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On the failure of Von Neumann's inequality

Anna-Maria Mantero
and
Andrew Tonge

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ON THE FAILURE OF VON NEUMANN'S INEQUALITY

by Anna Maria MANTERO and Andrew TONGE

Summary.- We examine von Neumann type inequalities for homogeneous polynomials in several commuting operators on a complex Hilbert space. Our results simplify considerably and improve slightly work of Varopoulos [7,8]. We also generalise theorems of Dixon [3].

Résumé.- Nous étudions des inégalités de type von Neumann pour des polynômes homogènes en plusieurs opérateurs commutants sur un espace d'Hilbert complexe. Nos résultats apportent une simplification considérable et une amélioration légère au travail de Varopoulos [7,8]. Nous généralisons également des théorèmes de Dixon [3].

I. INTRODUCTION.

In 1951, J. von Neumann [9] proved that if T is a (linear) contraction on a complex Hilbert space, then

$$\|Q(T)\| \leq \sup\left\{ |Q(z)| : z\in\mathbb{C},\ |z| \leq 1\right\}$$

whenever Q is a complex polynomial. This result was generalised by many people, and in particular by Brehmer [1], whose method shows that if $T_1, \ldots, T_N$ are commuting operators on a complex Hilbert space H such that

$$(\sum_{n=1}^{N} \|T_n h\|^2)^{1/2} \le \|h\| \qquad \forall\, h \in H$$

and if Q is a complex polynomial in N variables, then

$$\|Q(T_1, \ldots, T_N)\| \le \sup\{|Q(z_1, \ldots, z_N)| : |z_n| \le 1,\ 1 \le n \le N\}.$$

It was Varopoulos [7] who first discovered that the more natural inequality

$$\|Q(T_1, \ldots, T_N)\| \le \sup\{|Q(z_1, \ldots, z_N)| : \sum_{n=1}^{N} |z_n|^2 \le 1\}$$

is in general false. More precisely, he proved

THEOREM A [8]. _For every_ $K > 0$, _there exist commuting operators_ $T_1, \ldots, T_N$ _on some finite dimensional complex Hilbert space_ H _and a complex homogeneous polynomial_ $Q(z_1, \ldots, z_N)$ _of degree_ 3 _such that_

$$(\sum_{n=1}^{N} \|T_n h\|^2)^{1/2} \le \|h\| \qquad \forall\, h \in H$$

and $\|Q(T_1, \ldots, T_N)\| > K \sup\{|Q(z_1, \ldots, z_N)| : \sum_{n=1}^{N} |z_n|^2 \le 1\}$.

In this paper, we shall give a simpler proof of the following more general result.

THEOREM 1. _Let_ $2 \le p \le \infty$. _For all positive integers_ S _and_ N, _there exist commuting operators_ $T_1, \ldots, T_N$ _on some finite dimensional complex Hilbert space_ H, _and a complex polynomial_ $Q(z_1, \ldots, z_N)$ _of degree_ S _such that_

$$(\sum_n \|T_n h\|^p)^{1/p} \le \|h\| \qquad \forall\, h \in H$$

and $\|Q(T_1, \ldots, T_N)\| \ge AN^{\Phi} \sup\{|Q(z_1, \ldots, z_N)| : (\sum_n |z_n|^p)^{1/p} \le 1\}$

$$\equiv AN^{\Phi} \|Q\|_p$$

where A _is a constant independent of_ N _and_ $\Phi = \frac{1}{2}\left[\frac{S-1}{2}\right]$.

Here we have adopted the usual convention that $(\sum_n |a_n|^p)^{1/p}$ be interpreted as

$\sup_n |a_n|$ when $p = \infty$. The symbol $[.]$ means "integer part of ."

We note, following Varopoulos [8], that the theorem for $p = \infty$ follows easily from the case $p = 2$. However, we de not pursue this point, since our method of proof presents the same degree of difficulty for both cases.

Let us observe that a similar theorem may be proved for $1 \leq p \leq 2$. In this case, the exponent Φ is $(\frac{3}{2} - \frac{2}{p})\left[\frac{S-1}{2}\right] + (\frac{1}{2} - \frac{1}{p})$. The proof is the same.

Setting $p = \infty$, we are able to throw some light on the precision of Brehmer's theorem. A simple reductio ad absurdam argument proves the

COROLLARY. If $p > 4$ and $K \geq 1$, <u>there exist commuting operators</u> $T_1, \ldots, T_N$ <u>on some complex Hilbert space</u> H <u>and a complex polynomial</u> $Q(z_1, \ldots, z_N)$ <u>such that</u>

$$(\sum_n \|T_n h\|^p)^{1/p} \leq \|h\| \qquad \forall\, h \in H$$

<u>and</u>

$$\|Q(T_1, \ldots, T_N)\| > K\|Q\|_\infty .$$

The case $p = \infty$ of theorem 1 was proved by Dixon [3] by a different method. In this case, he also established an upper bound for the growth of the norm of a homogeneous polynomial of contractions. We shall prove a similar result for arbitrary p, $1 \leq p \leq \infty$.

THEOREM 2. <u>Suppose that</u> $1 \leq p \leq \infty$ <u>and</u> $\frac{1}{p} + \frac{1}{p'} = 1$. <u>Let</u> $T_1, \ldots, T_N$ <u>be commuting operators on a complex Hilbert space</u> H <u>satisfying</u>

$$(\sum_n \|T_n h\|^p)^{1/p} \leq \|h\| \qquad \forall\, h \in H.$$

<u>Then for every homogeneous complex polynomial</u> $Q(z_1, \ldots, z_N)$ <u>of degree</u> $S \geq 2$ <u>we have</u>

$$\text{(a)} \quad \|Q(T_1, \ldots, T_N)\| \leq GK(S)(2N)^{\frac{S-2}{2}} \|Q\|_p \qquad (2 < p \leq \infty)$$

(b) $\|Q(T_1, \ldots, T_N)\| \le K(S)\, N^{\frac{S-2}{p'}} \|Q\|_p \qquad (1 \le p \le 2)$

where G *is Grothendieck's constant* $(\le 1.527,$ *see* [6]), *and* $K(S) \le (2e)^S$ *is the symmetrisation constant of Davie* [2].

In fact, part (a) is an obvious consequence of Dixon's theorem if we note that the complex Littlewood constant is $\sqrt{2}$ (see [4]).

It should be noted that the case $p = 1$ is trivial. However, it allows us to deduce the pleasing, though superficial

COROLLARY. *If* $T_1, \ldots, T_N$ *are commuting operators on a complex Hilbert space* H *such that*

$$\sum_n \|T_n h\| \le \frac{1}{4e} \|h\| \qquad \forall\, h \in H,$$

then for every complex polynomial $Q(z_1, \ldots, z_N)$, *we have*

$$\|Q(T_1, \ldots, T_N)\| \le 2\|Q\|_1.$$

We conjecture that the "correct" value of Φ in theorem 1 is in fact the exponent of N in theorem 2. This would show Brehmer's theorem to be sharp.

The main tool that we use in the proof of theorem 1 is a probabilistic estimate of certain norms of symmetric random tensors. We must first establish some notation.

Let $\{\xi_{k_1,\ldots,k_S}\ ;\ 1 \le k_s \le N,\ 1 \le s \le S\}$ be random variables such that $\mathrm{prob}(\xi_{k_1,\ldots,k_S} = 1) = \mathrm{prob}(\xi_{k_1,\ldots,k_S} = -1) = \frac{1}{2}$, such that $\xi_{k_1,\ldots,k_S} = \xi_{\sigma(k_1,\ldots,k_S)}$ for every permutation σ, and such that the family $\{\xi_{k_1,\ldots,k_S} : 1 \le k_1 \le k_2 \le \ldots \le k_S \le N\}$ is independent. If $1 \le p \le \infty$, we shall denote by $\|\xi\|_{L(p;N;S)}$ the injective tensor product norm :

$$\|\xi\|_{\ell_N^p \check{\otimes} \ell_N^p \check{\otimes} \cdots \check{\otimes} \ell_N^p} = \sup \left\{ \left| \sum_{k_1,\ldots,k_S} \xi_{k_1,\ldots k_S} x^{(1)}_{k_1} \cdots x^{(S)}_{k_S} \right| \right\}$$

where the supremum is taken over all S-tuples $(x^{(1)}, \ldots, x^{(S)})$ of elements of the unit ball of $\ell_N^{p'}$ $(\frac{1}{p}+\frac{1}{p'}=1)$. Here $\ell_N^{p'}$ denotes the Banach space of complex N-tuples $(z_1,\ldots,z_N)$ with the norm $(\sum_n |z_n|^{p'})^{1/p'}$. If there is no possibility of confusion, we shall write $\|\xi\|_{L(p)} = \|\xi\|_{L(p;N)} = \|\xi\|_{L(p;N;S)}$.

We can now state :

PROPOSITION 3. Let S be a positive integer and take $1 \le p \le \infty$. Then for all $1 > \delta > 0$, and for all N, we have

$$\text{prob}(\|\xi\|_{L(p;N;S)} \le BN^{\Psi}) \ge \delta .$$

Here B is a constant independent of N, and

$$\Psi = \Psi(S,p) = \begin{cases} \frac{1}{2} & (p \ge 2) \\ \frac{1}{2}+S(\frac{1}{p}-\frac{1}{2}) & (1 \le p \le 2). \end{cases}$$

We shall see in section 5 that this proposition either contains or implies easily all the probabilistic estimates of [7] and [8].

Finally, we urge the reader not to be too frightened by the necessarily cumbersome notation. (S) he should not hesitate to imagine throughout that $S = 3$. This is moreover the only case in which our results are precise

2. THE PROBABILISTIC ESTIMATE.

In this section, we prove proposition 3. The proof is essentially due to Varopoulos

[7,8], but as we must make certain modifications, we give the details.

Proof of proposition 3. We retain the notations established in the introduction. First define

$$\Xi(x^{(1)}, \ldots, x^{(S)}) = \sum_{k_1,\ldots,k_S} \xi_{k_1,\ldots,k_S} x^{(1)}_{k_1} \cdots k^{(S)}_{k_S},$$

and note that

$$\|\xi\|_{L(p;N;S)} \leq 2^S \sup\{|\Xi(x^{(1)},\ldots,x^{(S)})|\} \equiv 2^S |||\xi|||$$

where the supremum is taken over all S-tuples $(x^{(1)}, \ldots, x^{(S)})$ of real elements of the unit ball of $\ell^{p'}_N$.

We observe that it is possible to cover the real unit ball of $\ell^{p'}_N$ by $M \leq (\frac{2+\varepsilon}{\varepsilon})^N$ real balls of radius $\varepsilon < 1$, whose centres $a^{(m)}$, $1 \leq m \leq M$, also lie in the real unit ball.

Now, if we fix $x^{(1)}, \ldots, x^{(S)}$ in the real unit ball of $\ell^{p'}_N$, we can choose $a^{(r_1)}, \ldots, a^{(r_S)}$ such that $\|x^{(s)} - a^{(r_s)}\| \leq \varepsilon$, $1 \leq s \leq S$. Then, using the appropriate generalisation of the identity

$$(xy - ab) = (x-a)(y-b) + a(y-b) + b(x-a),$$

we obtain

$$|\Xi(x^{(1)},\ldots, x^{(S)}) - \Xi(a^{(r_1)}, \ldots, a^{(r_S)})|$$

$$= \Big| \sum_{k_1,\ldots,k_S} \xi_{k_1,\ldots,k_S} \Big[x^{(1)}_{k_1} \cdots x^{(S)}_{k_S} - a^{(r_1)}_{k_1} \cdots a^{(r_S)}_{k_S}\Big] \Big|$$

$$\leq C(S)\, \varepsilon |||\xi|||,$$

where $C(S) > 1$ depends only on S.

On choosing $\varepsilon = \frac{1}{2C(S)}$, this yields

$$(1) \qquad |||\xi||| \leq 2 \sup\{|\Xi(a^{(r_1)}, \ldots, a^{(r_S)})|\}$$

where the supremum is taken over all possible choices of the $a^{(r_s)}$.

Now we claim that if $x^{(1)}, \ldots, x^{(S)}$ are in the real unit ball of $\ell^{p'}_N$, then

(2) $$\text{prob}\left[\left|\Xi(x^{(1)},\ldots,x^{(S)})\right| \ge \alpha\right] \le \begin{cases} 2\exp(-\alpha^2/2S!) & (p \ge 2) \\ 2\exp(-\alpha^2 N^{S(1-2/p)}/2S!) & (1 \le p \le 2). \end{cases}$$

To prove this, write

$$u_{k_1,\ldots,k_S} = \Sigma\, x^{(1)}_{j_1} \ldots x^{(S)}_{j_S}$$

where the sum is taken over all $(j_1,\ldots,j_S)$ which are permutations of $(k_1,\ldots,k_S)$.

Then
$$\Xi(x^{(1)},\ldots,x^{(S)}) = \sum_{k_1\le\ldots\le k_S} \xi_{k_1,\ldots,k_S}\, u_{k_1,\ldots,k_S}.$$

If now $\lambda \in \mathbb{R}$, we have, on taking expectations

$$\mathbb{E}\exp\left[\lambda\,\Xi(x^{(1)},\ldots,x^{(S)})\right] = \prod_{k_1\le\ldots\le k_S} \cos h(\lambda u_{k_1,\ldots,k_S})$$

$$\le \exp\left[\frac{1}{2}\lambda^2 \sum_{k_1\le\ldots\le k_S} (u_{k_1,\ldots,k_S})^2\right]$$

$$\le \exp\left[\frac{S!}{2}\lambda^2 \|x^{(1)}\|^2_{\ell^2_N} \ldots \|x^{(S)}\|^2_{\ell^2_N}\right]$$

$$\le \begin{cases} \exp\left[S!\,\lambda^2/2\right] & (p \ge 2) \\ \exp\left[S!\,\lambda^2 N^{S(\frac{2}{p}-1)}/2\right] & (1\le p\le 2). \end{cases}$$

This, together with Chebyshev's inequality and a suitable choice of λ, implies (2).

Thus, we arrive at

$$\text{prob}\left[\|\xi\|_{L(p;N;S)} \ge 2^{S+1}\alpha\right] \le \text{prob}\left[|||\xi||| \ge 2\alpha\right]$$

$$\le \text{prob}\left[\sup_{\substack{1\le r_s\le M \\ 1\le s\le S}} \left|\Xi(a^{(r_1)},\ldots,a^{(r_S)})\right| \ge \alpha\right] \quad \text{by (1)}$$

$$\le \begin{cases} 2\left[1+4C(S)\right]^{NS} \exp\left[-\alpha^2/2S!\right] & (p \ge 2) \\ 2\left[1+4C(S)\right]^{NS} \exp\left[-\alpha^2 N^{S(1-\frac{2}{p})}/2S!\right] & (1\le p\le 2). \end{cases}$$

Now, setting

$$\alpha^2 = \begin{cases} (4S)S!\ \log\left[\frac{1+4C(S)}{1-\delta}\right] N & (p \geq 2) \\ (4S)S!\ \log\left[\frac{1+4C(S)}{1-\delta}\right] N^{1-S(1-\frac{2}{p})} & (1 \leq p \leq 2) \end{cases}$$

we have the conclusion of the proposition.

3. THE PROOF OF THEOREM 1.

We base our proof on a construction of Dixon [3] and on proposition 3. First we state a lemma ; the theorem will follow as a simple consequence.

Since it is enough to prove theorem 1 for odd integers S, we shall suppose throughout this section that $S = 2R + 1$. If $1 \leq p \leq \infty$, we know, by proposition 3 that there is a symmetric tensor ξ such that

$$\xi_{k_1,\ldots,k_S} = \pm 1 \qquad \forall\ 1 \leq k_s \leq N \qquad \forall\ 1 \leq s \leq S$$

and

$$\|\xi\|_{L(p';N;S)} \leq B\, N^{\Psi(p')},$$

where B is a constant independent of N.

LEMMA 3.1. There exist a complex Hilbert space H and commuting contractions $U_1,\ldots,U_N$ on $\mathcal{H} = [e] \oplus H \oplus [f]$ such that

$$U_{k_1} \ldots U_{k_S}\, e = C^{-1} N^{-R/2}\, \xi_{k_1,\ldots,k_S} f \equiv \tau_{k_1,\ldots,k_S} f$$

and $U_{k_1} \ldots U_{k_S} h = 0 \quad \forall\, h \in H \oplus [f]$

where C is a constant independent of N.

(Here $[e]$ denotes the one dimensional complex Hilbert generated by e, $\|e\| = 1$).

Proof of theorem 1. With τ and $U_1, \ldots, U_N$ as in lemma 3.1, we set

$$Q(z_1, \ldots, z_N) = \sum_{k_1, \ldots, k_S} \tau_{k_1, \ldots, k_S} z_{k_1} \ldots z_{k_S} ,$$

and write $T_n = N^{-1/p} U_n$ $(1 \leq n \leq N)$. Clearly we have

$$(\sum_n \|T_n h\|^p)^{1/p} \leq \|h\| \qquad \forall\, h \in \mathcal{H} .$$

But

$$\begin{aligned}
\|Q(T_1, \ldots, T_N)\| &= \Big| \sum_{k_1, \ldots, k_S} \tau_{k_1, \ldots, k_S} \langle T_{k_1} \ldots T_{k_S} e, f \rangle \Big| \\
&= \Big| \sum_{k_1, \ldots, k_S} (\tau_{k_1, \ldots, k_S})^2 N^{-S/p} \Big| \\
&= C^{-2} N^{S/p'} . N^{-R} \\
&\geq (BC)^{-1} N^{(S/p')-(R/2)-\Psi(p')} \|\tau\|_{L(p';N;S)} \\
&\geq (BC)^{-1} N^{\Phi} \|Q\|_p
\end{aligned}$$

where a direct calculation shows that
$$\Phi = \begin{cases} R/2 & (p \geq 2) \\ (\frac{3}{2} - \frac{2}{p})R + (\frac{1}{2} - \frac{1}{p}) & (1 \leq p \leq 2) \end{cases} .$$

Observing that $R = \left[\frac{S-1}{2}\right]$, we have the result required.

We now follow Dixon's construction to give the

Proof of lemma 3.1. First of all, we define

$$H = E_1 \oplus \ldots \oplus E_R \oplus F_R \oplus \ldots \oplus F_1$$

where E_r, F_r are complex Hilbert spaces with bases $\{e_{k_1, \ldots, k_r} ; 1 \leq k_1 \leq \ldots \leq k_r \leq N\}$ and $\{f_{j_1, \ldots, j_r} ; 1 \leq j_1 \leq \ldots \leq j_r \leq N\}$ respectively. Now, if $1 \leq k_1, \ldots, k_r \leq N$, let us write $[k_1, \ldots, k_r]$ for the non-decreasing rearrangement. We define, for $1 \leq n \leq N$, the operators $U_n : \mathcal{H} \to \mathcal{H}$ by

$$U_n e = e_n$$

$$U_n e\left[k_1,\dots,k_r\right] = e\left[n,k_1,\dots,k_r\right] \qquad (1 \le r \le R-1)$$

$$U_n e\left[k_1,\dots,k_R\right] = \sum_{j_1\le\dots\le j_R} \tau_{n,k_1,\dots,k_R,j_1,\dots,j_R}\, f_{j_1,\dots,j_R}$$

$$U_n\, f\left[j_1,\dots,j_r\right] = \begin{cases} 0 & \text{if } n \notin \{j_1,\dots,j_r\} \\ f\left[j_1,\dots,j_{S-1},j_{S+1},\dots,j_r\right] & \text{if } n = j_S \end{cases} \qquad (2 \le r \le R)$$

$$U_n f_j = \delta_{nj}\, f$$

$$U_n f = 0.$$

By the symmetry of ξ, the U_n's are commuting operators. They will be contractions if

$$\|\xi\|_{\ell^2_N \check{\otimes} \ell^2_{N^R} \check{\otimes} \ell^2_{N^R}} \le C\, N^{R/2}.$$

But, by the proof of proposition 3, we see that it is possible to choose ξ in such a way that we have simultaneously

$$\|\xi\|_{L(p',N;S)} \le B\, N^{\Psi}$$

and

$$\|\xi\|_{\ell^2_N \check{\otimes} \ell^2_{N^R} \check{\otimes} \ell^2_{N^R}} \le C\, N^{R/2}$$

where C is independent of N. Since the products $U_{k_1} \dots U_{k_S}$ evidently have the required property, the proof is complete.

4. THE PROOF OF THEOREM 2 AND ITS COROLLARY.

It will be convenient to isolate two lemmas. The first resembles Grothendieck's inequality, but lies much more on the surface.

LEMMA 4.1. Consider $1 \le p \le 2$ and suppose that x_n $(1 \le n \le N)$ and y_m $(1 \le m \le M)$ are elements of a Hilbert space satisfying

$$\sum_n \|x_n\|^p \le 1 \quad \text{and} \quad \sum_m \|y_m\|^p \le 1.$$

Then for any matrix (a_{mn}) we have

$$\left|\sum_n \sum_m a_{nm} \langle x_n, y_m\rangle\right| \le \sup \left|\sum_n \sum_m a_{nm}\, s_n\, t_m\right|$$

where the supremum is taken over all $s = (s_n)$ and $t = (t_m)$ with

$$\left(\sum_n |s_n|^p\right)^{1/p} \le 1 \quad \text{and} \quad \left(\sum_m |t_m|^p\right)^{1/p} \le 1.$$

Proof. There is no loss of generality in working with the Hilbert space of finite dimension D generated by the x_n's and the y_m's. In an orthonormal basis of this space, our hypotheses may be rewritten as

$$\sum_n \left(\sum_d |x_{nd}|^2\right)^{p/2} \le 1 \quad \text{and} \quad \sum_m \left(\sum_d |y_{md}|^2\right)^{p/2} \le 1.$$

Suppose now that $\|a\|_{\ell_N^{p'} \check{\otimes} \ell_M^{p'}} \le 1$. Then

$$\left|\sum_n \sum_m a_{nm} \langle x_n, y_m\rangle\right| = \left|\sum_n \sum_m \sum_d a_{nm}\, x_{nd}\, \overline{y_{md}}\right|$$

$$\le \sum_d \left(\sum_n |x_{nd}|^p\right)^{1/p} \left(\sum_m |y_{md}|^p\right)^{1/p} \qquad \text{by hypothesis}$$

$$\le \left[\sum_d \left(\sum_n |x_{nd}|^p\right)^{2/p}\right]^{1/2} \left[\sum_d \left(\sum_m |y_{md}|^p\right)^{2/p}\right]^{1/2}$$

$$\le \left[\sum_n \left(\sum_d |x_{nd}|^2\right)^{p/2}\right]^{1/p} \left[\sum_m \left(\sum_d |y_{m_d}|^2\right)^{p/2}\right]^{1/p}$$

by Minkowski's inequality (since $p \le 2$)

≤ 1 by the conditions on the x_n's and the y_m's.

It should be noticed that a similar lemma is valid for all $1 \le p \le \infty$ - except that we must introduce Grothendieck's constant into the inequality when $p > 2$.

LEMMA 4.2. Let $1 \leq p \leq \infty$. Then if $I : \ell_N^p \check{\otimes} \ell_M^p \to \ell_{NM}^p$ is the identity mapping, we have

$$\|I\| \leq \min(N^{1/p}, M^{1/p}).$$

Proof. This follows immediately from the observation that if $\|a_{nm}\|_{\ell_N^p \check{\otimes} \ell_M^p} \leq 1$ then $\sum_m |a_{nm}|^p \leq 1 \quad \forall n$ and $\sum_n |a_{nm}|^p \leq 1 \quad \forall m$.

We may pass to the

Proof of theorem 2. We have already observed that it suffices to prove (b). Let us then fix $1 \leq p \leq 2$, and let us write

$$Q(z_1, \ldots, z_N) = \sum_{k_1,\ldots,k_S} a_{k_1,\ldots,k_S} z_{k_1} \cdots z_{k_S}$$

where $a_{k_1,\ldots,k_S}$ is a symmetric tensor.

If $g, h \in H$, then it is clear that

$$\sum_{k_2,\ldots,k_S} \|T_{k_2} \cdots T_{k_S} h\|^p \leq \|h\|^p \quad \text{and} \quad \sum_k \|T_{k_1}^* g\|^p \leq \|g\|^p.$$

Using lemma 4.1, we see that

$$|\langle Q(T_1,\ldots,T_N) h, g\rangle| = \Big|\sum_{k_1\ldots k_S} a_{k_1,\ldots,k_S} \langle T_{k_2} \cdots T_{k_S} h, T_{k_1}^* g\rangle\Big|$$

$$\leq \|a\|_{\ell_{N^{S-1}}^{p'} \check{\otimes} \ell_N^{p'}} \|h\| \, \|g\|.$$

By a repeated application of lemma 4.2, we have

$$\|Q(T_1,\ldots,T_N)\| \leq N^{(S-2)/p'} \|a\|_{L(p')}$$

$$= N^{(S-2)/p'} \sup_{\sum_n |x_n^{(s)}|^p \leq 1} \Big|\sum_{k_1,\ldots,k_S} a_{k_1,\ldots,k_S} x_{k_1}^{(1)} \cdots x_{k_S}^{(S)}\Big|$$

$$\leq N^{(S-2)/p'} K(S) \sup_{\sum_n |x_n|^p \leq 1} \left| \sum_{k_1,\ldots,k_S} a_{k_1,\ldots,k_S} x_{k_1} \ldots x_{k_S} \right|$$

by Davie's symmetrisation process [2].

This is exactly what was required.

To prove the corollary, we simply have to express the polynomial Q as a sum of homogeneous polynomials Q_S of degree S, and note that under the hypothesis, we shall have

$$\|Q_S(T_1,\ldots,T_N)\| \leq 2^{-S}\|Q_S\|_1 \leq 2^{-S}\|Q\|_1 .$$

The last inequality may be deduced easily from the well-known fact (see [2] for example) that $\|Q_S\|_\infty \leq \|Q\|_\infty$.

5. THE DEDUCTION OF VAROPOULOS' ESTIMATES.

Since many of the proofs in [7] and [8] are either involved or use the Kahane-Salem-Zygmund theorem, we feel that it is of interest to show how to deduce them simply from proposition 3. Accordingly, we shall fix $S = 3$ throughout.

PROPOSITION 5.1. <u>For every integer</u> N, <u>there exists a symmetric tensor</u> ξ <u>such that</u>

$$\xi_{ijk} = \pm 1 \qquad \forall\ 1 \leq i,j,k \leq N$$

<u>for which</u>

(1) $$\|\xi\|_{\ell^2_N \check{\otimes} \ell^2_N \check{\otimes} \ell^2_N} \leq KN^{1/2}$$

(2) $$\|\xi\|_{\ell^1_N \check{\otimes} \ell^1_N \check{\otimes} \ell^1_N} \leq KN^2$$

<u>and</u>

(3) $$\|\xi\|_{\ell^\infty_N \hat{\otimes} \ell^\infty_N \hat{\otimes} \ell^\infty_N} > \frac{1}{K} N$$

for some constant K independent of N.

Here $\hat{\otimes}$ denotes the projective tensor product.

Proof. Choose $\delta > \frac{1}{2}$, $p = 2$ and $p = 1$ in proposition 3. This yields (1) and (2). (3) follows, as in [7], from the observation that

$$N^3 = \sum_{i,j,k} 1 = |\langle \xi, \xi \rangle| \leq \|\xi\|_{\ell_N^\infty \hat{\otimes} \ell_N^\infty \hat{\otimes} \ell_N^\infty} \|\xi\|_{\ell_N^1 \check{\otimes} \ell_N^1 \check{\otimes} \ell_N^1} .$$

One may deduce proposition 1.1 of [8] as an immediate corollary.

If now a and b are elements of the algebraic tensor product $\ell^2 \otimes \ell^2 \otimes \ell^2$, wa may define a multiplication by

$$ab = (a_{ijk}\, b_{ijk})_{1 \leq i,j,k \leq \infty} .$$

It is known [8] that when $\ell^2 \otimes \ell^2 \otimes \ell^2$ is given the injective tensor product norm, this multiplication is not continuous. Let us prove a more precise result.

PROPOSITION 5.2. Under the above multiplication, we have

(1) $$\|ab\|_{L(2;N)} \leq N^{1/2} \|a\|_{L(2;N)} \|b\|_{L(2;N)} \qquad \forall\, a,b \in \ell_N^2 \check{\otimes} \ell_N^2 \check{\otimes} \ell_N^2$$

and

(2) There exists a tensor $a \in \ell_N^2 \check{\otimes} \ell_N^2 \check{\otimes} \ell_N^2$ such that

$$\|a^2\|_{L(2;N)} \geq \Lambda N^{1/2} \|a\|^2_{L(2;N)}$$

where Λ is a constant independent of N.

Proof. For (2), we need only choose, as in proposition 3, a random tensor ξ with $\|\xi\|_{L(2)} \leq B N^{1/2}$. Then

$$\|\xi^2\|_{L(2)} = N^{3/2} \geq \frac{1}{B^2} N^{1/2} \|\xi\|^2_{L(2)} .$$

We pass to the proof of (1). Suppose then that $\|a\|_{L(2)} \leq 1$ and that $\|b\|_{L(2)} \leq 1$.

Take
$$\sum_i |s_i|^2 \leq 1 \quad , \quad \sum_j |t_j|^2 \leq 1 \quad \text{and} \quad \sum_k |u_k|^2 \leq 1 .$$

Then $\left|\sum_i \sum_j \sum_k a_{ijk}\, b_{ijk}\, s_i\, t_j\, u_k\right| \leq \sum_i |s_i| \left(\sum_{j,k} |a_{ijk}\, u_k|^2\right)^{1/2} \left(\sum_{j,k} |b_{ijk}\, t_j|^2\right)^{1/2}$

and this will be $\leq \sqrt{N}$ if we can show that

$$\sum_{j,k} |a_{ijk}\, u_k|^2 \leq 1 \qquad \text{and} \qquad \sum_{j,k} |b_{ijk}\, t_j|^2 \leq 1 .$$

However, it follows at once from the hypothesis that $\sum_j \left|\sum_k a_{ijk}\, u_k\right|^2 \leq 1$ whenever $\sum_k |u_k|^2 \leq 1$. But, replacing u_k by $\pm u_k$ and averaging over all possible choices of $\pm$, we obtain the desired result.

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