (5.2.6-7) we obtain

$$\mu(K) \leq \frac{1}{p^2} \nu^{-2} \lambda_1^{-1}|f|^2 + \mu(K(t)) ,$$

whence (5.2.4) follows at once (by letting $p \longrightarrow \infty$) . This finishes the proof.

5.3. <u>Theorem</u>. <u>If</u>

(5.3.1) supp μ <u>is bounded in</u> H^1 <u>for any</u> $\mu \in \mathcal{S}(f;\nu)$ (resp. $\mathcal{S}_a(f;\nu)$) <u>then</u>

(5.3.2) supp $\mu \subset S(f;\nu)$ <u>for any</u> $\mu \in \mathcal{S}(f;\nu)$ (resp. $\mathcal{S}_a(f;\nu)$) .

[In other words if every stationary statistical (resp. accretive stationary statistical) solution has a bounded support in H^1, then actually the supports are included in the set of the stationary individual solutions. Let us also emphasize that the assumption (5.3.1) always holds for the case $n = 2$, i.e. for plane fluids ; see (2.5.2).]

Proof. Under the assumption (5.3.1), all measures $\mu \in \mathcal{S}(f;\nu)$ (resp. $\mathcal{S}_a(f;\nu)$) are invariant with respect to $\{RS(t)\}_{t\geq 0}$ (see sec. 3.7-8). But then the extremal elements of $\mathcal{S}(f;\nu)$ (resp. $\mathcal{S}_a(f;\nu)$) are these measures $\mu \in \mathcal{S}(f;\nu)$ (resp. $\mathcal{S}_a(f;\nu)$) for which the functional flow $\{RS(t)\}_{t\geq 0}$ is ergodic (see [7], p.). Let μ be such a measure. In virtue of sec. 4.2 and 4.6, for every $u \in \operatorname{supp} \mu$ we can extend $RS(t)u$ to a $\mathcal{D}_{A_{\mathbb{C}}}$ -valued analytic function $RS(\zeta)u$ defined in a strip $\sigma = \{\zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < \eta\}$ (where η depends on Ω, ν; f and $b_1 = \{\|u\| : u \in \operatorname{supp} \mu\}$, but is independent of $u \in \operatorname{supp} \mu$) satisfying also the following properties :

$$(5.3.3) \qquad |A_{\mathbb{C}} RS(\zeta)u| \leq b_2 + 1 \qquad (\zeta \in \sigma)$$

(see sec. 4.4 and again the proof in sec. 4.2) and

$$(5.3.4) \qquad RS(t)u \in \operatorname{supp} \mu \qquad (-\infty < t < \infty) .$$

By analyticity it follows at once

$$RS(t)u = (RS(-t)|\operatorname{supp} \mu)^{-1}u \qquad (t<0,\ u \in \operatorname{supp} \mu) ,$$

so that $\{RS(t) \operatorname{supp} \mu\}_{-\infty<t<\infty}$ forms a group of homeomorphic maps of $\operatorname{supp} \mu$, with respect to which μ is still invariant. We shall consider now some remarkable functions (the first one suggested by [3], [4]), namely

$$(5.3.5) \qquad \phi(t) = \int_{\operatorname{supp} \mu} (RS(t)u,u)\, d\mu(u) \qquad (-\infty<t<\infty) ,$$

$$(5.3.4) \qquad \phi_1(t) = \int_{\operatorname{supp} \mu} ((RS(t)u,u))\, d\mu(u) \qquad (-\infty<t<\infty)$$

and

$$(5.3.5) \qquad \psi(s,t) = \int_{\operatorname{supp} \mu} (B(RS(t)u,RS(t)u),u) \qquad (-\infty<s,t<\infty) .$$

In virtue of the properties of $RS(\zeta)u$ for $u \in \operatorname{supp} \mu$, these functions are analytic. Moreover, by the invariance property of μ we have

$$\phi(-t) = \int_{\text{supp } \mu} (RS(-t)u,u)\, d\mu(u)$$

$$= \int_{\text{supp } \mu} (RS(-t)\, RS(t)u,\, RS(t)u)\, d\mu(u)$$

$$= \int_{\text{supp } \mu} (u,RS(t)u)\, d\mu(u) = \phi(t)$$

(and analogously for Φ_1), that is

(5.3.6) $$\phi(t) = \phi(-t),\ \phi_1(t) = \phi_1(-t) \quad (-\infty<t<\infty) ;$$

also $$\psi(s-t,-t) = \int_{\text{supp } \mu} (B(RS(s-t)u,RS(-t)u),u)\, d\mu(u)$$

$$= \int_{\text{supp } \mu} (B(RS(s-t)RS(t)u,RS(-t)RS(t)u),RS(t)u)\, d\mu(u)$$

$$= \int_{\text{supp } \mu} (B(RS(s)u,u),RS(t)u)\, d\mu(u)$$

$$= - \int_{\text{supp } \mu} (B(RS(s)u,RS(t)u),u)\, d\mu(u) = -\psi(s,t) ,$$

(where we used (2.2.2) and (2.2.6))

(5.3.7) $$\psi(s-t,t) - \psi(s,t) = 0 \qquad (-\infty<s,t<\infty) .$$

From (5.3.7) we infer, by recurrence,

(5.3.8) $$\frac{\partial^K}{\partial t^K}\psi(s,t) - (-1)^K \sum_{j=o}^{K} \binom{K}{j} \frac{\partial^K}{\partial \xi^j\, \partial \eta^{K-j}} \psi(\alpha,\beta) = 0 \qquad \alpha = s-t,\ \beta = -t$$

for all $-\infty<s,t<\infty$. But (5.3.1) also yields $\psi(s,0) \equiv 0$, so that for $t=0$ the relation (5.3.8) becomes

(5.3.9) $$\left(\frac{\partial^K}{\partial t^K}\psi\right)(s,0)\left[1+(-1)^K\right] = 0 \qquad (K = 0,1,2,\dots,\ -\infty<s<\infty) .$$

In particular the relations (5.3.9) (for $s=0$) show that the analytic function $\psi(0,t)$ is odd. Thus

$$\psi(t,t) = -\psi(0,-t)$$

is also an odd function ; on the other hand (5.3.6) shows that the functions $\phi(t)$

and $\phi_1(t)$ are even. But, for $t \geq 0$, we have

$$\int_0^t [(f,u) - \nu\phi_1(\tau) - \psi(\tau,\tau)]\, d\tau = \int_0^t \Big[\int_{\text{supp } \mu} (\tfrac{d}{dt} RS(\tau)u,u)\, d\mu(u)\Big]\, d\tau$$
$$= \int_{\text{supp } \mu} \Big[\int_0^t (\tfrac{d}{dt} RS(\tau)u,u)d\tau\Big]\, d\mu(u)$$
$$= \int_{\text{supp } \mu} (RS(t)u,u)\, d\mu(u) - \int |u|^2 d\mu(u)$$

where again we used the analytical properties of $RS(\tau)u$ and sec. 4.4, but also the form (2.3.1), (2.3.6) of the Navier-Stokes equations.

It results that

$$\phi'(t) + \nu\phi_1(t) + \psi(t,t) = (f,\int u\, d\mu(u)) \quad (-\infty<t<\infty) . \tag{5.3.10}$$

Since the function involved in (5.3.10) are analytic, some odd and other even, the equation (5.3.10) splits into its odd and even part. This latter part is

$$\nu\phi_1(t) = (f,\int u\, d\mu(u)) \qquad (-\infty<t<\infty) ,$$

i.e.

$$\nu\int_{\tilde{\text{s}}\text{upp } \mu} ((RS(t)u,u))\, d\mu(u) = (f,\int u\, d\mu(u)) \qquad (-\infty<t<\infty) . \tag{5.3.11}$$

For $t=0$, (5.3.11) yields

$$\nu\int \|u\|^2\, d\mu(u) = (f_1, \int u\, d\mu(u)) . \tag{5.3.12}$$

Moreover, from (5.3.11) it follows easily that

$$\nu\int_{\text{supp } \mu} ((\frac{1}{t}\int_0^t RS(\tau)u\, d\tau,u))d\mu(u) = (f,\int u\, d\mu(u)) \qquad (0\leq t<\infty) . \tag{5.3.13}$$

Now because of the ergodicity we have on supp μ that for every $v \in H^1$

$$\frac{1}{t}\int_0^t ((RS(\tau)u,v))\, d\tau \longrightarrow ((\int w\, d\mu(w),v)) \tag{5.3.14}$$

for all $u \notin E_v$ where $E_v \subset \text{supp } \mu$, $\mu(E_v) = 0$. But, for $u \in \text{supp } \mu$,

$$\{\frac{1}{t}\int_0^t RS(\tau)\, u\, d\tau;\ t\geq 0\}$$

is bounded in H^1 . Therefore if $E = \bigcup_{j=1}^{\infty} E_{w_j}$ we can easily infer that for $t \to \infty$

$$\frac{1}{t}\int_0^t RS(\tau)u \longrightarrow \int w\, d\mu(w) \quad \text{weakly in } H^1 \text{ for all } u \in \operatorname{supp} \mu \setminus E .$$

Thus from (5.3.13) we obtain by Lebesgue's dominated convergence theorem

$$\nu \left\| \int w\, d\mu(w) \right\|^2 = \nu \int \left(\left(\int w\, d\mu(w), u \right) \right) d\mu(u) = \left(f, \int w\, d\mu(w)\right) . \tag{5.3.15}$$

Substracting (5.3.15) form (5.3.12) we obtain

$$\int \|u\|^2\, d\mu = \left\|\int w\, d\mu(w)\right\|^2 , \text{ i.e. } \int \left\|u - \int w\, d\mu(w)\right\|^2 d\mu(u) = 0 .$$

This shows that μ is the Dirac measure concentrated in $u_o = \int w\, d\mu(w)$; consequently (see sec. 4.6, for instance) $u_o \in S(f;\nu)$ (i.e. is an individual stationary solution. The theorem follows now directly from the general barycentric representation theorem of Choquet (see [7], p.).

5.4. <u>Theorem</u>. <u>If the assumption</u> (5.3.1) <u>holds for</u> $\mathcal{S}_a(f;\nu)$ (in particular if $n=2$, i.e. if the fluid is two dimensional), <u>then</u> :

(i) <u>Every asymptotic attractor is included in the set</u> $S(f;\nu)$ <u>of the stationary individual solutions</u>.

(ii) <u>Every individual solution is weakly asymptotically convergent in</u> H <u>to</u> $S(f;\nu)$.

(iii) <u>Every weakly almost periodic individual solution</u> (<u>in particular every periodic or quasi-periodic solution</u>) <u>is stationary</u> (i.e. time-independant).

(An individual solution $u(t)$ is called weakly almost periodic if it is defined on all $(-\infty,\infty)$ and for every $v \in H$, the function $(u(t),v)$ is a real almost periodic function ; see [2]).

<u>Proof</u>. If $X \subset H$ is an symptotic attractor then, by Theorem 3.9, X is the weak closure of $\operatorname{supp} \mu$, $\mu \in \mathcal{S}_a(f;\nu)$, thus, by Theorem 5.3, X is included in the weak closure of $S(f;\nu)$, which being compact in H (see, for instance, (2.3.7)) is also weakly closed, thus $X \subset S(f;\nu)$. This proves the statement (i).

Assume now that there exists an individual solution $u(t)$ which is not asymptotically convergent to $S(f;\nu)$. Then, if $d(u)$ denotes the distance in H^{-1} from $u \in H$ to $S(f;\nu)$, we have

$$\chi = \limsup_{t\to\infty} \frac{1}{t}\int_0^t d(u(\tau))\, d\tau > 0 .$$

Since $d(u) \in \mathscr{C}$, from sec. 3.1, 3.2 and 3.4 it follows that there exists $\mu \in \mathscr{S}_a(f;\nu)$ such that

$$\int d(u)\, d\mu(u) = \chi > 0$$

which shows that $\operatorname{supp} \mu \not\subset S(f;\nu)$, in contradiction with Theorem 5.3 . This proves the statement (ii).

Concerning (iii), the shortest proof runs as follows : let $u(t)$ denote also the extension to the whole $(-\infty,\infty)$ of our individual solution enjoying the property indicated before the proof. Then $u(t)$ extends by continuity to a uniquely determined weakly continuous H-valued function $U(\theta)$ defined on the Bohr compactification $\mathbb{R}^{\wedge\wedge}$ of $\mathbb{R} = (-\infty,\infty)$ (see for instance [2]). It is easy to check that if t measure μ on H is defined by

$$(5.4.1) \qquad \int \phi(u)\, d\mu(u) = \int_{\mathbb{R}^{\wedge\wedge}} \phi(U(\theta))\, d\theta \qquad (\phi \in \mathscr{C}) ,$$

then (see [2]),

$$(5.4.2) \qquad \lim_{t\to\infty} \frac{1}{t} \int_0^t \phi(u(\tau))\, d\tau = \int \phi(u)\, d\mu(u) \qquad (\phi \in \mathscr{C}) ,$$

so that, in virtue of sec. 3.1, 3.2 and 3.4, we can infer that $\mu \in \mathscr{S}_a(f;\nu)$. Therefore, by Theorem 5.3, $\operatorname{supp} \mu \subset S(f;\nu)$, whence

$$(5.4.3) \qquad \int_{\mathbb{R}^{\wedge\wedge}} d(U(\theta))\, d\theta = 0 .$$

It follows that $U(\theta) \in S(f;\nu)$ for all $\theta \in \mathbb{R}^{\wedge\wedge}$; in particular $u(0) \in S(f;\nu)$. But in this case $u(t) = RS(t)\, u(0) = u(0)\,(t \in [0,\infty))$ so that $u(t) \equiv u(0)$ is a stationary individual solution. This finishes the proof of the Theorem.

5.5. Corollary. In case $n=2$, every individual solution is strongly asymptotically convergent in H^1 to $S(f;\nu)$.

Proof. In the case $n=2$, it is known (see [13]) that for every individual solution $u(t)$ the following holds :

$$\limsup_{t\to\infty} \|u(t)\| < \infty$$

From Lemma 4.2 we can thus infer that

(5.5.1) $$\limsup_{t\to\infty} |Au(t)| = \gamma < \infty \quad .$$

Let us denote by $d_1(u)$ the distance in H^1 from $u \in H$ to $S(f;\nu)$ (where we set $d_1(u) = \infty$ if $u \in H\backslash H^1$). Since instead of $u(t)$ we can consider $u(t+1)$, we can assume from the beginning that $u(0) \in H^1$; thus

$$\delta(t) = \frac{1}{t}\int_0^t d_1(u(\tau))\, d\tau \qquad (t\geq 0)$$

will make sense. It will be sufficient to prove that

(5.5.2) $$\delta(t) \longrightarrow 0 \quad \text{for} \quad t \longrightarrow \infty \; .$$

To this purpose we notice that for $u(t)$ there exists $v_t \in S(f;\nu)$ such that $\|u(t) - v_t\|_{H^{-1}} = d(u(t))$ (see the proof of the statement (ii) in sec. 5.4). Therefore

$$d_1(u(t)) \leq \|u(t)-v_t\| \leq \|u(t)-v_t\|_{H^{-1}}^{1/4} \, |A(u(t)-v_t)|^{3/4}$$

$$\leq \; d(u(t))^{1/4}[|Au(t)|+\sup\{|Av| \; : \; v \in S(f;\nu)\}]^{3/4}$$

whence, by Corollary 4.3 and formula (5.5.1), (4.10.1),

$$d_1(u(t)) \leq \gamma_1 \; d(u(t))^{1/4} \qquad (\text{for} \quad t\geq 1)$$

where γ_1 is constant with respect to the time t .

We can infer now that

$$0 \leq \delta(t) \leq \frac{1}{t}\int_0^1 d_1(u(\tau))\, d\tau + \left(\frac{1}{t}\int_0^t d(u(\tau))\, d\tau\right)^{1/4} \gamma_1 \; ,$$

where the latter term tends to 0 (for $t \longrightarrow \infty$) , by Theorem 5.4 (ii). Hence (5.5.2) is valid.

5.6. As we already pointed out the results in sec. 5.3-5 are always valid in case $n=2$, i.e. for two dimensional fluids. In the more interesting case $n=3$, i.e. that of three dimensional fluids (the real ones!) we will now show that the basic assumption (5.3.1) for $\mathcal{S}_a(f;\nu)$ is connected with the preservation of the regularity for the individual solutions. Precisely, we have the following.

5.7. Theorem. Let $n=3$ and assume that

$$t(u) = \infty \quad \text{for every} \quad u \in H^1 \tag{5.7.1}$$

(see (3.6.1)). Then

(i) This assumption (5.3.1) holds for $\mathcal{S}_a(f;\nu)$.

(ii) Every individual solution is strongly asymptotically convergent in H^1 to $S(f;\nu)$.

Proof. Since the proof of statement (ii) is similar to that of Corollary 5.6 (once the statement (i) is proved) we shall omit it. In order to prove the statement (i) we will firstly prove that (5.7.1) implies

$$\sup\{\|RS(t)u\| : 0 \leqslant t \leqslant T,\ \|u\| \leqslant R\} < \infty \tag{5.7.2}$$

for every fixed $T, R \in (0,\infty)$. Let us assume the contrary.Then there exists $u_j \in H^1$, $\|u_j\| \leqslant R$ and $t_j \in (0,T_j]$ $(j=1,2,\dots)$ such that

$$\|RS(t_j)u_j\| \longrightarrow \infty \quad \text{for} \quad j \longrightarrow \infty . \tag{5.7.3}$$

In virtue of Lemma 4.2, there exists $t_o \in (0,T)$ such that

$$\sup\{\|RS(t)u_j\| : 0 \leqslant t \leqslant t_o,\ j=1,2,\dots\} < \infty \tag{5.7.4}$$

$$\sup\{|ARS(t_o)u_j| : j=1,2,\dots\} < \infty . \tag{5.7.5}$$

Plainly, we can also assume that $t_j \longrightarrow t_\infty \in [0,T]$ and that u_j converges weakly in H^1 to some $u_\infty \in H^1$, $\|u_\infty\| \leqslant R$. From (5.7.4) it follows $t_\infty \in [t_o,T]$. On the other hand, by an argument similar to that in sec. 3.5 [16], we can infer that a subsequence of $\{RS(t)u_j\}_{j=1}^{\infty}$ (which will again be denoted as the initial one) is weakly convergent in H , uniformly on every compact interval $\subset [0,\infty)$ (in particular on $[0,T]$) to an individual solution $u(t)$ such that $u(0) = u_\infty$. By the uniqueness theorem for the regular solution and the assumption (5.7.1) we have $u(t) = RS(t)u_\infty$ (for al $t \geqslant 0$) . But since by (5.7.5) $\{RS(t_o)u_j : j=1,2,\dots\}$ is bounded in $\mathcal{D}_A$ it is relatively compact in H^1 . This fact joint to the weak convergence in H of $RS(t_o)u_j$ to $u(t_o)$ yields that

$$\|RS(t_o)u_j - u(t_o)\| \longrightarrow 0 \quad \text{for} \quad j \longrightarrow \infty . \tag{5.7.6}$$

Moreover for $t \in [t_o,T]$ and $v_j(t) = RS(t)u_j - u(t)$ we obtain easily (from (2.3.6), (2.2.3) and (2.2.5))

$$(5.7.7) \qquad \frac{1}{2}\frac{d}{dt}\|v_j\|^2 + \nu|Av_j|^2 = -\langle B(u,v_j),Av_j\rangle - \langle B(v_j,u),Av_j\rangle - \langle B(v_j,v_j),Av_j\rangle$$

$$\leqslant (c_3+c_4)\|u\|\|v_j\|^{1/2}|Av_j|^{3/2} + c_3\|v_j\|^{3/2}|Av_j|^{3/2}$$

$$\leqslant \nu|Av_j|^2 + c(\nu,u_\infty)\|v_j\|^2(1+\|v_j\|^4)$$

where $c(\nu,u_\infty)$ is a constant depending on Ω (by the c_{3-4}), ν and $\max\{\|RS(t)u_\infty\|: t_o \leqslant t \leqslant T\}$. Integrating (5.7.7) we obtain

$$(5.7.8) \qquad \frac{\|v_j(t)\|^4}{1+\|v_j(t)\|^4} \leqslant \frac{\|v_j(t_o)\|^4}{1+\|v_j(t_o)\|^4} \cdot \exp[2c(\nu,u_\infty)(T-t_o)] \quad (t \in [t_o,T]) .$$

Since $t_\infty \in [t_o,T]$, in (5.7.8) we can take $t = t_j$ for j large enough. Letting afterwards $j \longrightarrow \infty$, we obtain in virtue of (5.7.6) that for $j \longrightarrow \infty$

$$\|RS(t_j)u - u(t_\infty)\| \leqslant \|v(t_j)\| + \|u(t_j) - u(t_\infty)\| \longrightarrow \infty$$

in contradiction with (5.7.3). Thus, we conclude that (5.7.2) is valid.

We return now to the proof of the statement (i). Let thus $\mu \in \mathcal{S}_a(f;\nu)$, and let ρ denote the upper bound in (5.7.2) corresponding to the choice $T = 1$ and

$$(5.7.9) \qquad R = (2\lambda_1^{-1}, \nu^{-1}|f|^2 + 1)^{1/2} .$$

Let moreover $u(t)$ be any individual solution starting from $\operatorname{supp}\mu$, i.e. such that $u(0) \in \operatorname{supp}\mu$. Then from (2.3.4) and (2.4.6) it follows plainly

$$\int_0^1 \|u(t)\|^2\,dt \leqslant 2\lambda_1^{-1}\nu^{-1}|f|^2$$

so that the set

$$\{t \in (0,1) : \|u(t)\| \leqslant R\}$$

is of positive measure. Thus we can chose a point t_o in this set at which $u(t)$ is strongly continuous (in H) from the right (see the definition of an individual solution in sec. 2.3). Then, since $u(t+t_o)$ (for $t \geqslant 0$) is also an individual solution (with initial value $u(t_o)$) we have $u(t+t_o) = RS(t)\,u(t_o)$ for all $t \geqslant 0$

and consequently, by (5.7.2),

$$\|u(1)\| = \|RS(1-t_o)\ u(t_o)\| \leq \rho \ .$$

It follows

$$(5.7.10) \qquad (\text{supp}\ \mu)(1) \subset \{u \in H^1 : \|u\| \leq \rho\} \ .$$

Recalling that μ is accretive, we have firstly

$$\mu(\{u \in H^1 : \|u\| \leq \rho\}) = 1$$

and then, secondly

$$\text{supp}\ \mu \subset \{u \in H^1 : \|u\| \leq \rho\} \ ,$$

achieving our proof.

§.6. Structure of the set of stationary individual solutions.

6.1. We want now to describe the structure of the set $S(f;\nu)$ of the stationary individual solutions, i.e. the set of $u \in H^1$ such that

(6.1.1) $$A(u) = \nu Au + B(u,u) - f = 0 .$$

The main results are Theorems 6.3 and 6.8.1.
We recall that for $n = 2$ or 3, by reiteration of the regularity results for the Stokes problem [6] [40] it can be shown (see for instance [41]) that any solution $u \in H^1$ of (6.1.1) actually belongs to D_A ; thus

(6.1.2) $$S(f;\nu) \subset D_A .$$

This result can be also deduced from Lemma 4.2 : let $u_o \in S(f;\nu)$; then $u_o \in \mathbb{H}^1$ and $u_o \equiv RS(t)u_o$, so that $u_o \in D_A$.

6.2. Lemma. *There exists a constant c_1 depending only on Ω and n (n=2,3) such that if*

(6.2.1) $$(m+1)^{2/n} \geq c_1 \frac{|f|^4}{\nu^8}$$

then the restriction of P_m to $S(f;\nu)$ is a one to one mapping.

Proof. We first show that $S(f;\nu)$ is bounded in D_A . Let u belongs to $S(f;\nu)$; taking the scalar product of (6.1.1) with Au , we obtain

$$\nu|Au|^2 = (f,Au) - b(u,u,Au) .$$

If $n=2$ we apply (2.2.4) with $\alpha_1 = \gamma_1 = \gamma_2 = 0$, $\alpha_2 = \beta_2 = \frac{1}{2}$, $\beta_1 = 1$:

$$|b(u,u,Au)| \leq c_2 \|u\|^{3/2} |Au|^{3/2} .$$

For $n=3$, we deduce the same inequality from (2.2.3) written with $\alpha = \frac{1}{2}$, $\beta = \frac{3}{4}$, $\gamma = 0$ and the inequality

$$|A^{3/4}u| \leq c \; |A^{1/2}u|^{1/2}|Au|^{1/2} \leq c'\|u\||Au|^{1/2} .$$

Whence

$$\nu\ |Au|^2 \leqslant |f|\,|Au| + c_2\|u\|^{3/2}|Au|^{3/2}$$

$$\leqslant \text{(from (2.3.7))}$$

$$\leqslant |f|\,|Au| + c\,\frac{|f|^{3/2}}{\nu^{3/2}}\,|Au|^{3/2}$$

$$\leqslant \frac{\nu}{4}\,|Au|^2 + \frac{1}{\nu}|f|^2 + \frac{\nu}{4}|Au|^2 + c'\,\frac{|f|^6}{\nu^9}$$

so that

$$|Au| \leqslant c_3\,\frac{|f|}{\nu}\,(1 + \frac{|f|^2}{\nu^4}), \tag{6.2.2}$$

with some suitable constant c_3 depending only on Ω .

Let now $u, v \in S(f;\nu)$, and let $w = u - v$. We take the scalar product in H of $A(u) - A(v)$ with $Q_m w$ $(Q_m = I - P_m)$ and we obtain (m an arbitrary integer) :

$$\|Q_m w\|^2 = - b(u,u,Q_m w) + b(v,v,Q_m w)$$

$$= - b(u,w,Q_m w) - b(w,v,Q_m w)$$

$$= + b(u,Q_m w,P_m w) + b(P_m w,Q_m w,v) + b(Q_m w,v,Q_m w) \ .$$

For the first two terms we apply the first inequality (2.2.5) ; the third term is majorized with (2.2.3) $(\alpha = \gamma = \frac{3}{8}\ ,\ \beta = \frac{1}{2}\ ,\ |A^{3/8}u| \leqslant c|u|^{1/4}|A^{1/2}u|^{3/4}$ $\leqslant c'|u|^{1/4}\|u\|^{3/4})$.

We arrive to

$$\nu\,\|Q_m w\|^2 \leqslant c(|u|^{1/4}|Au|^{3/4}+|v|^{1/4}|Av|^{3/4})\,\|Q_m w\|\,|P_m w| + c\|v\|\,\|Q_m w\|^{3/2}\ |Q_m w|^{1/2}\ .$$

$$\leqslant \text{(from (2.3.7) and (6.2.2))}$$

$$\leqslant c\,\frac{|f|}{\nu}(1 + \frac{|f|^2}{\nu^4})^{3/4}\,\|Q_m w\|\,|P_m w| + c_4\,\frac{|f|}{\nu}\|Q_m w\|^{3/2}|Q^m w|^{1/2}$$

$$\leqslant \frac{\nu}{2}\,\|Q_m w\| + c'\,\frac{|f|^2}{\nu^3}\,(1 + \frac{|f|^2}{\nu^4})^{3/2}\ |P_m w|^2 + \frac{c_4}{\lambda_{m+1}^{1/4}}\,\frac{|f|}{\nu}\,\|Q_m w\|^2\ ,$$

and thus,

$$(6.2.3)\qquad (\frac{\nu}{2} - c_4 \frac{|f|}{\nu\, \lambda_{m+1}^{1/4}}) \|Q_m w\|^2 \leqslant c'' \frac{|f|^2}{\nu^3} (1 + \frac{|f|^2}{\nu^4})^{3/2} |P_m w|^2 .$$

For

$$(6.2.4)\qquad \lambda_{m+1}^{1/4} \geqslant 4c_4 \frac{|f|}{\nu^2}$$

we obtain

$$(6.2.5)\qquad \|Q_m(u-v)\|^2 \leqslant 4c_5 \frac{|f|^2}{\nu^4} (1 + \frac{|f|^2}{\nu^4})^{3/2} |P_m(u-v)|^2 .$$

Plainly (6.2.4) together with (2.1.4) show that if m satisfies (6.2.1) with a suitable constant c_1 depending only on Ω, then P_m is injective on $S(f;\nu)$.

More generally, for any $u, v \in S(f;\nu)$

$$(6.2.6)\qquad |u-v|^2 \leqslant \left[1 + \frac{c_6}{(m+1)^{2/n}} \frac{|f|^2}{\nu^4} (1 + \frac{|f|^2}{\nu^4})^{3/2}\right] |P_m(u-v)|^2 .$$

6.3. Theorem. $S(f;\nu)$ *is homomorphic to a compact set of* $\mathbb{R}^m$, m *sufficently large so that* (6.2.1) *is fulfilled.*

Proof. $S(f;\nu)$ is a compact set of H, and $P_m S(f;\nu)$, for m satisfying (6.2.1) is compact too. Hence P_m^{-1} is continuous on the image and the result follows.

6.4. Before continuing, it will be usefull to establish a majoration similar to (6.2.5) for the H^2-norm. Let $u, v \in S(f;\nu)$ and $w = u - v$. We have

$$\nu\, Aw + B(w,u) + B(v,w) = 0$$

$$\nu\, Aw + B(P_m w + Q_m w, u) + B(v, P_m w + Q_m w) = 0$$

and taking the scalar product in H with $A\,Q_m w$, we find

$$\begin{aligned}
\nu\, |A\, Q_m w|^2 &\leqslant |(B(w,u), A\, Q_m w)| + |(B(v,w), A\, Q_m w)| \\
&\leqslant c|P_m w|^{1/2} \|P_m w\|^{1/2} \|u\|^{1/2} |Au|^{1/2} |A\, Q_m w| \\
&+ c|Q_m w|^{1/2} |A\, Q_m w|^{1/2} \|u\|^{1/2} |A\, Q_m w| \\
&+ c|v|^{1/2} |Av|^{1/2} \|P_m w\|\, |A\, Q_m w|
\end{aligned}$$

$$+ c|v|^{1/2}|Av|^{1/2}\|Q_m w\||A\, Q_m w|$$

$$\leq \frac{c'}{\lambda_1} |AP_m w||A\, Q_m w|(|Au|+|Av|)$$

$$+ \frac{c'}{\lambda_{m+1}^{1/2}} |A\, Q_m w|^2 \; (|Au|+|Av|) \; .$$

By virtue of (2.1.5) and (6.2.2) we easily deduce that

$$(6.4.1) \qquad |A\, Q_m(u-v)| \leq c_7 \frac{|f|}{\nu^2} (1 + \frac{|f|^2}{\nu^4}) \; |A\, P_m(u-v)| \quad , \quad u,v, \in S(f;\nu)$$

provided

$$(6.4.2) \qquad (m+1)^{2/n} \geq c_8 \frac{|f|^2}{\nu^4} (1 + \frac{|f|^2}{\nu^4})^2$$

6.5. We introduce the complexified space of H denoted H_C and let A_C , B_C , $P_{m,C}$,..., be the operators in H_C characterised as the linear extensions of the operators A, B, P_m,... . All the relations (2.2.3-4-5) are still valid if we replace B by $\frac{1}{8} B_C$.

Lemma 1. Let $\nu_o, \rho_o > 0$ be fixed and let

$$(6.5.1) \qquad (m+1)^{2/n} \geq \frac{c_9}{\nu_o} (\rho_o + \frac{|f|+\rho_o^2}{\nu_o^2} + (|f|+\rho_o^2)^{1/2})$$

Then for each $\nu \in C$, $|\nu| \geq \nu_o$, and $a \in \mathcal{D}_{A_C}$, $|A_C a| \leq \rho_o$, the equation

$$(6.5.2) \qquad \nu\, A_C w + Q_{m,C}\, B_C(w+a,w+a) = Q_{m,C} f \; ,$$

possesses one and only one solution $w = w_m(a)$, such that

$$(6.5.3) \qquad Q_{m,C} w = w \in \mathcal{D}_{A_C} \; , \; |A_C w| \leq \frac{c_{10}^2}{\nu_o^2} (|f|+|\rho_o|^2) \; .$$

Proof. (For simplicity we omit in this proof the subscripts C , and we note A, B, Q_m,..., instead of A_C, B_C, $Q_{m,C}$,...).

Let us consider the function

$$w \longmapsto F(w) = \nu^{-1} A^{-1}[-Q\, B(w+a,w+a) + Q\, f] \; ,$$

from $Q\, H_C \cap \mathcal{D}_A$ into itself, this space being equiped with the norm $|Au|$.
We infer from (2.2.5) :

$$
\begin{aligned}
|AF(w)| &\leq \frac{1}{\nu_o}(|f|+|B(a,a)|+|B(a,w)|+|B(w,a)|+|B(w,w)|)\\
&\leq \frac{1}{\nu_o}(|f|+c|a|^{1/2}\|a\||Aa|^{1/2}+c|a|^{1/2}|Aa|^{1/2}\|w\|+c|w|^{1/2}\|w\|^{1/2}\|a\|^{1/2}|Aa|^{1/2}\\
&\qquad +c|w|^{1/2}\|w\||Aw|^{1/2})\\
&\leq \frac{1}{\nu_o}(|f|+\frac{c\rho_o^2}{\lambda_1}+\frac{c\rho_o}{\lambda_1^{1/2}}\|w\|+\frac{c\rho_o}{\lambda_1^{1/4}}|w|^{1/2}\|w\|^{1/2}+c|w|^{1/2}\|w\||Aw|^{1/2})\\
&\leq \frac{c'}{\nu_o}(|f|+\rho_o^2+\frac{\rho_o}{\lambda_{m+1}^{1/2}}|Aw|+\frac{\rho_o}{\lambda_{m+1}^{3/4}}|Aw|+\frac{1}{\lambda_{m+1}}|Aw|^2)\\
&\leq \frac{c_{11}}{\nu_o}(|f|+\rho_o^2+\frac{1}{\lambda_{m+1}}|Aw|^2)
\end{aligned}
$$

(where as always c, c', c_i, are various constants depending only on Ω). Therefore using (2.1.4)

$$
(6.5.4) \qquad |AF(w)| \leq \frac{c_{12}}{\nu_o}(|f|+\rho_o^2+\frac{1}{(m+1)^{2/n}}|Aw|^2)
$$

where c_{12} is an appropriate constant ≥ 1. In a similar manner, we write

$$
\begin{aligned}
|AF(w)-AF(w')| &\leq \frac{1}{\nu_o}(|B(a,w-w')|+|B(w-w',a)|+|B(w-w',w)|+|B(w',w-w')|)\\
&\leq \frac{c}{\nu_o}(|a|^{1/2}|Aa|^{1/2}\|w-w'\|+|w-w'|^{1/2}|A(w-w')|^{1/2}\|a\|\\
&\qquad +|w-w'|^{1/2}|A(w-w')|^{1/2}\|w\|+|w'|^{1/2}|Aw'|^{1/2}\|w-w'\|)\\
&\leq \frac{c'}{\nu_o}\left(\frac{\rho_o}{\lambda_1^{1/2}\lambda_{m+1}^{1/2}}+\frac{|Aw|+|Aw'|}{\lambda_{m+1}}\right)|Aw-Aw'|
\end{aligned}
$$

Thus

$$
(6.5.5) \qquad |AF(w)-AF(w')| \leq \frac{c_{13}}{\nu_o(m+1)^{1/n}}(\rho_o+|Aw|+|Aw'|)|A(w-w')| .
$$

We take $c_{10} = (2c_{12})$. Then, because of (6.5.4), if

$$
(6.5.6) \qquad (1+m)^{2/n} \geq \frac{c_{10}^2}{\nu_o^2}(|f|+\rho_o^2)
$$

the function F maps the following set W_o into itself

$$W_o = \{w \in H_C \cap \mathcal{D}_A \,|\, |Aw|^2 \leqslant \frac{c_{10}^2}{\nu_o}(|f|+\rho_o^2)\} .$$

With an appropriate constant $c_{14} \geqslant c_{13}$, (6.5.5) implies

$$(6.5.7) \qquad |A(F(w)-F(w'))| \leqslant \frac{c_{14}}{\nu_o(m+1)^{1/n}}\left(\rho_o + \frac{|f|+\rho_o^2}{\nu_o^2}\right)|A(w-w')|$$

for each $w, w \in W_o$. Chosing the constant $c_9 \geqslant \max(c_{10}, c_{14})$, we immediately see that if m satisfies (6.5.1) then F is a strict contraction from W_o into itself and thus there exists a unique $w \in W_o$ such that $F(w) = w$.

This complete the proof of the Lemma.

<u>Lemma 2. Under the assumptions of Lemma 1, let</u>

$$(6.5.8) \qquad (m+1)^{1/n} \geqslant \frac{c_{15}}{\nu_o}\left(\rho_o + \frac{|f|+\rho_o^2}{\nu_o^2} + (|f|+\rho_o^2)^{1/2}\right)$$

<u>where $c_{15} \geqslant c_9$ is a constant depending only on Ω . Then the mapping $a \mapsto w_m(a)$ from $\{a \in \mathcal{D}_{A_C} \,|\, |A_C a| < \rho_o\}$ into $\mathcal{D}_{A_C}$ given by Lemma 1 is analytic.</u>

<u>Proof</u>. (In this proof again we omit the subscripts C).

For m satisfying (6.5.1), $a, b \in \mathcal{D}_A$, $|Aa| \leqslant \rho_o$ and $\nu \in C$, $|\nu| \geqslant \nu_o$, we consider the following equation with unknown $z \in Q\, H_C \cap \mathcal{D}_A$:

$$(6.5.9) \qquad \nu Az + Q\, B(b+z,a+w) + Q\, B(a+w,b+z) = 0 ,$$

where $w = w_m(a)$. For the mapping

$$z \mapsto G(z) = -\nu^{-1} A^{-1}[Q\, B(b+z,a+w) + Q\, B(a+w,b+z)]$$

from $Q\, H_C \cap \mathcal{D}_A$ into itself, we can show exactly as for F that

$$(6.5.10) \qquad |AG(z)| \leqslant \frac{c_{16}}{\nu_o}\left[\rho_o + \frac{c_{10}^2(|f|+\rho_o^2)}{\nu_o^2}\right]\left[|Ab| + \frac{|Az|}{(m+1)^{1/n}}\right]$$

$$(6.5.11) \qquad |A(G(z)-G(z'))| \leqslant \frac{c_{17}}{(m+1)^{1/n}\nu_o}\left[\rho_o + \frac{c_{10}^2(|f|+\rho_o^2)}{\nu_o^2}\right]|A(z-z')| .$$

Now if $c_{18} = \max(c_{16}, c_{17})$ and if

$$(6.5.12) \qquad (m+1)^{2/n} \geq \frac{2c_{18}}{\nu_o^2}\left(\rho_o + \frac{c_{10}^2(|f|+\rho_o^2)}{\nu_o^2}\right)$$

we easily deduce from (6.5.10-11) that there exists a unique $z = z_m(a)b$ which is solution of (6.5.9), i.e. $G(z) = z$, and such that

$$(6.5.13) \qquad |Az| \leq \frac{2c_{18}}{\nu_o}\left[\rho_o + \frac{c_{10}^2(|f|\rho_o^2}{\nu_o^2}\right]|Ab| .$$

It follows from (6.5.11) that z is uniquely determined by (6.5.9), if we impose to z to satisfy furthermore (6.5.13). We conclude that $b \mapsto z_m(a)b$ defines a linear continuous mapping from $Q\, H_C \cap \mathcal{D}_A$ into itself (the space being normed by $|Au|$).

Now we will show that if m is sufficently large, then $a \mapsto w = w_m(a)$ is Frechet differentiable in $\mathcal{D}_A$, at each $a \in \mathcal{D}_A$, $|Aa| < \rho_o$, and that $w'_m(a) = z_m(a)$. In virtue of [28], this implies that $w_m(a)$ is analytic in a. First we observe that we can prove without any difficulty (the technics are similar to those leading to (6.5.1)) that

$$(6.5.14) \qquad |A\, w_m(a+b) - A\, w_m(a)| \leq \frac{c'_{18}}{\nu_o}\left[\rho_o + \frac{c_{10}^2(|f|+\rho_o^2}{\nu_o^2}\right]|Ab| ,$$

for every $a, b \in \mathcal{D}_A$, $|Aa| \leq \rho_o$ and m satisfying (6.5.1). For m satisfying (6.5.1) and (6.5.12), let $a, b \in \mathcal{D}_A$, $|Aa| < \rho_o$, $|A(a+b)| \leq \rho_o$; we set $w_1 = w_m(a+b)$, $w = w_m(a)$ and $z = z_m(a).b$. Then

$$|\nu|\,|A(w_1-w-z)| = |Q[B(b,b)+B(w_1-w,w_1-w)+B(b,w_1-w)+B(w_1-w,b)+B(a,w_1-w-z)+B(w_1-w-z,a)$$
$$+B(w,w_1-w-z)+B(w_1-w-z,w)]|$$
$$\leq c|b|^{1/2}\|b\|\,|Ab|^{1/2}+c|w_1-w|^{1/2}\|w_1-w\|\,|A(w_1-w)|^{1/2}$$
$$+c|w_1-w|^{1/2}|A(w_1-w)|^{1/2}\|b\|+c|b|^{1/2}\|b\|^{1/2}\|w_1-w\|$$
$$+c(|w|^{1/2}|Aw|^{1/2}+|a|^{1/2}|Aa|^{1/2})\|w_1-w-z\|$$
$$\leq c_{19}|Ab|^2+c_{19}|A(w_1-w)|^2+2c_{19}|Ab|\,|A(w_1-w)|$$
$$+ c_{19}(|Aw|+|Aa|)(|w_1-w-z|^{1/2}|A(w_1-w-z)|^{1/2}$$

Using (2.1.5), (6.5.3) and (6.5.14) we obtain

(6.5.15)
$$\nu_o |A(w_1-w-z)| \leqslant c_{19} |Ab|^2 \left\{ \frac{c'_{18}}{\nu_o} \left[\rho_o + \frac{c_{10}^2(|f|+\rho_o^2)}{\nu_o^2}\right] + 1\right\}^2 + \frac{c_{20}}{(m+1)^{1/n}} |A(w_1-w-z)| \left\{\rho_o + \frac{c_{10}^2}{\nu_o^2}(|f|+\rho_o^2)\right\}$$

Thus taking in (6.5.8), $c_{15} = c_9 + 2c_{18}, c_{10}^2 + 2c_{20}(1+c_{10}^2)$, the relation (6.5.15) can be writen

$$|A[w_m(a+b)-w_m(a)-z_m(a).b]| \leqslant \frac{c_{21}}{\nu_o^3} (\rho_o + \frac{|f|+\rho_o^2}{\nu_o}) |Ab|^2$$

as is verified as soon as m verifies (6.5.8). This implies that $w'_m(a)$ exists and is equal to $z_m(a)$, and completes the proof of the Lemma.

Let now $D_m(\nu_o,\rho_o) \subset C^{m+1}$ be the open set

(6.5.16) $$\{\nu \in C, |\nu| > \nu_o\} \times \{\zeta \in C^m, |A_C p_m \zeta| < \rho_o\}$$

where for $\zeta = \{\zeta_1,\ldots,\zeta_m\} \in C^m$, we set

(6.5.17) $$p_m\zeta = \zeta_1 w_1 + \ldots + \zeta_m w_m .$$

<u>Lemma 3</u>. <u>Under the assumptions of Lemma 2, we define the mapping</u> $\Theta(\nu,\zeta) = \Theta_m(\nu,\zeta,\nu_o,\rho_o)$ <u>from</u> $D_m(\nu_o,\rho_o)$ <u>in</u> C^m <u>by setting</u>

(6.5.18) $$\Theta(\nu,\zeta) = P_{m,C}[\nu A_C\, p_m\zeta + B_C(p_m\zeta + w_m(\nu,p_m\zeta), p_m\zeta + w_m(\nu,p_m\zeta)) - f]$$

<u>where</u> $w_m(\nu,p_m\zeta)$ <u>represents the solution of</u> (6.5.2) <u>given by Lemma 1 for</u> $a = p_m\zeta$.

<u>Then the function</u> $\{\nu,\zeta\} \mapsto \Theta(\nu,\zeta)$ <u>is analytic</u>.

<u>Proof</u>. We have shown in Lemma 2 that $w_m(\nu,a)$ is differentiable with respect to a . It is much easier to show with similar methods that, under the assumptions of Lemma 2, $w_m(\nu,a)$ is also differentiable with respect to ν, $|\nu| > \nu_o$. Hence, $\Theta(\nu,\zeta)$ is separately analytic in ν and ζ . It follows then from a classical theorem of Hartogs that $\theta(\nu,\zeta)$ is analytic in $\{\nu,\zeta\}$.

6.6. Lemma (Representation of stationary solutions).
We assume that

(6.6.1) $$0 < \nu_o < c_{10}^2 \quad \text{and} \quad \rho_o > c_3 \frac{|f|}{\nu}\left(1 + \frac{|f|^2}{\nu^4}\right)$$

where m satisfies to (6.5.8) and $\nu > \nu_o$.

Then $u \in S(\nu)$ if and only if

(6.6.2) $$u = p_m\zeta + w_m(\nu,p_m\zeta) \ ,$$

where

(6.6.3) $$\zeta \in D_m(\nu_o,\rho_o) \cap \mathbb{R}^m \quad \text{and} \quad \theta_m(\nu,\zeta\ ,\nu_o,\rho_o) = 0 \ .$$

Proof. Let $u \in S(f;\nu)$ and let $\zeta \in \mathbb{R}^m$ be defined by $P_m u = \zeta_1 w_1 + \ldots + \zeta_m w_m$. Because of (6.2.2) $|Au| < \rho_o$, and the assumptions (6.6.1) imply that

(6.6.4) $$|A\ P_m u| \leq |Au| < \rho_o \ ,$$

(6.6.5) $$|A\ Q_m u| < \rho_o < \frac{c_{10}}{\nu_o^{1/2}}\left(|f|+\rho_o^2\right)^{1/2} \ ;$$

Applying the operator Q_m to (6.1.1) we obtain the equation (6.5.2) with $a = P_m u = p_m\zeta$ and $w = Q_m u$. Due to Lemma 6.5.1, and (6.6.5), we see that $Q_m u = w_m(p_m\zeta)$. Introducing this value of w_m in (6.5.2) and applying P_m to the relation that we obtain, we arrive to $\theta_m(\nu,\zeta;\nu_o,\rho_o) = 0$.

Conversely if u satisfies (6.6.2-3) then it is obvious that $u \in S(f;\nu)$ since the mappings p_m, w_m , θ_m are real (i.e. $p_m\phi$, $w_m(\nu,\psi) \in H$, $\theta_m(\nu,\zeta) \in H$, if $\phi \in \mathbb{R}^m$, resp. $\{\nu,\psi\} \in R \times H$, $\{\nu,\zeta\} \in \mathbb{R}^{m+1}$) .

Taking into account Lemma 6.6 and Lemmas 6.5.2-3, we obtain the following

6.7. Corollary. The set $S(\nu) = S(f;\nu)$ is for a fixed ν (and $f \in H$) a real analytical set of finite dimension in H (*)

Moreover, proving as in Lemma 6.5.3 that $w_m(\nu,p_m\zeta)$ is analytic with

(*) We use the usual definition of an analytical set ; see for instance [28], ch.V.

respect to $\{\nu,\zeta\}$, $\{\nu,\zeta\} \in D_m(\nu_o,\rho_o)$, we can prove as well that $\bigcup_{\nu>\nu_o} S(f;\nu)$ is a real analytical set (in H) of finite dimension which has only one irreductibl unbounded component (of dimension 1).

6.8. After Theorem 6.3, our main result on the structure of the set $S(f;\nu)$ will be now the following.

Theorem 1. Let $\nu > 0$ be fixed. Then for every generic f (i.e. for each $f \in H \setminus E$, where E is some rare set of H), the set $S(f;\nu)$ of stationary solutions of Navier-Stokes equations (cf. (6.1.1)) is finite.

In fact we will prove a more precise result. Before stating this result, let us make preliminary remarks. In order to emphasize the dependance in f of the solution w of (6.5.2), we will denote this solution $w_n(\nu,a;f)$ or $w_n(a;f)$ instead of $w_n(\nu,a)$. For $\rho > 0$, let us define ρ_o by

(6.8.1) $$\rho_o = 1 + c_3 \frac{\rho}{\nu_o}\left(1 + \frac{\rho}{\nu_o^4}\right)$$

where $\nu_o = \frac{1}{2}\min\{c_{10}^2,\nu\}$. With this choice of ρ_o , let m be the first integer m such that

(6.8.2) $$(m+1)^{2/n} \geq \frac{c_8\,\rho^2}{\nu^4}\left(1 + \frac{\rho^2}{\nu^4}\right)^2 + \frac{c_{15}}{\nu_o} + \frac{c_{15}}{\nu_o^2}\left(\rho_o^2 + \frac{\rho+\rho_o^2}{\nu_o}\right)$$

(see (6.4.2) and (6.5.8).

We set then

(6.8.3) $$B(\rho) = \{f \in H,\ |f| < \rho\}$$

and let $B_m(\rho)$ be the subset of $f \in B(\rho)$ such that $S(\nu,f)$ is finite and such that for each $u \in S(\nu,f)$, the following condition holds

$$\left\{\begin{array}{l} \text{If } v \in \mathcal{D}_A \\ (6.8.4) \qquad \nu Av + B(u,v) + B(v,u) = 0 \\ \text{and} \\ (6.8.5) \qquad Q_m v = \left[\frac{D}{D\zeta} w_m(p_m\zeta;f)\Big|_{p_m\zeta=P_m u}\right]\cdot\eta \ , \\ \text{where } p_m\eta = P_m v \text{ , then } v = 0 \ . \end{array}\right.$$

Let $B_{reg}(\rho) = \bigcup_{m=m\rho}^{\infty} B_m(\rho)$. Then

__Theorem 2__. $B(\rho)\setminus B_{reg}(\rho)$ __is rare in__ $B(\rho)$ __equiped with the strong topology of__ H .

Theorem 2 provides immediately a proof of Theorem 1 for which we can take

$$E = \bigcup_{k=1}^{\infty} [B(k)\setminus B_{reg}(k)] \quad .$$

For this reason the remaining part of this section is devoted to the proof of Theorem 2.

6.9. We first prove that if the set $S(\nu,f)$ is not finite for some $f \in B(\rho)$, then necessarily there exists $u_o \in S(f;\nu)$ and $v_o \in \mathcal{D}_A$, $v_o \neq 0$, such that $u = u_o$, $v = v_o$ satisfy (6.8.4-5).

Assume that $S(f;\nu)$ is not finite. Since it is a compact set of H , there exists a sequence $u_j \in S(\nu)$, $(j=1,2,\ldots)$ with mutually different elements, such that u_j converges strongly in H to some element u_o of $S(f;\nu)$. The relations (6.8.2) and (6.4.1-2) give

$$(6.9.1)\quad \begin{cases} |A(u_j-u_o)|^2 = |AP_m(u_j-u_o)|^2 + |AQ_m(u_j-u_o)|^2 \\ \qquad \leq \left[1 + c_7^2 \dfrac{|f|^2}{\nu^4} \left(1 + \dfrac{|f|^2}{\nu^4}\right)^2\right] |AP_m(u_j-u_o)|^2 \\ \qquad \leq \left[1 + c_7^2 \dfrac{\rho^2}{4} \left(1 + \dfrac{\rho^2}{\nu^4}\right)^2\right] \lambda_m^2 |u_j-u_o|^2 \ . \end{cases}$$

Thus if $v_j = (u_j-u_o)\ |u_j-u_o|^{-1}$, the sequence $\{v_j\}_{j=1}^{\infty}$ is bounded in $\mathcal{D}_A$. We can extract a subsequence (still denoted v_j) which converges weakly in $\mathcal{D}_A$ to some element v_o , i.e. $A\,v_j$ converges weakly in H to Av_o . By compacity, v_j converges to v_o strongly in H^1 and H , and $|v_o| = 1$. On the other hand

$$(6.9.2)\quad \begin{aligned} |\nu\, Av_j + B(u_o,v_j) + B(v_j,u_o)| &= |u_j-u_o|\,|B(v_j,v_j)| \\ &\leq c|u_j-u_o|\,|Av_j|^{1/2}\|v_j\|^{1/2} \\ &\leq c\, \lambda_1^{-1/2}|u_j-u_o|\,|Av_j| \end{aligned}$$

and this converges to zero since $|Av_j|$ is bounded (cf. (6.9.1)). We infer from the 3rd and 5th inequalities (2.2.5) that $\phi \mapsto B(u_o,\phi)$ and $\phi \mapsto B(\phi,u_o)$ are linear continuous mappings from H^1 into H. Thus $B(u_o,v_j) \to B(u_o,v_o)$, $B(v_j,u_o) \to B(v_o,u_o)$ strongly in H, and, from (6.9.2) νA_j converges strongly in H to

$$\nu Av_o = - B(u_o,v_o) - B(v_o,u_o) .$$

There remains to prove that $u = u_o$, and $v = v_o$ satisfy (6.8.5) too.

Let $\xi_j \in \mathbb{R}^m$, $(j=0,1,2,\ldots)$ be fefined by $P_m u_j = p_m \xi_j$. Then, in virtue of Section 6.6

$$(6.9.3)\quad \begin{cases} Q_m v_j = |u_j-u_o|^{-1}(Q_m u_j - Q_m u_o) = |u_j-u_o|^{-1}[w_m(p_m \xi_j,f)-w_m(p_m\xi_o,f)] \\ \quad = |u_j-u_o|^{-1} \left\{[\frac{D}{D\xi_o} w_m(p_m \xi_o,f)](\xi_j-\xi_o) + \varepsilon_j|p_m \xi_j-p_m \xi_o|\right\} \end{cases}$$

where $|\varepsilon_j| \to 0$ as $j \to \infty$. Let us set $\eta_j = (\xi_j-\xi_o)|u_j-u_o|^{-1}$ and let $\eta_o \in \mathbb{R}^m$ be defined by $p_m \eta_o = P_m v_o$. Then $|p_m(\eta_j-\eta_o)| = |P_m(v_j-v_o)| \to 0$ as $j \to \infty$. Hence, letting $j \to \infty$ in (6.9.3), we obtain (6.8.5) with $u = u_o$, $v = v_o$, and this completes the proof of our assertion.

6.10. We now prove that $B(\rho) \setminus B_m(\rho)$ is closed in $B(\rho)$ (for the strong topology of H).

Let f_j be a sequence of $B(\rho) \setminus B_m(\rho)$ strongly convergent in H to some element f. As a consequence of the results of Section 6.9, for each j there exists $u_j \in S(f_j,\nu)$ and $v_j \in \mathcal{D}_A$, $|v_j| = 1$, such that $u = u_j$ and $v = v_j$ satisfy (6.8.4-5). Due to (6.2.2), we have

$$(6.10.1)\qquad M = \sup_j |Au_j| < + \infty.$$

Then, by extracting perhaps a subsequence, we can assume u_j is weakly convergent in $\mathcal{D}_A$ to some limit u_o, i.e. $Au_j \to Au_o$ weakly in H. On the other hand because of (2.2.2), (2.2.5) and (6.8.2)

$$\nu|Av_j|^2 = -(B(v_j,u_j),Av_j) - (B(u_j,v_j),Av_j)$$

$$\leq c\,\|v_j\|^{1/2}|Av_j||Au_j| + c|Au_j|\|v_j\||Av_j|$$

$$\leq (\text{ by } (6.10.1))$$

$$\leq c'\,M|Av_j|^{3/2}$$

Whence

$$(6.10.2) \qquad M_1 = \sup_j |Av_j| \leq (c'\,M\nu^{-1})^2 < +\infty.$$

After extraction of a subsequence, we can also assume that v_j converges weakly in $\mathcal{D}_A$ to some limit v_o (i.e. $Av_j \longrightarrow Av_o$ weakly in H). Now by compacity, u_j (resp. v_j) converges to u_o (resp. v_o) strongly in H^1 and H . Using again (2.2.5) and (6.10.1-2), we can write :

$$|B(v_j,u_j) - B(v_o,u_o)| \leq |B(v_j-v_o,u_j)| + |B(v_o,u_j-u_o)|$$

$$\leq c|v_j-v_o|^{1/2}\|v_j-v_o\|^{1/2}\|u_j\|^{1/2}\,M^{1/2} + c\,M_1^{1/2}\|u_j-u_o\| .$$

Thus $B(v_j,u_j) \longrightarrow B(v_o,u_o)$, strongly in H , and in the same manner we can prove that $B(u_j,v_j) \longrightarrow B(u_o,v_o)$ for the norm of H . This implies that

$$\nu\,Av_j = -B(v_j,u_j) - B(u_j,v_j)$$

converges strongly in H to its limit $\nu\,Av_o$,

$$\nu\,Av_o = -B(v_o,u_o) - B(u_o,v_o) ,$$

and consequently u_o, v_o satisfy (6.8.4).

Finally it is easy to deduce from Lemmas 6.5.1-2, that

$$(w_m(a,f))'_a = z_m(\nu;a,f) ,$$

is continuous from $\{a\in\mathcal{D}_A, |a|<\rho_o\}\times B(\rho)$ into the space of linear continuous operators on H (for instance we prove this fact for the mapping $\{a,f\} \longrightarrow w(a,f$ which is analytic on $\{a\in\mathcal{D}_{A_C}, |a|<\rho_o\}\times\{f\in H_C, |f|<\rho\}$, and we apply the

vectoriel form of the classical theorem of Cauchy ; see [4]).
Setting then $f = f_j$, $u = u_j$ and $v = v_j$ in (6.8.5), we can easily let $j \longrightarrow \infty$, and conclude at the limit that $u = u_o$, $v = v_o$ satisfy (6.8.5).

Hence $f \in B(\rho) \setminus B_m(\rho)$ and this completes the proof of this second assertion ($B(\rho) \setminus B_m(\rho)$ is closed).

6.11. In order to achieve the proof of Theorem 2 it will be sufficent to show that the set $B_{reg}(\rho)$ is dense in $B(\rho)$. But since

$$\bigcup_{N=1}^{\infty} (B(\rho) \cap P_N H)$$

is dense in $B(\rho)$ and since for $m \geqslant \max\{N, m_\rho\}$, we have

$$B(\rho) \cap P_N H \subset B(\rho) \cap P_m H \quad \text{and} \quad B_m(\rho) \subset B_{reg}(\rho)$$

it will suffice to prove that for such an m , $B(\rho) \cap P_m H$ is in the closure (in $B(\rho)$) of $B_m(\rho)$.

For proving that we first observe that since the mapping

$$\mathcal{A} : u \longmapsto \nu Au + B(u,u)$$

is continuous form $\mathcal{D}_A$ into H , the set $\mathcal{A}^{-1} B_m(\rho)$ is open in $\mathcal{D}_A$. On the other hand, in virtue of section 6.6, $u \in \mathcal{A}^{-1}[B(\rho) \cap P_m H]$ if and only if for one $g \in B(\rho) \cap P_m H$, we have

$$(6.11.1) \qquad P_m u = p_m \varphi \quad \text{and} \quad Q_m u = w_m(p_m \varphi, g) ,$$

where φ satisfies (6.6.4). Since $Q_{m,C}\, g = Q_m g = 0$, w_m is independant of $g \in B(\rho) \cap P_m H$ and the condition (6.6.5) becomes (see (6.5.1))

$$(6.11.2) \qquad g = P_m[\nu A p_m \varphi + B(p_m \varphi + w_m(p_m \varphi, g), p_m \varphi + w_m(p_m \varphi, g))] .$$

Hence the mapping $u \longrightarrow \varphi = \Pi u$, where u and φ are linked by (6.11.1), is an homeomorphism from $\mathcal{A}^{-1}[B(\rho) \cap P_m H]$ on some open set G of $\{\varphi \in \mathbb{R}^m, |A p_m \varphi| < \rho_o\}$
Furthermore if $\mathcal{U}$ denotes the mapping Π^{-1} form G into $\mathcal{A}^{-1}[B(\rho) \cap P_m H]$ we will have

(6.11.3) $$\mathcal{U}(\zeta) = p_m\zeta + w_m(p_m\zeta, g) \;,$$

and, by Lemma 5.5.2, $\mathcal{U}$ is analytic from G into $\mathcal{D}_A$. It is then clear that the function $\zeta \longmapsto \mathcal{A}\,\mathcal{U}(\zeta)$ is differentiable with differential (apply the chain rule differentiation)

(6.11.4) $$\frac{D}{D\zeta}\,\mathcal{A}(\mathcal{U}(\zeta)).\eta = \nu\, A\, \mathcal{U}'(\zeta).\eta + B(\mathcal{U}(\zeta), \mathcal{U}'(\zeta)\eta) + B(\mathcal{U}'(\zeta).\eta, \mathcal{U}(\zeta))$$

$$(\zeta \in G,\ \eta \in \mathbb{R}^m) \;.$$

Let now Φ denote the analytic function $\zeta \longmapsto \eta$ from G into $\mathbb{R}^m$, where η is determined by $p_m\zeta = P_m\mathcal{A}(\mathcal{U}(\zeta))$. Because of (6.11.2) we have

(6.11.5) $$\mathcal{A}(\mathcal{U}(\zeta)) = p_m\, \phi(\zeta) \qquad\qquad (\zeta \in G)$$

and therefore

(6.11.6) $$p_m\, \Phi(G) = B(\rho) \cap P_m H = p_m\{\eta \in \mathbb{R}^m, \mid p_m\zeta \mid < \rho\} \;;$$

Now let G_1 be the subset of points of G where the Jacobian $\det \phi'(\zeta)$ of ϕ is 0 . We infer from the classical theorem of Sard (see [39], p.13) that the Lebesgue measure in $\mathbb{R}^m$ of $\phi(G_1)$ is 0 . Consequently $p_m[\phi(G) \setminus \Phi(G_1)]$ is dense in $B(\rho) \cap P_m H$, and the proof will be complete if we show that

(6.11.7) $$p_m[\phi(G) \setminus \phi(G_1)] \subset B_m(\rho) \;.$$

If this inclusion is not true, Section 6.9 shows us the existence of $f \in p_m[\phi(G) \setminus \phi(G_1)]$, $u \in S(\nu, f)$, $v \in \mathcal{D}_A$, $v \neq 0$, satisfying (6.8.4-5). Then (6.8.5) and (6.11.3) imply that $v = \mathcal{U}'(\zeta).\eta$, and because of (6.11.3-5) and (6.8.4) we have

$$p_m\, \phi'(\zeta).\eta = \nu\, Av + B(u,v) + B(v,u) = 0$$

and necessarily $\zeta \in G_1$, $f = p_m\, \phi(\zeta) \in \phi(G_1)$ which contradicts the definition of f .

The proof of Theorem 2 is complete.

§.7. Connections with the theory of turbulence.

7.1. There are several distinct mathematical points of view on the turbulence. Some do not involve the Navier-Stokes equations (example : [25]; see also [26], [27]) others do involve the Navier-Stokes equations, but essentially without any boundary conditions (as for instance in the case of the theory of homogeneous turbulence [5], [30]). Finally there are some points of view which are a priori suited also for the Navier-Stokes equations on bounded fixed domains. Since this is the boundary problem considered in the present paper we will try now to discuss some of these last points of view in the light of our previous results and methods.

First of all these last points of view can be further divided into two groups. The first group contains those views which consider that the irregularity and the randomness of the turbulent flows are due to the same character of their initial states. The second group contains those views which consider that these irregularity and randomness are produced by the Navier-Stokes equations even if the initial states are neither irregular nor random. As we will show below the results of this paper are pertinent to this second group . (For the first group we refer to [17], [29], [10], [14],...)

7.2. The oldest mathematical attempt is due to Reynolds [34], who proposed the study of the time averages

$$\frac{1}{t}\int_0^t u(\tau)\, d\tau = \{\frac{1}{t}\int_0^t u_j(\tau,x)d\tau\}_{j=1}^{n}$$

of a flow $u(t) = \{u(t,x)\}_{j=1}^{n}$ (n=2 or 3), for convenient large t . Plainly, in virtue of the results in sections 3.1-4, this study is equivalent to that of the means

$$\overline{u} = \int u\, d\mu(u)$$

where μ stands for an accretive stationary statistical solution. Thus the study of these statistical solutions will include any Reynolds type theory.

Another old point of view on turbulence belongs to Leray [22], for whom an individual solution $u(t)$ is turbulent whenever it is not regular on any interval $[0,T] \subset [0,\infty]$ though $u(0) = u_0 \in H^1$. In other words, turbulence in Leray's sense exists if and only if $t(u_0) < \infty$ (see sections 2.6 and 3.6) for some $u_0 \in H^1$.

7.3. Other views on the occurence of turbulence was proposed by Landau (see for instance [20]) and Hopf [16]. Essentially these views can be reduced to the following : for given $\nu>0$ there exists an asymptotic attractor (see sec. 3.8) M_ν , uniquely determined, enjoying the following properties :

(i) M_ν is a finite dimensional manifold.

(ii) M_ν is the closure of its almost periodic trajectories (i.e. the closure in H of the union of the sets $\{U(t) : -\infty<t<\infty\}$, where $U(t)$ is an almost periodic H-valued function and $U(t+t_o)$ is an individual solution for any $t_o\in(-\infty,\infty)$

(iii) If $d_\nu = \dim M_\nu > 0$ there exists at least one non stationary individual solution lying in M_ν .

(iv) $d_\nu\nearrow\infty$ for $\nu\searrow 0$ (this assumption corresponds to the development of more turbulence with the increase of the Reynolds number).

We will refer in the sequel to this kind of behaviour as turbulence in the sense of Landau-Hopf.

7.4. A related view on turbulence was proposed by Ruelle and Takens [35], [36], [37], which in our frame can be sumarized as follows : for $\nu>0$ enough small there exists an asymptotic attractor A with a number of strange properties, among which we quote the following : A is the closure of its non-stationary periodic trajectories. This behaviour will be called in the sequel, turbulence in the sense of Ruelle-Takens.

7.5. It is well known that in the case $n=2$ (i.e. the case of plane fluids), turbulence in the sense of Leray does not occur. Our Theorems 5.4 (i), (iii) and 5.7 plainly implies that in this case also turbulences in the sense of Landau-Hopf or Ruelle-Takens do not occur.

These mathematical facts seem to confirm the experimental point of view that in laboratory no plane turbulence can occur.

7.6. One of the hardest open problem in the study of the Navier-Stokes equations is that of the existence of the turbulence in some of the above senses, for three dimensional fluids (i.e. the case $n=3$). There seems to exist a strong believe that one of the turbulence in the sense of Landau-Hopf or Ruelle-Takens does not exist in this case (see for instance [9]). It is plain that our Theorem 5.4 (i), (iii) and 5.7 yields also the following : In case $n=3$, if turbulence in the sense of Landau-Hopf or in the sense of Ruelle-Takens does ~~not~~ exist, then there exists also turbulence in the sense of Leray.

7.7. We wish now to discuss also another view on turbulence due to Bass [4]. In order to present it, let us call <u>pseudo-random</u> an $\mathbb{R}^n$-valued bounded function $\phi(t) = (\phi_j(t))_{j=1}^n$ (n=2 or 3) defined on $[0,\infty)$ satisfying the following conditions

(i) $$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi_i(t+\tau)\,\phi_j(t)\,dt = \Phi_{ij}(\tau)$$ exists for any $\tau \in [0,\infty)$ and $1\leq i,j\leq n$;

(ii) $(\Phi_{ij}(\tau))_{i,j=1}^n \not\equiv 0$ on $(0,\infty)$;

(iii) $\Phi_{ij}(\tau) \longrightarrow 0$ for $\tau \longrightarrow \infty$

for any $1\leq i,j\leq n$. For Bass a flow $u(t) = \{u_j(t,x)\}_{j=1}^n$ is turbulent if the $\mathbb{R}^n$-valued function $t \longmapsto \{u_j(t,x_o)\}_{j=1}^n$ is pseudo-random for at least some $x_o \in \Omega$. We will refer to such a flow as <u>turbulence in the sense of Bass</u> (Actually the definition of Bass is more restringent, but for our purposes the above one is sufficient.)

7.8. <u>Proposition</u>. (i) <u>In case</u> n=2 , <u>there exists no turbulence in the sense of Bass</u>.

(ii) <u>In case</u> n=3 , <u>if turbulence in the sense of Bass exists, there exists also turbulence in the sense of Leray</u>.

<u>Proof</u>. Let us assume that turbulence in the sense of Leray does not exist. (In the case n=2 this assumption is automatically satisfied.) Then if $C_1(T,R)$ denotes the supremum (5.7.2) we have for any individual solution $u(t)$, such that $u(0) = u_o \in H^1$, the following global estimate

$$\|u(t)\| \leq C_1(1,R_{u_o}) \tag{7.8.1}$$

where

$$R_{u_o}^2 = \max\{\|u_o\|^2,\ \frac{1}{\nu}|u_o|^2 + \frac{|f|^2}{\nu^2\lambda_1}\} . \tag{7.8.2}$$

Indeed for $t \in [0,1]$ we have $\|u(t)\| \leq C_1(1,\|u_o\|) \leq C(1,R_{u_o})$. Moreover for $t \in [1,\infty]$ we have (by (2.3.5))

$$\int_{t-1}^t \|u(\tau)\|^2\,d\tau \leq \frac{1}{\nu}\,e^{-\lambda_1 t}|u(0)|^2 + \frac{|f|^2}{\nu^2\lambda_1} \leq R_{u_o}^2$$

henceforth there exists $t_o \in (t-1,t)$ such that $\|u(t_o)\| \leq R_{u_o}$ and consequently $\|u(t)\| = \|RS(t-t_o)\, u(t_o)\| \leq C_1(1,R_{u_o})$. Thus (7.8.1-2) is checked. Using Lemma 4.2 we can now easily infer that

$$(7.8.3) \qquad u(t) \in C((0,\infty), \mathscr{D}_A)$$

and

$$(7.8.4) \qquad |Au(t+1)| \leq C_2(R_{u_o}) \qquad (t \geq 0)$$

where $C_2(R)$ is a finite real function defined on $[0,\infty]$.

Let now $u_o(t) = \{u_{oj}(t,x)\}_{j=1}^n$ be an individual solution turbulent in the sense of Bass and let $x_o \in \Omega$ be a point such that $\{u_{oj}(t,x_o)\}_{j=1}^n$ is pseudo-random (in the sense defined in sec. 7.7). This in particular means that

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T u_{oj}(t+\tau,x_o)\, u_{oj}(t,x_o)\, dt = \beta_{ij}(\tau)$$

exists for any $\tau \in [0,\infty]$ and $1 \leq i,j \leq n$, or equivalently

$$(7.8.5) \qquad \Big|\sum_{i,j=1}^n (\beta_{ij}(\tau,T)-\beta_{ij}(\tau))\xi_i\, \eta_j\Big| \leq \varepsilon_1(T)\Big(\sum_{i=1}^n \xi_i^2\Big)^{1/2}\Big(\sum_{i=1}^n \eta_i^2\Big)^{1/2}$$

where $\xi, \eta \in \mathbb{R}^n$ are arbitrary, $\varepsilon_1(T) \longrightarrow 0$ for $T \longrightarrow \infty$ and

$$(7.8.6) \qquad \beta_{ij}(\tau,T) = \frac{1}{T} \int_0^T u_{oi}(t+\tau,x_o)\, u_{oj}(t,x_o)\, dt \qquad (T \geq 1) .$$

Let now $\xi, \eta \in \mathbb{R}^n$ be fixed and let $\Delta_p \in L^2(\Omega)$ $(p=1,2,\dots)$ be such that

$$(7.8.7) \qquad \|\delta_{x_o} - \Delta_p\|_{(H^2(\Omega))'} = \varepsilon_2(p) \longrightarrow 0 \text{ for } p \longrightarrow \infty ,$$

where δ_{x_o} denotes the Dirac functional in x_o which obviously $\in (H^2(\Omega))'$.
Set $h_p = P(\xi \otimes \Delta_p)$, $k_p = P(\eta \otimes \Delta_p)$, where as usual P denotes the orthogonal projection of $(L^2(\Omega))^n$ on H and where, for instance, $\xi \otimes \Delta_p$ denotes the $\mathbb{R}^n$-valued function $\{\xi_j\, \Delta_p(x)\}_{j=1}^n$, obviously belonging to $(L^2(\Omega))^n$. Finally let us se

$$\beta_p(\tau,T) = \frac{1}{T} \int_1^T (u_o(t+\tau),h_p)(u_o(t),k_p)\, dt \qquad (T \geq 1) .$$

Then (for $T\geq 1$)

$$|\beta_p(\tau,T) - \sum_{i,j=1}^{n} \beta_{ij}(\tau)\ \xi_i\eta_j| \leq \frac{1}{T}\left|\int_1^T \sum_{i,j=1}^{n} [<\Delta_p,u_{oi}(t+\tau)><\Delta_p,u_{oj}(t)> - u_{oi}(t+\tau,x_o) \cdot u_{oj}(t,x_o)]\xi_i\ \eta_j\ dt\right|$$

$$+ \left|\sum_{i,j=1}^{n} (\beta_{ij}(\tau,T) - \beta_{ij}(\tau))\xi_i\ \eta_j\right|$$

$$\leq 2\varepsilon_2(p)\ C_2(R_{u_o(0)})^2\ \sup\|\Delta_p\|_{(H^2(\Omega))'} \sum_{i,j=1}^{n}|\xi_i||\eta_j|$$

$$+ \varepsilon_1(T)(\sum_{i=1}^{n}\xi_i^2)^{1/2}(\sum_{j=1}^{n}\eta_j^2)^{1/2}$$

that is

$$|\beta_p(\tau,T) - \sum_{i,j=1}^{n} \beta_{ij}(\tau)\xi_i\ \eta_j| \leq (\varepsilon_3(p) + \varepsilon_1(T))(\sum_{i=1}^{n}\xi_i^2)^{1/2}(\sum_{j=1}^{n}\eta_j^2)^{1/2} \tag{7.8.8}$$

where $\varepsilon_3(p) \longrightarrow 0$ for $p \longrightarrow \infty$, and neither ε_3 nor ε_1 do not depend on $\tau \in [0,\infty]$. On the other hand, if for $m,p = 1,2,\ldots$ and $\tau \in [0,\infty)$ fixed we set

$$\phi(u) = (RS(\tau)\ P_m u,\ h_p)(u,k_p)\ , \tag{7.8.9}$$

the functional Φ belongs to $\mathscr{C}$ (see sec. 3.1). Thus if μ is any time average of the individual solution $u_o(t)$ (see sec. 3.1), there exists a sequence $1 \leq T_1 < T_2 < \ldots \longrightarrow \infty$ such that

$$\int\Phi(u)\ d\mu(u) = \lim_{j\to\infty}\frac{1}{T_j}\int_o^{T_j}\Phi(u_o(t))\ dt\ . \tag{7.8.10}$$

But, in virtue of sections 3.2-5, $\mu \in \mathscr{S}_a(f;\nu)$, so that, by Theorems 5.3 and 5.7,

$$\operatorname{supp}\mu \subset S(f,\nu)$$

It results

$$\int\phi(u)\ d\mu(u) = \int(P_m u,h_p)(u,k_p)\ d\mu(u)\ . \tag{7.8.11}$$

Since

$$\left|\frac{1}{T_j}\int_1^{T_j}\Phi(u_o(t))dt - \beta_p(\tau,T_j)\right| \leqslant \frac{1}{T_j}(C_2(R_{u_o(0)}))^2 \left(\sum_{i,j=1}^{n}|\xi_i||\eta_j|\right)(\sup_p\|\Delta_p\|_{(H^2(\Omega))'})^2 \cdot \int_1^{T_j}|A[RS(\tau)\ u_o(t) - RS(\tau)\ P_m\ u_o(t)]|\ dt$$

$$\leqslant C\left(\sum_{i=1}^{n}\xi_i^2\right)^{1/2}\left(\sum_{j=1}^{n}\eta_j^2\right)^{1/2}\frac{1}{T_j}\int_o^{T_j}|A[RS(\tau)\ u_o(t) - RS(\tau)\ P_m\ u_o(t)|\ dt$$

$$\leqslant C_o\left(\sum_{i=1}^{n}\xi_i^2\right)^{1/2}\left(\sum_{j=1}^{n}\eta_j^2\right)^{1/2} C(\tau,m)\ ,$$

where C_o is a constant with respect to j, p, m and τ, while

(7.8.12) $$C(\tau,m) = \sup\{|A[RS(\tau)u - RS(\tau)P_m u| : |Au| \leqslant C_2(R_{u_o(0)})\}\ ,$$

from (7.8.8-11), we can plainly infer

$$\left|\int(P_m u,h_p)(u,k_p)\ d\mu(u) - \sum_{i,j=1}^{n}\beta_{ij}(\tau)\xi_i\ \eta_j\right|$$
$$\leqslant [\varepsilon_3(p) + C_o\cdot C(\tau,m)]\left(\sum_{i=1}^{n}\xi_i^2\right)^{1/2}\left(\sum_{j=1}^{n}\eta_j^2\right)^{1/2}$$

whence

(7.8.13) $$|\beta_{ij}(\tau_1) - \beta_{ij}(\tau_2)| \leqslant 2\ \varepsilon_3(p) + C_o[C(\tau_1,m) + C(\tau_2,m)]$$

for any τ_1, $\tau_2 \in [0,\infty]$ and $1\leqslant i$, $j\leqslant n$. In (7.8.13), p and m (= 1,2,...) are at our disposal. Letting $p, m \longrightarrow \infty$, and using the fact that

$$C(\tau,m) \longrightarrow 0 \text{ for } m \longrightarrow \infty \quad (\tau>0)$$

(which results readily from (5.7.4-5) and Corollary 1.9, (1.9.3) of [ ] , Ch.III), we obtain

(7.8.14) $$\beta_{ij}(\tau_1) = \beta_{ij}(\tau_2) \qquad (\tau_1,\tau_2 \in (0,\infty),\ 1\leqslant i,\ j\leqslant n)\ .$$

Since $\{u_{oj}(t,x_o)\}_{j=1}^{n}$ is pseudo-random we must have (see sec. 7.7.(iii)), $\beta_{ij}(\tau) \to 0$ for $\tau \to \infty$, thus, by (7.8.14),

$$(7.8.15) \qquad (\beta_{ij}(\tau))_{i,j=1}^{n} \equiv 0 \quad \text{on} \quad (0,\infty) ,$$

in contradiction with the property (ii) in sec. 7.7. Thus, if turbulence in the sense of Leray does not exist, neither does exist turbulence in the sense of Bass.

7.9. The results in §.5 explain why whenever Bifurcation Theory was successfully applied to produce periodic, or more general almost periodic, individual solutions for the Navier-Stokes equations (see for instance [18] and [38]), the boundary value problem involved was different of the classical Dirichlet type one we considered in this paper. Indeed for this classical boundary value problem our results show that, at least in case $n=2$, no non stationary periodic or almost periodic individual solutions exist. In particular this implies, via the Bifurcation Theory (see for instance [38]), a very peculiar spectral behaviour of the operator

$$(7.9.1) \qquad A_u v = \nu\, Au + B(u,v) + B(v,u) \qquad (v \in \mathcal{D}_A)$$

in H, where $u \in \mathcal{D}_A$ plays the role of a parameter. Therefore this raises the question if, again in case $n=2$, the set $S(f,\nu)$ of all stationary individual solutions is not a singleton. This would constitute a substantial improvement of our results in §.6. However this seems rather improbable to happen.

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