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ON THE FRACTIONAL PARTS OF x/n AND RELATED SEQUENCES

B. SAFFARI

and

R.C. VAUGHAN

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B. SAFFARI and R.C. VAUGHAN

1. Introduction.

1. Throughout this paper $\{x\} = x-[x]$ denotes the fractional part of the real number x. We write $\|x\| = \min_{k\in\mathbf{Z}} |x-k|$ and $e(x) = e^{2\pi ix}$.

Also, the implied constants in the O symbol of Landau and the $\gg$ and $\ll$ symbols of Vinogradov are absolute.

Finally, by a distribution function we always mean a distribution function in the sense of probability theory, defined on the real line.

2. Let (x_n) be a sequence of real numbers. The usual study of the distribution modulo 1 of (x_n) is essentially that of the distribution of the sequence $(e(x_n))$ on the circle $\mathbf{T}$. The main problems are those of investigating

(i) the existence of the asymptotic (or limit) distribution measure

$$\mu = \lim_{k\to\infty} \mu_k \tag{1.1}$$

where

$$\mu_k = \frac{1}{k}\sum_{n=1}^{k} \delta_{e(x_n)} \tag{1.2}$$

with δ_v denoting the Dirac measure at $v \in \mathbf{T}$, and

(ii) the size of the discrepancy

$$\sup_{\omega} |\mu_k(\omega)-\mu(\omega)| \tag{1.3}$$

where ω runs through those arcs of $\mathbf{T}$ whose endpoints have μ-measure zero.

It is classical that the existence of μ together with the assumption that the point $1 \in \mathbf{T}$ has μ-measure zero is equivalent to the existence of a distribution function F such that

$$(1.4) \qquad F(0+) = 0, \quad F(1-) = 1$$

and

$$(1.5) \qquad F(\alpha) = \lim_{k \to \infty} \frac{1}{k} A([0,\alpha),k,(x_n))$$

at every α at which F is continuous, the counting function

$$(1.6) \qquad A([\alpha,\beta),k,(x_n)) = \operatorname{Card}\{n: 1 \leqslant n \leqslant k,\ \alpha \leqslant \{x_n\} < \beta\}$$

being here defined for all real numbers α and β. The conditions (1.4) mean that F is continuous at 0 and 1, and imply that F is constant on the intervals $(-\infty, 0]$ and $[1, \infty)$. In this case F is called the asymptotic (or limit) distribution function modulo 1 of the sequence (x_n), and the discrepancy (1.3) is equal to

$$(1.7) \qquad \sup_{0 \leqslant \alpha < \beta \leqslant 1} \left|\frac{1}{k} A([\alpha,\beta),k,(x_n)) - (F(\beta)-F(\alpha))\right|$$

where α and β run through the continuity points of F.

In some situations it may be more appropriate to consider the existence of the A-asymptotic distribution function modulo 1, namely the existence (outside a countable set), and the continuity at $\alpha = 0$ and $\alpha = 1$, of

$$(1.8) \qquad \lim_{k \to \infty} \sum_{n=1}^{k} a_{k,n} c_\alpha(x_n)$$

where

$$(1.9) \qquad c_\alpha(u) = \begin{cases} 1 & 0 \leqslant \{u\} < \alpha \\ 0 & \text{otherwise} \end{cases}$$

is the characteristic function modulo 1 of $[0,\alpha)$, and $A = (a_{k,n})$ is a positive Toeplitz matrix. Here by a positive Toeplitz matrix we mean that $a_{k,n} \geqslant 0$, $\sum_{n=1}^{\infty} a_{k,n} < \infty$ and $\lim_{k \to \infty} \sum_{n=1}^{\infty} a_{k,n} = 1$.

3. The sequence (x_n) is, of course, independant of k. Our object is to investigate the distribution modulo 1 of $xh(n)$ with x a large real number, $h(n)$ an arithmetical function, and the integer n belonging to $S \cap [1,k]$ where $S \subset \mathbb{N}$ and k depends on x. For our purposes it is somewhat more convenient to replace k by a real parameter y. We call $\mathcal{A} = (a_n(y) : y \in [1, \infty),\ n = 1,2,\ldots)$ a positive Toeplitz transformation if $a_n(y) \geqslant 0$ for all n and y, $\sum_{n=1}^{\infty} a_n(y) < \infty$ for every y, and $\lim_{y \to \infty} \sum_{n=1}^{\infty} a_n(y) = 1$. We are particularly interested in the special case where the $a_n(y)$ are the simple Riesz means (R, λ_n) given by

$$(1.10) \qquad \lambda_n \geqslant 0 \quad (n = 1,2,\ldots), \qquad \lambda_1 > 0$$

and

$$(1.11) \qquad a_n(y) = \begin{cases} \lambda_n \Big/ \sum_{m \leqslant y} \lambda_m & (m \leqslant y) \\ 0 & (m > y) \end{cases}$$

which we assume henceforward, although several of our proofs go through in the general case (see Appendix). Let

$$(1.12) \qquad \Phi_{x,y}(\alpha, h) = \sum_{n=1}^{\infty} a_n(y) c_\alpha(xh(n)).$$

A good deal of our attention will be taken up with $h(n) = 1/n$ and we write

$$(1.13) \qquad \Phi_{x,y}(\alpha) = \sum_{n=1}^{\infty} a_n(y) c_\alpha(x/n).$$

The problems arising from the study of $\Phi_{x,y}(\alpha)$ as x and $y = y(x)$ tend together to infinity are closely related to the Dirichlet divisor problem.

If there exists a distribution function Φ_h such that

$$(1.14) \qquad \Phi_h(0+) = 0, \quad \Phi_h(1-) = 1$$

and

$$(1.15) \qquad \Phi_h(\alpha) = \lim_{x \to \infty} \Phi_{x,y(x)}(\alpha,h)$$

at every α at which Φ_h is continuous, then we call Φ_h the $\mathcal{A}$- asymptotic distribution function modulo 1. This situation is equivalent to the existence on the circle $\mathbb{T}$ of the $\mathcal{A}$- limit (or $\mathcal{A}$- asymptotic) distribution measure

$$(1.16) \qquad \nu = \lim_{x \to \infty} \sum_{n=1}^{\infty} a_n(y)\delta_{e(xh(n))}$$

together with the fact that the point $1 \in \mathbb{T}$ has ν-measure zero. However, if there exists no distribution function Φ_h satisfying both (1.14) and (1.15), then it is more appropriate to investigate the distribution modulo 1 of $xh(n)$ via (1.16).

4. Our interest in this problem arose from investigating the asymptotic behaviour of

$$\sum_{n \leqslant y} c_\alpha(x/n).$$

During our investigation it became obvious that there were methods which could be applied in a much more general situation. In this paper we present these methods, deferring to the sequel the study of special methods.

As an example of the application of Theorem 2, consider a subset A of $\mathbb{N}^*$ such that the counting function

$$A(x) = \sum_{\substack{a \leqslant x \\ a \in A}} 1$$

satisfies

$$A(x) = x^{\sigma}L(x)$$

where σ is a constant with $0 < \sigma \leqslant 1$ and L is a slowly varying function, that is

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1$$

for any positive constant c. Then

$$(1.17) \qquad \lim_{x \to \infty} \frac{1}{A(x)} \sum_{\substack{a \leqslant x \\ a \in A}} c_{\alpha}(x/a) = \sum_{n=1}^{\infty} (n^{-\sigma} - (n+\alpha)^{-\sigma}).$$

Moreover, there exists a function $y_0(x)$ such that if $y > y_0(x)$ and $y = o(x)$ as $x \to \infty$, then

$$(1.18) \qquad \lim_{x \to \infty} \frac{1}{A(y)} \sum_{\substack{a \leqslant y \\ a \in A}} c_{\alpha}(x/a) = \alpha.$$

Relation (1.18) means that the fractional parts $\{x/a\}$, where a runs over $[0,y] \cap A$, are asymptotically uniformly distributed, whereas (1.17) means that if a runs over the whole of $[0,x] \cap A$, then the $\{x/a\}$ have the asymptotic distribution function

$$\sum_{n=1}^{\infty} (n^{-\sigma} - (n+\alpha)^{-\sigma}).$$

2. Theorems and proofs.

1. We first of all state a theorem which gives a sufficient condition for the (R,λ_n)- asymptotic distribution to be uniform. This is essentially due to Erdös and Turan [1], [2] and is a finite form of Weyl's criterion. It is also possible, of course, to give a necessary condition corresponding to Weyl's criterion, and to give results when the asymptotic distribution is non-uniform but continuous, but we have no applications in mind for these.

Theorem 1 is somewhat divorced from the following theorems. However, it clearly applies to the general situation. As an application we have in mind the case

$$h(n) = \log n. \tag{2.1}$$

THEOREM 1. *Let the discrepancy* $D_{x,y}(h)$ *be defined by*

$$D_{x,y}(h) = \sup_{0\leq\alpha<\beta\leq 1} |\Phi_{x,y}(\beta,h)-\Phi_{x,y}(\alpha,h)-(\beta-\alpha)|. \tag{2.2}$$

Then, for any positive integer m,

$$D_{x,y}(h) < \frac{6}{m+1}+\frac{4}{\pi}\sum_{k=1}^{m}\left(\frac{1}{k}-\frac{1}{m+1}\right)\left|\sum_{n=1}^{\infty}a_n(y)e(kxh(n))\right|. \tag{2.3}$$

Theorem 1 is a generalization of Theorem 2.2.5 of Kuipers and Niederreiter [3], and can be proved in exactly the same way.

2. The following theorem (together with the observations made in Lemmas 2, 3, 4) shows that the (R,λ_n) asymptotic distribution function modulo 1 of x/n can exist under very general conditions provided that y is not too small compared with x.

Whenever $\xi \geq 1$ and $\sigma \geq 0$ define

$$(2.4)\qquad F(\alpha\ ;\ \xi,\sigma) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha\ ;\ \xi)(1-\xi^{\sigma}([\xi]+\alpha)^{-\sigma})+\xi^{\sigma}\sum_{k>\xi}(k^{-\sigma}-(k+\alpha)^{-\sigma}) & (0 < \alpha < 1,\ \sigma > 0) \\ \alpha & (0 < \alpha < 1,\ \sigma = 0) \end{cases}$$

where

$$(2.5)\qquad \theta(\alpha\ ;\ \xi) = \begin{cases} 1 & \text{if } (\xi-\alpha,\xi] \cap \mathbb{N} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

<u>THEOREM 2.</u> <u>Suppose that for every real number</u> t <u>with</u> $0 < t < 1$ <u>the limit</u>

$$(2.6)\qquad \lim_{y\to\infty} \sum_{n\leqslant ty} a_n(y)$$

<u>exists and for at least one value of</u> t <u>is non-zero. Then there is a non-negative real number</u> σ <u>such that for every real number</u> ε <u>with</u> $0 < \varepsilon < \frac{1}{2}$ <u>there is a real number</u> $y_0(\varepsilon,\sigma) \geqslant 1$ <u>so that whenever</u> $y_0(\varepsilon,\sigma) \leqslant y \leqslant x$ <u>we have</u>

$$(2.7)\qquad \Phi_{x,y}(\alpha) = F(\alpha\ ;\ x/y,\sigma) + O(\varepsilon^{1+\sigma}xy^{-1}) + O(2^{\sigma}\varepsilon^{\sigma}).$$

Lemma 1 below will show that the limit (2.6) is t^{σ}, which defines σ. We observe that when $\sigma = 0$ Theorem 2 fails to give non-trivial information. Very likely $\Phi_{x,y}(\alpha) \to \alpha$ still holds in this case, at least when $\sum_{n\leqslant y} \lambda_n \to \infty$, but even when $\lambda_n = 1/n$ this is a deep result.

Before proceeding with the proof of Theorem 2 we state a corollary concerning the case when the integer n is allowed only to run through a shorter interval $[y,z]$.

<u>COROLLARY 2.1.</u> <u>With the assumptions of Theorem 2, if</u> $y_o(\varepsilon,\sigma) \leqslant y < z \leqslant x/2$, $(y/z)^\sigma < 1-\varepsilon^{2+\sigma}$, $\varepsilon z \leqslant y$, <u>and</u> $\sum_{y<n\leqslant z} \lambda_n > 0$, <u>then</u>

$$(2.8) \qquad \frac{\sum_{y<n\leqslant z} \lambda_n c_\alpha(x/n)}{\sum_{y<n\leqslant z} \lambda_n} - \alpha \ll$$

$$\ll (\sigma 2^\sigma z x^{-1} + \varepsilon^{1+\sigma} x y^{-1} + 2^\sigma \varepsilon^\sigma)(1 - y^\sigma z^{-\sigma} - \varepsilon^{2+\sigma})^{-1}.$$

We remark that, in this case, the asymptotic distribution is always the uniform one, at least when $\sigma > 0$.

3. The proof of Theorem 2 requires the following lemma.

<u>LEMMA 1.</u> <u>On the hypothesis of Theorem 2 there is a non-negative real number</u> σ <u>such that for every real number</u> ε <u>with</u> $0 < \varepsilon < 1/2$ <u>there is a real number</u> $y_o(\varepsilon,\sigma) \geqslant 1$ <u>so that whenever</u> $y \geqslant y_o(\varepsilon,\sigma)$ <u>we have, for every</u> t <u>with</u> $\varepsilon \leqslant t \leqslant 1$,

$$(2.9) \qquad \Big| t^\sigma - \sum_{n\leqslant ty} a_n(y) \Big| < \varepsilon^{2+\sigma}.$$

<u>Proof.</u> The existence of (2.6) for every real number t with $0 < t < 1$ together with the assumption that for some t in this range the limit is non-zero imply that there is a non-negative real number σ such that for every t with $0 < t \leqslant 1$ we have

$$\lim_{y\to\infty} \sum_{n\leqslant ty} a_n(y) = t^\sigma.$$

Let

$$N = [2e^{\varepsilon^{-2-\varepsilon}} \max(1,\sigma)]+1$$

and choose $y_o(\varepsilon,\sigma) \geqslant 1$ so that if $y \geqslant y_o(\varepsilon,\sigma)$, then for every integer r with $1 \leqslant r \leqslant N$ we have

$$(2.10)\qquad \left|\left(\frac{r}{N}\right)^{\sigma}-\sum_{n\leqslant ry/N} a_n(y)\right| < \frac{1}{2}\varepsilon^{2+\sigma}.$$

Now choose an integer q such that

$$(2.11)\qquad \frac{1}{N}\leqslant \frac{q}{N} < t \leqslant \frac{q+1}{N} \leqslant 1,$$

which is always possible if $\varepsilon \leqslant t \leqslant 1$. Note that

$$\left(\frac{q+1}{N}\right)^{\sigma}-\left(\frac{q}{N}\right)^{\sigma} = \int_{q/N}^{(q+1)/N} \sigma u^{\sigma-1}du \leqslant \frac{\sigma}{N}\max\left(\left(\frac{q+1}{N}\right)^{\sigma-1},\ \left(\frac{q}{N}\right)^{\sigma-1}\right)$$

$$\leqslant \sigma\max(N^{-1},N^{-\sigma}) \leqslant \max\left(\sigma N^{-1},(e\log N)^{-1}\right)$$

$$< \frac{1}{2}\varepsilon^{2+\sigma}.$$

Thus, by (2.10) and (2.11),

$$\sum_{n\leqslant ty} a_n(y) \leqslant \sum_{n\leqslant (q+1)y/N} a_n(y) < \left(\frac{q+1}{N}\right)^{\sigma}+\frac{1}{2}\varepsilon^{2+\sigma} < \left(\frac{q}{N}\right)^{\sigma}+\varepsilon^{2+\sigma} \leqslant t^{\sigma}+\varepsilon^{2+\sigma}$$

and

$$\sum_{n\leqslant ty} a_n(y) \geqslant \sum_{n\leqslant qy/N} a_n(y) > \left(\frac{q}{N}\right)^{\sigma}-\frac{1}{2}\varepsilon^{2+\sigma} > \left(\frac{q+1}{N}\right)-\varepsilon^{2+\sigma} \geqslant t^{\sigma}-\varepsilon^{2+\sigma}.$$

These last two inequalities give (2.9) as required.

4. <u>Proof of Theorem 2.</u> Since (2.7) is trivially true when $\alpha \leqslant 0$ or $\alpha \geqslant 1$, we may assume $0 < \alpha < 1$. Let $K = [\frac{x}{\varepsilon y}-\alpha]$. Then, by (1.13), (1.11), (1.9), Lemma 1 and (2.5),

$$\Phi_{x,y}(\alpha) = \sum_{\frac{x}{K+\alpha}<n\leqslant y} a_n(y)c_{\alpha}(x/n) + O\left(\sum_{n\leqslant 2\varepsilon y} a_n(y)\right)$$

$$= \sum_{k=1}^{K} \sum_{\substack{n\leqslant y\\ x/(k+\alpha)<n\leqslant x/k}} a_n(y) + O\left(2^{\sigma}\varepsilon^{\sigma}\right)$$

$$= \theta(\alpha\ ;\ x/y)\left(\sum_{n\leqslant y} a_n(y) - \sum_{n\leqslant x/([x/y]+\alpha)} a_n(y)\right)$$

$$+\sum_{x/y\leqslant k\leqslant K}\left(\sum_{n\leqslant x/k} a_n(y) - \sum_{n\leqslant x/(k+\alpha)} a_n(y)\right) + O\,(2^\sigma\varepsilon^\sigma).$$

Hence, by Lemma 1 and (2.4),

$$\Phi_{x,y}(\alpha) = F(\alpha\ ;\ x/y,\sigma) + O\,(\varepsilon^{2+\sigma}K) + O\,(2^\sigma\varepsilon^\sigma) + O\left(\sum_{k>K}\left(\frac{x}{y}\right)^\sigma (k^{-\sigma}-(k+\alpha)^{-\sigma})\right).$$

The proof of (2.7) is completed by observing that $\varepsilon K \leqslant x/y$ and

$$\sum_{k>K}(k^{-\sigma}-(k+\alpha)^{-\sigma}) = \sum_{k>K}\int_k^{k+\alpha}\sigma u^{-\sigma-1}du \leqslant \sum_{k>K}\int_k^{k+1}\sigma u^{-\sigma-1}du$$

$$= (K+1)^{-\sigma} < (2\varepsilon y/x)^\sigma.$$

5. <u>Proof of Corollary 2.1.</u> We use (2.7) and Lemma 1. The condition that $(y/z)^\sigma < 1-\varepsilon^{2+\sigma}$ means that we can assume that $\sigma > 0$. Suppose that $\xi > 1$. Then, by (2.4),

$$F(\alpha\ ;\ \xi,\sigma) \leqslant \xi^\sigma\int_{[\xi]}^{[\xi]+\alpha} u^{-\sigma}du + O\left(\theta(\alpha\ ;\ \xi)\int_{\xi/([\xi]+\alpha)}^{1}\sigma u^{\sigma-1}du\right)$$

$$\leqslant \alpha(\xi/[\xi])^\sigma + O\left(\theta(\alpha\ ;\ \xi)\sigma\left(1-\xi/([\xi]+\alpha)\right)\max\left(1,\left(\frac{\xi}{[\xi]+\alpha}\right)^{\sigma-1}\right)\right)$$

$$= \alpha + O\,(\sigma 2^\sigma\xi^{-1}).$$

Similarly

$$F(\alpha\ ;\ \xi,\sigma) \geqslant \xi^\sigma\int_{\xi+1}^{\xi+1+\alpha} u^{-\sigma}du \geqslant \alpha\left(1+\frac{1+\alpha}{\xi}\right)^{-\sigma} \geqslant \alpha - \frac{\sigma\alpha(1+\alpha)}{\xi}.$$

Hence, if $y_0(\varepsilon,\sigma) \leqslant y \leqslant x/2$, then by (1.11), (1.13) and (2.7),

$$\sum_{n\leqslant y}\lambda_n c_\alpha(x/n) = \left(\alpha + O\,(\sigma 2^\sigma y x^{-1} + x\varepsilon^{1+\sigma}y^{-1} + 2^\sigma\varepsilon^\sigma)\right)\sum_{n\leqslant y}\lambda_n.$$

Thus, if $y_0(\varepsilon,\sigma) \leqslant y < z \leqslant \frac{1}{2}$, then

$$\sum_{y<n\leqslant z} \lambda_n c_\alpha(x/n) = \alpha \sum_{y<n\leqslant z} \lambda_n + \underline{O}\left((\sigma 2^\sigma y x^{-1} + x\varepsilon^{1+\sigma} y^{-1} + 2^\sigma \varepsilon^\sigma) \sum_{n\leqslant y} \lambda_n\right).$$

We complete the proof of (2.8) by observing that by (1.11) and Lemma 1,

$$\left(\sum_{n\leqslant z} \lambda_n\right) \Big/ \sum_{y<n\leqslant z} \lambda_n = \left(1 - \left(\sum_{n\leqslant y} \lambda_n\right) \Big/ \sum_{n\leqslant z} \lambda_n\right)^{-1} < \left(1-(y/z)^\sigma - \varepsilon^{2+\sigma}\right)^{-1}.$$

6. In this section we make some observations concerning the nature of $F(\alpha\ ;\ \xi,\sigma)$.

<u>LEMMA 2.</u> <u>Suppose that</u> $0 \leqslant \alpha \leqslant 1$ <u>and</u> $\xi \geqslant 1$. <u>Then</u>

(2.12) $$F(\alpha\ ;\ \xi,\sigma) = \alpha + \underline{O}\,(\sigma 2^\sigma \xi^{-1}) \qquad (\sigma > 0),$$

(2.13) $$\lim_{\sigma \to 0+} F(\alpha\ ;\ \xi,\sigma) = \alpha = F(\alpha\ ;\ \xi,0)$$

<u>and</u>

(2.14) $$F(\alpha\ ;\ 1,\sigma) = \sum_{k=1}^{\infty} (k^{-\sigma} - (k+\alpha)^{-\sigma}) \qquad (\sigma > 0).$$

By (2.14) with $\sigma = 1$, $F(\alpha\ ;\ 1,1) = \Gamma'(\alpha)/\Gamma(\alpha)+\gamma+1/\alpha$ where Γ is the gamma function and γ is Euler's constant.

<u>Proof.</u> The asymptotic formula (2.12) was established in the proof of (2.8), (2.13) then follows trivially, and (2.14) is immediate from (2.4).

<u>LEMMA 3.</u> <u>For each</u> $\xi \geqslant 1$ <u>and</u> $\sigma > 0$ <u>the function</u> $F(\alpha\ ;\ \xi,0)$ <u>is a continuous function of</u> α <u>and is analytic on</u> $\mathbb{R}\setminus\{0,\{\xi\},1\}$ <u>with</u>

(2.15) $$F'(\alpha) = \begin{cases} 0 & (\alpha < 0,\ \alpha > 1) \\ \sigma\xi^\sigma \sum_{k>\xi} (k+\alpha)^{-\sigma-1} & (0 < \alpha < \{\xi\}) \\ \sigma\xi^\sigma([\xi]+\alpha)^{-\sigma-1} + \sigma\xi^\sigma \sum_{k>\xi} (k+\alpha)^{-\sigma-1} & (\{\xi\} < \alpha < 1). \end{cases}$$

<u>The points</u> 0, $\{\xi\}$ <u>and</u> 1 <u>are angular points of</u> F.

<u>LEMMA 4.</u> <u>Suppose that</u> $0 < \alpha < 1$ <u>and</u> $\sigma > 0$. <u>Then considered as a function of</u> ξ, $F(\alpha\,;\,\xi,\sigma)$ <u>is continuous on</u> $[1,\infty)\setminus\{2,3,4,\ldots\}$ <u>and for each integer</u> $n \geqslant 2$,

$$(2.16)\qquad \lim_{\xi\to n-} F(\alpha\,;\,\xi,\sigma) = n^{\sigma}\sum_{k=n+1}^{\infty}(k^{-\sigma}-(k+\alpha)^{-\sigma})$$

<u>and</u>

$$(2.17)\qquad \lim_{\xi\to n+} F(\alpha\,;\,\xi,\sigma) = n^{\sigma}\sum_{k=n}^{\infty}(k^{-\sigma}-(k+\alpha)^{-\sigma}) = F(\alpha\,;\,n,\sigma).$$

7. We now establish upper and lower bounds for the mean square of $\Phi_{x,y}(\alpha)-\alpha$ which in turn imply respectively

(i) that if y is small compared with x then the only possible (R,λ_n) asymtotic distribution modulo 1 is the uniform one, and

(ii) that the discrepancy cannot be too small.

<u>THEOREM 3.</u> <u>Suppose that</u> x_0 <u>and</u> x <u>are non-negative real numbers,</u> $y \geqslant 1$ <u>and</u> $0 < \alpha < 1$. <u>Then</u>

$$(2.18)\qquad \int_{x_0}^{x_0+x} \left|\Phi_{u,y}(\alpha)-\alpha\right|^2 du \leqslant \min(I_1,I_2)$$

<u>where</u>

$$(2.19)\qquad I_1 = \tfrac{1}{3}(x+y^2)\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\tfrac{1}{m}a_{mn}(y)\right)^2$$

<u>and</u>

$$(2.20)\qquad I_2 = \sum_{n=1}^{\infty}\left(\tfrac{1}{3}x+\tfrac{1}{2}yn\right)\left(\sum_{m=1}^{\infty}\tfrac{1}{m}a_{mn}(y)\right)^2.$$

This theorem can be thought of in a rather loose way as a law of the iterated logarithm. This will be discussed further in a later paper. (see [5]).

THEOREM 4. On the hypothesis of Theorem 3,

$$\int_{x_0}^{x_0+x} \left|\Phi_{u,y}(\alpha)-\alpha\right|^2 du \geqslant \max(J_1, J_2) \tag{2.21}$$

where

$$J_1 = \tfrac{1}{2}\pi^{-2}(x-y^2)\sum_{n=1}^{\infty} \left|\sum_{m=1}^{\infty} \tfrac{1}{m} a_{mn}(y) \left(1-e(\alpha m)\right) \right|^2 \tag{2.22}$$

and

$$J_2 = ((2\pi)^{-2} \sum_{n=1}^{\infty} (2x-3yn) \left|\sum_{m=1}^{\infty} \tfrac{1}{m} a_{mn}(y) \left(1-e(\alpha m)\right) \right|^2. \tag{2.23}$$

By taking the real part of the innermost sum in (2.22) and (2.23) and then discarding all the terms with $m > 1$ one obtains in (2.21) the particularly simple lower bound $\max(L_1, L_2)$, where

$$L_1 = 2\pi^{-2}(\sin \pi\alpha)^4(x-y^2)\sum_{n=1}^{\infty} a_n^2(y)$$

and

$$L_2 = \pi^{-2}(\sin \pi\alpha)^4\sum_{n=1}^{\infty} (2x-3yn)a_n^2(y).$$

However, in certain circumstances this loses a factor as large as loglog y.

COROLLARY 4.1. Let the discrepancy $D_{x,y}$ be given by

$$D_{x,y} = \sup_{0\leqslant\alpha<\beta\leqslant 1} \left|\Phi_{x,y}(\beta)-\Phi_{x,y}(\alpha)-(\beta-\alpha)\right|. \tag{2.24}$$

Then

$$\int_{x_0}^{x_0+x} D_{u,y}^2 du \geqslant \sup_{\alpha\in[0,1]} \max(J_1, J_2). \tag{2.25}$$

By analogous methods it is possible to obtain corresponding inequalities for

$$\sum_{n=M+1}^{M+N} |\Phi_{n,y}(\alpha)-\alpha|^2$$

but the bounds obtained are more complicated and not so illuminating.

8. To prove Theorems 3 and 4 we require the following lemma which is Theorem 2 of Montgomery and Vaughan [4].

__LEMMA 5.__ __Suppose that__ $x_1, x_2, \ldots, x_R$ __are__ R __distinct real numbers, and that__ $v_1, v_2, \ldots, v_R$ __are__ R __complex numbers. Also, let__

(2.26) $$\delta = \min_{\substack{r,s \\ r\neq s}} |x_r - x_s| \quad \underline{\text{and}} \quad \delta_r = \min_{\substack{s \\ s\neq r}} |x_r - x_s|.$$

__Then__

(2.27) $$\left| \sum_{\substack{r=1 \\ r\neq s}}^{R} \sum_{s=1}^{R} \frac{v_r \overline{v_s}}{x_r - x_s} \right| \leqslant \pi \min(K_1, K_2)$$

__where__

(2.28) $$K_1 = \delta^{-1} \sum_{r=1}^{R} |v_r|^2$$

__and__

(2.29) $$K_2 = \frac{3}{2} \sum_{r=1}^{R} |v_r|^2 \delta_r^{-1}.$$

__9.__ __Proofs of Theorems 3 and 4.__ Let K be a positive integer. Then it is easily seen that the function $c_\alpha(u)$ given by (1.9) can be written in the form

$$(2.30)\qquad c_\alpha(u) = \alpha + \sum_{0<|k|\leqslant K} \frac{1-e(-\alpha k)}{2\pi i k} e(uk)$$

$$+ O\left(\min\left(1, \frac{1}{K\|u\|}\right)\right) + O\left(\min\left(1, \frac{1}{K\|u-\alpha\|}\right)\right).$$

Clearly

$$(2.31)\qquad \int_{x_0}^{x_0+x} \min\left(1, \frac{1}{K\|\frac{u}{n}-\beta\|}\right) du \ll (x+n)\frac{\log K}{K} \qquad (0 \leqslant \beta \leqslant 1).$$

Hence, by (1.9) and (2.30),

$$(2.32)\qquad \int_{x_0}^{x_0+x} \left|\sum_{n=1}^{\infty} a_n(y) c_\alpha(u/n) - \alpha\right|^2 du = I + O\left((x+y)\frac{\log K}{K}\right)$$

where

$$(2.33)\qquad I = \int_{x_0}^{x_0+x} \left|\sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} \left(\sum_{m\leqslant K/|k|} \frac{a_{nm}(y)\ \left(1-e(-\alpha km)\right)}{}\right) e\left(\frac{uk}{n}\right)\right|^2 du.$$

Clearly, if $n_j \leqslant y$, $0 < |k_j| \leqslant K$, $(n_j, k_j) = 1$ for $j = 1,2$ and $k_1/n_1 \neq k_2/n_2$, then $|k_1/n_1 - k_2/n_2| \geqslant 1/(yn_1) \geqslant y^{-2}$.

Therefore, by (2.33) and Lemma 5,

$$(2.34)\qquad I = \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} (x+\theta_1 y^2) \left|\sum_{m\leqslant K/|k|} \frac{a_{nm}(y)\ \left(1-e(-\alpha km)\right)}{2\pi i k m}\right|^2$$

and

$$(2.35)\qquad I = \sum_{n=1}^{\infty} \sum_{\substack{0<|k|\leqslant K \\ (n,k)=1}} (x+\tfrac{3}{2}\theta_2 ny) \left|\sum_{m\leqslant K/|k|} \frac{a_{nm}(y)\ \left(1-e(-\alpha km)\right)}{2\pi i k m}\right|^2$$

where $|\theta_1| \leqslant 1$, $|\theta_2| \leqslant 1$. Theorem 3 now follows from (2.32) on letting $K \to \infty$. Theorem 4 follows in the same way on discarding all the terms with $|k| \neq 1$.

Sometimes, when the simple Riesz means (R,λ_n) are specified, it may be more appropriate to use (2.34) and (2.35) rather than appeal to Theorems 3 and 4.

10. By (2.7), (2.8) and (2.13) we see that if y is small compared with x but not too small, then under very general conditions

$$(2.36)\qquad \lim_{x\to\infty} \Phi_{x,y(x)}(\alpha) = \alpha.$$

We now show, as a consequence of Theorem 3, and again under very general conditions, that even if y is very small compared with x, then (2.36) still holds.

<u>THEOREM 5.</u> <u>Suppose that</u> $0 < \theta < 1,\ 0 < \alpha < 1,$

$$(2.37)\qquad \lim_{y\to\infty}\left(\left(1+y^{\frac{3\theta-1}{2\theta}}\right)\left(\sum_{n\leqslant y-y^{(3\theta-1)/2\theta}} \lambda_n\right)^{-2}\sum_{n\leqslant y}\left(\sum_{m\leqslant y/n}\frac{1}{m}\lambda_{mn}\right)^2\right) = 0$$

<u>and</u>

$$(2.38)\qquad \lim_{x\to\infty}\Phi_{x,x^\theta}(\alpha)$$

<u>exists. Then</u>

$$(2.39)\qquad \lim_{x\to\infty}\Phi_{x,x^\theta}(\alpha) = \alpha.$$

We remark that (2.37) is rather a weak condition. For instance, if $\lambda_n = 1$ for every n, then it holds for every θ with $0 < \theta < 1$.

<u>Proof.</u> Let y be large and define $z = y-y^{(3\theta-1)/2\theta}$. Then by Theorem 3, (1.13) and (1.11),

$$(2.40)\qquad \int_{z^{1/\theta}}^{y^{1/\theta}}\left|\sum_{n\leqslant z}\lambda_n\left(c_\alpha\left(\frac{u}{n}\right)-\alpha\right)\right|^2 du \ll \left(y^2+y^{1/\theta}-z^{1/\theta}\right)\sum_{n\leqslant y}\left(\sum_{m\leqslant y/n}\frac{1}{m}\lambda_{mn}\right)^2.$$

Furthermore, by Cauchy's inequality (inégalité de Schwarz en français !),

$$\int_{z^{1/\theta}}^{y^{1/\theta}} \Big|\sum_{z<n\leqslant u^{\theta}} \lambda_n \left(c_{\alpha}(\tfrac{u}{n})-\alpha\right)\Big|^2 du \ll \left(y^{1/\theta}-z^{1/\theta}\right)(1+y-z)\sum_{n\leqslant y}\lambda_n^2.$$

Hence, by (2.40),

$$(2.41)\qquad \int_{z^{1/\theta}}^{y^{1/\theta}} \Big|\sum_{n\leqslant u^{\theta}} \lambda_n \left(c_{\alpha}(\tfrac{u}{n})-\alpha\right)\Big|^2 du \ll$$

$$\ll \left(y^2+(y^{1/\theta}-z^{1/\theta})\left(1+y^{\frac{3\theta-1}{2\theta}}\right)\right)\sum_{n\leqslant y}\Big(\sum_{m\leqslant y/n}\tfrac{1}{m}\lambda_{mn}\Big)^2.$$

It is easily verified that

$$y^2 \ll (y^{1/\theta}-z^{1/\theta})y^{(3\theta-1)/2\theta}.$$

Thus, by (2.41) and (2.37),

$$\inf_{z^{1/\theta}\leqslant u\leqslant y^{1/\theta}} \left|\Phi_{u,u^{\theta}}(\alpha)-\alpha\right| \to 0 \quad \text{as} \quad y\to\infty.$$

This gives the desired result.

3. Appendix.

1. Theorem 1 does not require that the $a_n(y)$ be the simple Riesz means (R,λ_n). It is valid provided that $\sum_{n=1}^{\infty} a_n(y) = 1$.

2. Theorem 2 can be generalized in the following way. We say that the positive Toeplitz transformation $\mathcal{A} = (a_n(y))$ has asymptotic (or limit) distribution function φ with respect to the ordinary Cesàro method $(C,1)$ if there exists a distribution function φ such that

$$(3.1) \qquad \lim_{y \to \infty} \sum_{n \leqslant ty} a_n(y) = \varphi(t)$$

at every t at which φ is continuous. For example, if the $a_n(y)$ are the simple Riesz means (R,λ_n) and if φ exists, then by Lemma 1 it is either a continuous function given by

$$(3.2) \qquad \varphi(t) = \begin{cases} 0 & (t \leqslant 0) \\ t^{\sigma} & (0 < t < 1) \qquad (\text{with } \sigma > 0), \\ 1 & (t \geqslant 1) \end{cases}$$

or is one of the "Heaviside" functions Y_0 and Y_1, where $Y_a(t) = 0$ if $t < a$, $Y_a(t) = 1$ if $t \geqslant a$. (In the general case, necessarily $\varphi(t) = 0$ for $t < 0$). On examining the proof of Theorem 2, one sees that provided φ exists, is continuous and satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, then it is possible to replace Theorem 2 by a similar but more general statement. In particular $F(\alpha\ ;\ \xi,\sigma)$ is to be replaced by

$$(3.3) \quad G(\alpha\ ;\ \xi,\varphi) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha,\xi)\ \left(1-\varphi\left(\frac{\xi}{[\xi]+\alpha}\right)\right) + \sum_{k>\xi} \left(\varphi\left(\frac{\xi}{k}\right)-\varphi\left(\frac{\xi}{k+\alpha}\right)\right) & \\ & (\text{when } 0 < \alpha < 1), \end{cases}$$

but some care is needed with the error terms. Besides the above example where φ is given by (3.2), there are other interesting instances in which φ exists.

3. Theorems 3 and 4 do not require the $a_n(y)$ to be the simple Riesz means (R,λ_n). They remain valid without modification provided that $a_n(y) = 0$ for $n > y$. Otherwise, there are extra error-terms involving $\sum_{n>y} a_n(y)$. Thus one can still obtain meaningful information in case

$$\lim_{y \to \infty} \sum_{n>y} a_n(y) = 0.$$

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Université de Paris-Sud,
Orsay.

University of Michigan,
Ann Arbor.

Imperial College,
London.

On the fractional parts of x/n and related sequences. II

B. SAFFARI and R.C. VAUGHAN

1. Introduction and statement of theorems.

1.1. In this paper we assume the notation of [9]. Throughout, the implicit constants in the O, $\ll$ and $\gg$ notations are absolute unless otherwise indicated. In addition, we use the symbol $\asymp$. By $U \asymp V$ one means that $U \ll V$ and $V \ll U$. The letter p always designates a prime number.

1.2. The standard case. In this section we study the case $h(n) = 1/n$. We are primarily interested in the behaviour of

$$\theta_{x,y}(\alpha) = y^{-1} \sum_{n \leqslant y} c_\alpha(x/n) \tag{1.1}$$

where x and y tend to infinity together. We observe that this is essentially the same as taking the simple Riesz means (R, λ_n) with $\lambda_n = 1$ for $n \leqslant y$ and $\lambda_n = 0$ for $n > y$. In fact, we are considering the positive Toeplitz transformation

$$\mathcal{A} = (a_n(y) : y \in [1, \infty), n = 1,2,\ldots)$$

with $a_n(y) = y^{-1}$ for $n \leqslant y$ and $a_n(y) = 0$ for $n > y$.

We recall the definition of $F(\alpha,\xi,\sigma)$ (cf. [9], (2.4), (2.5)).

$$(1.2)\quad F(\alpha,\xi,\sigma) = \begin{cases} 0 & (\alpha \leqslant 0) \\ 1 & (\alpha \geqslant 1) \\ \theta(\alpha,\xi)(1 - \xi^{\sigma}([\xi] + \alpha)^{-\sigma}) + \xi^{\sigma}\sum_{k>\xi}(k^{-\sigma} - (k+\alpha)^{-\sigma}) & (0 < \alpha < 1,\ \sigma > 0) \\ \alpha & (0 < \alpha < 1,\ \sigma = 0) \end{cases}$$

where

$$(1.3)\quad \theta(\alpha,\xi) = \begin{cases} 1 & \text{if } (\xi-\alpha,\xi] \cap \mathbb{N} \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

and write

$$(1.4)\qquad F(\alpha,\xi) = F(\alpha,\xi,1).$$

The connection between $\theta_{x,y}$ and the Dirichlet divisor problem can be seen, for example, *via*

$$(1.5)\quad \Delta(x) = 2x^{\frac{1}{2}}\int_0^1 (\theta_{x,x^{1/2}}(\alpha) - \alpha)d\alpha + O(1)$$

or

$$(1.6)\quad \Delta(x) = x\int_0^1 (\theta_{x,x}(\alpha) - F(\alpha,1))d\alpha + O(1)$$

where

$$(1.7)\quad \Delta(x) = \sum_{n\leqslant x} d(n) - x\log x - (2\gamma-1)x$$

and as usual d is the divisor function and γ is Euler's constant.

THEOREM 1. *Suppose that* $1 \leqslant y \leqslant x$. *Then*

$$(1.8)\quad \theta_{x,y}(\alpha) = F(\alpha,x/y) + O(x^{\frac{1}{3}}y^{-1}\log x).$$

By adapting the Van der Corput method of trigonometric sums it would be possible to improve the error term here, much as in the Dirichlet divisor problem. However, we have carried out no detailed calculations in this direction, partly because we do not believe that the small improvements that could be obtained are anywhere near the truth. In fact, Theorem 2 below suggests that $\theta_{x,y}(\alpha) \to \alpha$ even when $y = x^{\varepsilon}$ where ε is any fixed number with $0 < \varepsilon < 1$.

There are three immediate consequences of Theorem 1.

COROLLARY 1.1. *As* $x \to \infty$,

$$(1.9) \qquad \theta_{x,x}(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha}{k(k+\alpha)} + O\left(x^{-\frac{2}{3}}\log x\right).$$

COROLLARY 1.2. *Let* t *be a fixed number with* $0 < t < 1$. *Then*

$$(1.10) \qquad \theta_{x,tx}(\alpha) = F(\alpha, \tfrac{1}{t}) + O\left(x^{-\frac{2}{3}}\log x\right).$$

COROLLARY 1.3. *Suppose that* $y/x \to 0$ *as* $x \to \infty$. *Then*

$$(1.11) \qquad \theta_{x,y}(\alpha) = \alpha + O\left(yx^{-1} + x^{\frac{1}{3}}y^{-1}\log x\right).$$

If y is quite close to x, the error term in (1.11) is not very good, and at first sight one might hope to do better. However, on inspecting $F(\alpha,\xi)$ one finds that the error can indeed be this large, and is essentially due to the irregular behaviour of $F(\alpha,\xi)$ as a function of ξ at the points 2,3,... (see Lemma 4 of [9]).

The next theorem suggests that Theorem 1 is some way from being best possible.

THEOREM 2. *Suppose that* $y = y(x)$ *is increasing,* $y = o(x)$ *and* $y \to \infty$ *as* $x \to \infty$. *Suppose further that* $0 < \alpha < 1$ *and* $\lim_{x \to \infty} \theta_{x,y}(\alpha)$ *exists. Then*

$$\lim_{x \to \infty} \theta_{x,y}(\alpha) = \alpha.$$

The next three theorems put some limitations on how good the error term can really be in (1.8) and on how small y can be for there to be an asymptotic distribution.

THEOREM 3. *Suppose that* $y(u)$ *is increasing and* $y(u) \to \infty$ *as* $u \to \infty$. *Let* $\delta = \delta(\alpha)$ *be sufficiently small, and suppose that* x *and* X *satisfy the inequalities*

(1.12) $$0 < X \leqslant x,$$

(1.13) $$(y(x+X) - y(x))(\log(y(x+X)))^4 < \delta y(x)$$

and

(1.14) $$2 \leqslant y(x) \leqslant \tfrac{1}{2}X^{\frac{1}{2}}.$$

Then, for $x > x_0(\alpha)$,

(1.15) $$(\sin \pi\alpha)^4 \frac{X}{y(x)} \ll \int_x^{x+X} |\theta_{u,y(u)}(\alpha) - \alpha|^2 \, du \ll \frac{X}{y(x)}.$$

As an immediate consequence we have

COROLLARY 3.1. *Suppose that* $0 < \alpha < 1$ *and* $0 < \beta < \frac{1}{2}$. *Then there are numbers* $\delta_1(\alpha)$ *and* $x_1(\beta)$ *such that, whenever* $x > x_1(\beta)$,

$$\int_x^{x + \delta_1(\alpha)x(\log x)^{-4}} |\theta_{u,u^\beta}(\alpha) - \alpha|^2 \, du \asymp_\alpha x^{1-\beta}(\log x)^{-4}. \tag{1.16}$$

Moreover

$$\limsup_{x \to \infty} x^{\beta/2}|\theta_{x,x^\beta}(\alpha) - \alpha| > 0. \tag{1.17}$$

THEOREM 4. *Suppose that the continuous function* $G(u)$ *satisfies the differential difference equation*

$$uG'(u) = -G(u-1) \quad (u > 1), \qquad G(u) = 1 \quad (0 \leqslant u \leqslant 1). \tag{1.18}$$

Then, for each $u > 0$,

$$\limsup_{x \to \infty} \theta_{x,y}(\alpha) \geqslant G(u) \quad (0 < \alpha < 1, \; y = (\log x)^u). \tag{1.19}$$

Theorem 5 is an immediate corollary of Theorems 2 and 4.

THEOREM 5. *Suppose that* $0 < \alpha < G(u)$. *Then*

$$\theta_{x,y}(\alpha) \qquad (y = (\log x)^u)$$

does not have a limit as $x \to \infty$.

The function G, often called Dickman's function, has been studied by a number of people (see references in Norton [7]), who have shown that it is monotone decreasing and satisfies

$$0 < G(u) \leqslant \Gamma(u+1)^{-1} \tag{1.20}$$

and

$$(1.21)\qquad \int_0^\infty G(u)du = e^\gamma.$$

It is easily seen that

$$(1.22)\qquad G(u) = 1 - \log u \qquad (1 < u \leqslant 2)$$

and

$$(1.23)\qquad G(u) = 1 - \log u + \int_2^u \log(v-1)\frac{dv}{v} \qquad (2 < v \leqslant 3).$$

1.3. The "logarithmic case". As one might expect, when one considers limit distributions of $\{x/n\}$ in the sense of the logarithmic density, things can be pushed a good deal further. Write

$$(1.24)\qquad \theta_{x,y}(\alpha) = (\log y)^{-1} \sum_{n \leqslant y} \frac{1}{n}\, c_\alpha(x/n).$$

There is a close connection between $\theta_{x,y}$ and the error term

$$E(x) = \sum_{n \leqslant x} \sigma(n) - \frac{\pi^2}{12}x^2$$

where σ is the sum of the divisors function. It is easily seen that

$$E(x) = x(\log z) \int_0^1 (\theta_{x,z}(\alpha) - \alpha)d\alpha +$$

$$+ \frac{1}{2}x^2z^{-2} \int_0^1 (\varphi_{x,x/z}(\alpha) - \alpha)d\alpha$$

$$+ \underset{\sim}{O}(x) + \underset{\sim}{O}(x^2z^{-3})$$

where $z \leqslant x$ and

$$\varphi_{x,y}(\alpha) = 2y^{-2} \sum_{n \leqslant y} nc_\alpha(x/n).$$

When $x^{\frac{1}{2}} \leqslant z \leqslant x$ this reduces to

$$E(x) = x(\log z)\int_0^1 (\theta_{x,z}(\alpha) - \alpha)d\alpha + O(x).$$

There is also a simple relation connecting $\varphi_{x,y}$ with E, namely

$$E(x) = \tfrac{1}{2}x^2\int_0^1 (\varphi_{x,x}(\alpha) - F(\alpha,1,2))d\alpha + O(x).$$

We do not study $\varphi_{x,y}$ in detail, since its general behaviour can be easily deduced from that of $\theta_{x,y}$.

The next theorem shows that, not only does one obtain the uniform distribution for $\theta_{x,y}$ when $y = x^{\varepsilon}$, but even when $y = x$.

<u>THEOREM 6.</u> <u>Suppose that</u> $y \leqslant x$. <u>Then</u>

(1.25) $$\theta_{x,y}(\alpha) = \alpha + O((\log x)^{\frac{2}{3}}(\log y)^{-1}).$$

We would conjecture that $\theta_{x,y}(\alpha) \to \alpha$ providing that $\log\log x = o(\log y)$.

<u>THEOREM 7.</u> <u>Suppose that</u> $y = y(x)$ <u>is increasing to infinity and</u> $y \leqslant x$. <u>Suppose that</u> $0 < \alpha < 1$. <u>Then, whenever</u> $\theta_{x,y}(\alpha)$ <u>tends to a limit as</u> $x \to \infty$, <u>the limit must be</u> α.

In the opposite direction we can do somewhat better than the analogue of Theorem 4. (Note that by (1.20) and (1.21), for $u > 1$,

$$G(u) \leqslant \Gamma(1+u)^{-1} < 1/u$$

whereas $\int_0^u G(v)dv > 1$ and $\int_0^u G(v)dv \to e^{\gamma}$ as $u \to \infty$).

THEOREM 8. *For each* $u > 0$,

$$(1.26)\quad \limsup_{x\to\infty} \theta_{x,y}(\alpha) \geqslant \frac{1}{u}\int_0^u G(v)dv \qquad (0 < \alpha < 1,\ \ y = (\log x)^u)$$

where G *is given by* (1.18).

As an immediate consequence of Theorems 7 and 8 we have

THEOREM 9. *Suppose that* $u > 0$ *and* $0 < \alpha < \frac{1}{u}\int_0^u G(v)dv$. *Then*

$$\theta_{x,y}(\alpha) \qquad (y = (\log x)^u)$$

does not have a limit as $x \to \infty$.

It is very likely that both Theorems 5 and 9 hold with the upper bounds 1 for α for every fixed u.

1.4. *The prime numbers.* The following theorem shows that the prime numbers, suitably normalized, behave in much the same way as the natural numbers. Let

$$(1.27)\quad \vartheta_{x,y}(\alpha) = y^{-1}\sum_{p\leqslant y} (\log p)c_\alpha(x/p).$$

THEOREM 10. *Suppose that* $\varepsilon > 0$ *and* $x^{\frac{6}{11}+\varepsilon} < y \leqslant x$. *Then*

$$(1.28)\quad \vartheta_{x,y}(\alpha) = F(\alpha,x/y) + O\left(\exp\left(-\,C(\varepsilon)\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right)\right)$$

where $C(\varepsilon)$ *is a positive number depending at most on* ε.

We remark that on the density hypothesis concerning the distribution of the zeros of the Riemann zeta function the $\frac{6}{11}$ could be replaced by $\frac{1}{2}$. The $\frac{6}{11}$ arises as $\frac{c}{c+2}$ where c is such that

$$N(\sigma,T) \ll T^{c(1-\sigma)+\varepsilon} \tag{1.29}$$

and where $N(\sigma,T)$ is the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $\beta \geqslant \sigma$ and $|\gamma| \leqslant T$. The ε in Theorem 10 could be made an explicit function of x, but there is little point in doing so.

As far as the un-normalized case is concerned, providing that the conditions of Theorem 10 are satisfied, partial summation gives

$$\sum_{p \leqslant y} c_\alpha(x/p) = \frac{y}{\log y} F(\alpha, x/y) + \int_2^y \frac{F(\alpha, x/v)}{(\log v)^2} dv + \tag{1.30}$$
$$+ O\left(y \exp\left(-C(\varepsilon)\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right)\right).$$

The asymptotic distribution is the same, but there is a second order term which has no very simple closed form, although the main terms can be combined to give

$$\int_{x/y}^{\infty} \frac{\alpha x du}{(u - (1-\alpha)\{u\})^2 \log(x/(u - (1-\alpha)\{u\}))}.$$

It is trivial that $\vartheta_{x,y}(\alpha)$ does not have an asymptotic distribution when $y = (\log x)^u$ with $0 < u \leqslant 1$. (Indeed, this is so for all choices of λ_n. We hope to discuss this further in a later paper). However, we have not been able to extend this to the region $u > 1$.

It is a simple application of Theorem 5 of [9], that if $0 < \theta < 1$, $y = x^\theta$ and $\vartheta_{x,y}(\alpha)$ has a limit as x tends to infinity, then the limit must be α. Moreover, this can be sharpened along the lines of Theorem 2.

1.5. A "law of the iterated logarithm". In all the applications of Theorems 3 and 4 of [9] hitherto, the expressions

$$\sum_n \Big(\sum_m \frac{1}{n} a_{mn}(y)\Big)^2$$

and

$$\sum_n \Big|\sum_m \frac{1}{m} a_{mn}(y)(1 - e(\alpha m))\Big|^2$$

have behaved very much like $\sum_n a_n^2(y)$. We now show that this is not always so, even under fairly reasonable conditions. In particular, the following theorem justifies our remark below Theorem 4 of [9] to the effect that taking $\sum_n a_n^2(y)$ in that theorem can lose a factor as large as $\log\log y$.

THEOREM 11. *There is an infinite subset* $\mathfrak{D}$ *of* $\mathbb{N}^*$ *with the following property. Let*

$$\mathcal{A} = (a_n(y) : y \in [1,\infty),\ n = 1,2,\ldots)$$

be the Toeplitz transformation where the $a_n(y)$ *are the simple Riesz means* (R,λ_n) *obtained by taking* λ_n *to be the characteristic function of* $\mathfrak{D}$. *Then there are arbitrarily large* y *such that whenever* $x_0 \geqslant 0$ *and* $x > 0$,

$$\max(0,\ x-y^2) \ll \frac{\sum_{n<y} \lambda_n}{\log\log y} \sup_\alpha \int_{x_0}^{x_0+x} (\Phi_{u,y}(\alpha) - \alpha)^2\, d\alpha \ll x+y^2.$$

1.6. In conclusion we mention an example with $h(n) = 1/n$ in which the asymptotic distribution function differs from $F(\alpha,\xi)$. Suppose that $k \in \mathbb{N}^*$ and let $\lambda_n = 1$ if n is a k th power and $\lambda_n = 0$ otherwise. Then trivially by the method of the hyperbola,

$$\Phi_{x,y}(\alpha) = F(\alpha, x/y), 1/k) + \underline{O}\,(x^{\frac{1}{k+1}}\, y^{-\frac{1}{k}}),$$

and deeper methods doubtless enable one to improve a little further the range of validity for y.

2. Proof of Theorem 1.

2.1. The following lemma is implied by Satz 566 of Landau [4].

LEMMA 1. Let

$$b(z) = \begin{cases} \{z\} - \frac{1}{2} & (z \notin \mathbb{Z}) \\ 0 & (z \in \mathbb{Z}). \end{cases}$$

Suppose that $u < w$, $f(v)$ is positive and twice differentiable for $u \leqslant v \leqslant w$ and $f''(v)$ is non-zero and always has the same sign. Suppose also that for $u \leqslant v \leqslant w$ we have $0 < \lambda \leqslant f'(v) \leqslant \mu$ and that ρ is any real number with $\rho > 1$, $\rho > \lambda^{-3}$ and

$$\rho \geqslant |f''(v)|^{-1} (1 + f'(v)^2)^{3/2} \qquad (u \leqslant v \leqslant w).$$

Let N be the number of pairs of integers m,n for wich $u \leqslant m \leqslant w$ and $0 \leqslant n \leqslant f(m)$ where any pair m,n for which either $m = u$, $m = w$, $n = 0$ or $n = f(m)$ is counted with weight $\frac{1}{2}$. Then

$$N = \int_u^w f(u)du - b(w)f(w) + b(u)f(u) + O(\rho^{\frac{2}{3}}\mu).$$

2.2. To prove Theorem 1, consider six sets S_1, S_2, S_2', S_3, S_3' and S_4 of pairs of integers m,n, defined as follows ;

S_1 : $\quad \frac{x}{y} < m \leqslant x^{1/3}, \quad \frac{x}{m+\alpha} < n \leqslant \frac{x}{m}$

S_2 : $\quad x^{1/2} < m \leqslant x^{2/3}, \quad \frac{x}{m+\alpha} < n \leqslant \frac{x}{m}$

S_2' : $\quad \frac{x}{y} < m \leqslant x^{2/3}, \quad \frac{x}{m+\alpha} < n \leqslant \frac{x}{m}$

S_3 : $\quad \frac{x}{x^{1/2}+\alpha} < m \leqslant x^{2/3}, \quad \frac{x}{m} - \alpha < n \leqslant \frac{x}{m}, \; x^{1/3} < n \leqslant x^{1/2}$

S_3' : $\quad \frac{x}{x^{1/2}+\alpha} < m \leqslant x^{2/3}, \quad \frac{x}{m} - \alpha < n \leqslant \frac{x}{m}, \quad \frac{x}{y} < n \leqslant x^{1/2}$

S_4 : $\quad m \leqslant x^{1/3}, \qquad \frac{x}{m} - \alpha < n \leqslant \frac{x}{m}, \quad x^{2/3} < n \leqslant x.$

Let $|S|$ denote the number of elements of the set S. By (1.1),

$$\theta_{x,y}(\alpha) = \left(\left[\frac{x}{y}\right] - \left[\frac{x}{y} - \alpha\right]\right)\left([y] - \frac{x}{\left[\frac{x}{y}\right] + \alpha}\right) + \sum_{j=1}^{4} M_j \tag{2.1}$$

where

$$M_1 = \begin{cases} |S_1| & \text{if } x^{2/3} < y \leqslant x \\ 0 & \text{if } y \leqslant x^{2/3} \end{cases} \tag{2.2}$$

$$M_2 = \begin{cases} |S_2| & \text{if } x^{1/2} < y \leqslant x \\ |S_2'| & \text{if } x^{1/3} < y \leqslant x^{1/2} \\ 0 & \text{if } y \leqslant x^{1/3} \end{cases} \tag{2.3}$$

$$M_3 = \begin{cases} |S_3| & \text{if } x^{2/3} < y \leqslant x \\ |S_3'| & \text{if } x^{1/2} < y \leqslant x^{2/3} \\ 0 & \text{if } y \leqslant x^{1/2} \end{cases} \tag{2.4}$$

and

$$M_4 = \begin{cases} |S_4| & \text{if } x^{1/3} < y \leqslant x \\ 0 & \text{if } y \leqslant x^{1/3}. \end{cases} \tag{2.5}$$

Suppose first of all that $x^{2/3} < y \leqslant x$. By (2.2) and (2.5),

$$M_1 = \sum_{\frac{x}{y} < m \leqslant x^{1/3}} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3}) \quad \text{and} \quad |M_4| \leqslant x^{1/3}. \tag{2.6}$$

If $x^{1/2} \leqslant m \leqslant x^{2/3}$, then there are $\ll 1$ integers n with

$$\frac{x}{m+\alpha} < n \leqslant \frac{x}{m},$$

and the number of pairs m,n with either $n(m+\alpha) = x$ or $mn = x$ is $\ll x^{\varepsilon}$. Hence, by (2.3),

$$M_2 = M_2' + O(x^{\varepsilon}) \quad \text{with} \quad M_2' = {\sum_{x^{1/2} \leqslant m \leqslant x^{2/3}}}' \; {\sum_{\frac{x}{m+\alpha} \leqslant n \leqslant \frac{x}{m}}}' 1 \tag{2.7}$$

where the dashes are used to indicate that if the pair m,n is on the "boundary" of the region under consideration, then it is counted with weight $\frac{1}{2}$. The same argument is applied to M_3. Note that there is at most one integer n in $[\frac{x}{x^{2/3}+\alpha}, x^{1/3}]$ and likewise in $[x/(x^{1/2}+\alpha), x^{1/2}]$. Hence, by (2.4),

$$M_3 = M_3' + O(x^{\varepsilon}) \quad \text{where} \quad M_3' = \sum_{x^{1/2} \leqslant m \leqslant x^{2/3}} \; {\sum_{\frac{x}{m} - \alpha \leqslant n \leqslant \frac{x}{m}}}' 1. \tag{2.8}$$

Now write

$$M_2' = N_2(0) - N_2(\alpha) \quad \text{and} \quad M_3' = N_3(0) - N_3(\alpha) \tag{2.9}$$

where, for β with $0 \leqslant \beta \leqslant 1$,

$$N_2(\beta) = {\sum_{-x^{2/3} \leqslant m \leqslant -x^{1/2}}}' \; {\sum_{0 \leqslant n \leqslant \frac{x}{\beta - m}}}' 1 \tag{2.10}$$

and

$$(2.11) \qquad N_3(\beta) = \sum_{-x^{2/3} \leqslant m \leqslant -x^{1/2}}{}' \; \sum_{0 \leqslant n \leqslant -\frac{x}{m} - \beta}{}' \; 1.$$

It is now a straightforward application of Lemma 1 to intervals of the kind $-2^{h+1}x^{1/2} \leqslant m \leqslant -2^h x^{1/2}$ to obtain

$$N_2(\beta) = \int_{x^{1/2}}^{x^{2/3}} \frac{x}{u+\beta}du + b(x^{\frac{1}{2}})\frac{x}{x^{1/2}+\beta} - b(x^{\frac{2}{3}})\frac{x}{x^{2/3}+\beta} + $$
$$+ O(x^{1/3}\log x)$$

and

$$N_3(\beta) = \int_{x^{1/2}}^{x^{2/3}} (\frac{x}{u} - \beta)du + b(x^{\frac{1}{2}})(x^{\frac{1}{2}} - \beta) - b(x^{\frac{2}{3}} - \beta) + $$
$$+ O(x^{1/3}\log x).$$

Therefore, by (2.7), (2.8), (2.9), (2.10) and (2.11),

$$M_2 = \sum_{m > x^{1/2}} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3}\log x)$$

and

$$M_3 = \alpha(x^{2/3} - x^{1/2}) + O(x^{\frac{1}{3}}\log x)$$
$$= \sum_{x^{1/3} < m \leqslant x^{1/2}} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3}\log x).$$

Hence, by (2.6),

$$\sum_{j=1}^{4} M_j = \sum_{m > x/y} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3}\log x)$$

and Theorem 1 in the case $x^{2/3} < y \leqslant x$ now follows from (2.1).

The cases $x^{1/2} < y \leqslant x^{2/3}$ and $x^{1/3} < y \leqslant x^{1/2}$ are treated similarly.

3. Proofs of Theorems 2, 3 and 7.

3.1. First of all we state a lemma which is a consequence of Theorems 3 and 4 of [9].

LEMMA 2. *Suppose that* x *and* X *are non-negative real numbers,* $y \geqslant 1$ *and* $0 < \alpha < 1$. *Then*

$$(\sin \pi\alpha)^4(X-y^2)y \ll \int_x^{x+X} \Big|\sum_{n\leqslant y}(c_\alpha(u/n) - \alpha)\Big|^2 du \ll (X+y^2)y.$$

3.2. We require a result in which in the integrand y can be made a function of u. In order to obtain this we first of all require some information concerning short intervals.

LEMMA 3. *Suppose that* x, z *and* X *are non-negative real numbers,* $y \geqslant 1$, $Y = \max(z,y)$ *and* $0 < \alpha < 1$. *Then*

$$\int_x^{x+X} \Big|\sum_{z<n\leqslant z+y}(c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du \ll (X+Y^2)y(\log 2Y)^2.$$

Proof. By Theorem 3 of [9], the left hand side is

$$\ll (X+Y^2)\sum_n\Big(\sum_{\substack{m \\ z<nm\leqslant z+y}} \frac{1}{m}\Big)^2$$

$$\ll (X+Y^2)\sum_n\Big(\sum_{z<nm\leqslant z+y} \frac{1}{m}\Big)\log 2Y$$

$$\ll (X+Y^2)\sum_{z<q\leqslant z+y}\ \sum_{m|q} \frac{1}{m}\log 2Y$$

$$\ll (X+Y^2)y(\log 2Y)^2,$$

as required.

LEMMA 4. On the hypothesis of Lemma 3,

$$\int_x^{x+X} \sup_{v \leqslant y} \Big| \sum_{z < n \leqslant z+v} (c_\alpha(\tfrac{u}{n}) - \alpha) \Big|^2 du \ll (X+Y^2)y(\log 2Y)^4,$$

where the supremum is taken over all non-negative real numbers with $v \leqslant y$.

Proof. This uses a technique which goes back to Menchov [5] and Rademacher [8]. It may certainly be supposed that the supremum is taken only over those numbers of the form

$$v = y \sum_{r=0}^{k} \varepsilon_r 2^{-r}$$

where $\varepsilon_r = 0$ or 1 and $k = [\log y/\log 2]$. For such a v let

$$m_r = m_r(v) = \sum_{j=0}^{r-1} \varepsilon_j 2^{r-j}, \qquad m_0 = 0.$$

Then

$$m_r < 2^r v/y \leqslant 2^r \leqslant y,$$
$$m_{r+1} 2^{-r-1} = m_r 2^{-r} + \varepsilon_r 2^{-r}$$

and

$$y m_{k+1} 2^{-k-1} = v.$$

Now for given u choose some $v = v(u)$ for which the supremum occurs. Then

$$\sup_{v \leqslant y} \Big| \sum_{z < n \leqslant z+v} (c_\alpha(\tfrac{u}{n}) - \alpha) \Big| \leqslant \sum_{r=0}^{k} \Big| \sum (c_\alpha(\tfrac{u}{n}) - \alpha) \Big|$$

where the inner summation is over those integers n such that

$$z+ym_r2^{-r} < n \leqslant z + (m_r + \varepsilon_r)y2^{-r}.$$

Hence,

$$(3.1) \qquad \int_x^{x+X} \sup_{v \leqslant y} \Big|\sum_{z<n\leqslant z+v} (c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du \ll$$

$$\ll (\log 2Y) \int_x^{x+X} \sum_{r=0}^{k} \Big|\sum (c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du$$

where the inner sum is over those n such that

$$z+ym_r2^{-r} < n \leqslant z + y(m_r+1)2^{-r}.$$

The right hand side of (3.1) is

$$(3.2) \qquad \leqslant (\log 2Y) \sum_{r=0}^{k} \sum_{m=0}^{2^r-1} \int_x^{x+X} \Big|\sum (c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du$$

where the inner summation is over those n such that

$$z+ym2^{-r} < n \leqslant z + y(m+1)2^{-r},$$

and, by Lemma 3, (3.2) is

$$\ll (\log 2Y) \sum_{r=0}^{k} 2^r(X+Y^2)y2^{-r}(\log 2Y)^2 \ll (X+Y^2)y(\log 2Y)^4,$$

as required.

3.3. First of all we prove Theorem 2. Observe that

$$\liminf_{x \to \infty} \frac{y(2x) - y(x)}{y(x)} \leqslant 1,$$

for otherwise $y(x) \gg x$. Therefore the set

$$(3.3) \qquad S = \{x > 1 : \frac{y(2x) - y(x)}{y(x)} \leqslant 2\}$$

is unbounded. By Lemma 4,

$$(3.4) \qquad \inf_{x \leqslant u \leqslant 2x} \Big|\sum_{n \leqslant y(u)} \frac{c_\alpha(u/n) - \alpha}{y(u)}\Big|^2 \leqslant$$

$$\leqslant \frac{1}{xy(x)^2} \int_x^{2x} \Big|\sum_{n \leqslant y(u)} (c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du$$

$$\ll \frac{x + y(2x)^2}{xy(x)^2} y(2x)(\log 2y(2x))^4.$$

If S contains an unbounded subset S^* such that

$$(3.5) \qquad y(2\xi) \leqslant \xi(\log \xi)^{-5} \quad \text{whenever} \quad \xi \in S^*,$$

then by (3.3), (3.4) and (3.5)

$$\liminf_{x \to \infty} \Big|\sum_{n \leqslant y(x)} \frac{c_\alpha(x/n) - \alpha}{y(x)}\Big| = 0.$$

This gives the desired conclusion if such an S^* exists. Otherwise there is a constant $x_0 > 1$ such that

$$(3.6) \qquad y(2x) > x(\log x)^{-5} \quad \text{whenever} \quad x \in S, \ x > x_0.$$

Then, by (3.6) and Corollary 1.3,

$$\lim_{\substack{x \to \infty \\ x \in S}} \theta_{2x, y(2x)} = \alpha.$$

This completes the proof of Theorem 2.

3.4. To prove Theorem 3 we use both Lemma 2 and Lemma 4. By Lemma 2,

$$(\sin \pi\alpha)^4 Xy(x) \ll \int_x^{x+X} \Big|\sum_{n \leqslant y(x)} (c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du \ll Xy(x),$$

and by Lemma 4,

$$\int_x^{x+X} \Big|\sum_{y(x) < n \leqslant y(u)} (c_\alpha(\tfrac{u}{n}) - \alpha)\Big|^2 du \ll Xy(x).$$

Thus, if y is sufficiently small in terms of x, then

$$(\sin \pi\alpha)^4 \frac{X}{y(x)} \ll \int_x^{x+X} |\Theta_{u,y(u)}(\alpha) - \alpha \frac{[y(u)]}{y(u)}|^2 du \ll \frac{X}{y(x)}.$$

This gives (1.15), provided that $x > x_0(\alpha)$.

3.5. The proof of Theorem 7 follows the same pattern as that of Theorem 2. We observe that Theorem 3 of [9] gives

$$\int_x^{2x} |\sum_{n \leq y(u)} \frac{1}{n}(c_\alpha(\frac{u}{n}) - \alpha)|^2 du \ll x.$$

Thus

$$(3.7) \qquad \inf_{x \leq u \leq 2x} |\sum_{n \leq y(u)} \frac{c_\alpha(u/n) - \alpha}{n \log y(u)}|^2 \leq$$

$$\leq \frac{1}{x(\log y(x))^2} \int_x^{2x} |\sum_{n \leq y(u)} \frac{1}{n}(c_\alpha(\frac{u}{n}) - \alpha)|^2 du$$

$$\ll (\log y(x))^{-2}(1 + \log \frac{y(2x)}{y(x)})^2.$$

If there exists an unbounded set of real numbers $x > 1$ on which $y(2x)/y(x)$ is bounded, then Theorem 7 follows at once from (3.7). Otherwise

$$(3.8) \qquad y(x) \gg x,$$

and Theorem 7 follows from (3.8) and Theorem 6, which we shall prove in Section 5.

4. Proofs of Theorems 4 and 8.

Let

(4.1) $$x_n = \exp\left(\sum_{r=1}^{n} \Lambda(r)\right)$$

where Λ is Von Mangoldt's function, and

(4.2) $$y_n = (\log x_n)^u.$$

Then

(4.3) $$\sum_{m \leqslant y_n} c_\alpha(x_n/m) = \sum_{m \leqslant n^u} c_\alpha(x_n/m) + O\left(\frac{n^u}{\log n}\right)$$

$$\geqslant \sum_{m \leqslant n^u}{}' c_\alpha(x_n/m) + O\left(\frac{n^u}{\log n}\right)$$

where $\sum'$ means that the sum is restricted to those m which have no prime divisor exceeding n. (Very probably the part of the sum thrown away contributes an amount infinitely often as large as $(\alpha - \varepsilon)(1 - G(u))$, and if this is so, then Theorem 5 also holds when $G(u) \leqslant \alpha < 1$). By (4.1), the number of these m not exceeding n^u and not divising x_n is at most

$$\sum_{\substack{p,k \\ k \geqslant 2,\ p > n^{1/k}}} n^u p^{-k} \ll n^{u - \frac{1}{2}}.$$

Thus we have

(4.4) $$\sum_{m \leqslant y_n} c_\alpha(x_n/m) \geqslant \sum_{m \leqslant n^u}{}' 1 + O\left(\frac{n^u}{\log n}\right).$$

de Bruijn [1] has shown that if $\Psi(X,Y)$ is the number of natural numbers not exceeding X which have no prime factor exceeding Y, then

$$\Psi(Y^u,Y) = G(u)Y^u + O\left(Y^{u-1}(u+1)^2 \max_{2\leqslant x\leqslant Y} |R(x)|\right) \tag{4.5}$$

uniformly for $Y \geqslant 2$, $u \geqslant 0$, where $R(x) = \pi(x) - \mathrm{li}\, x$ is the error term in the prime number theorem. This with (4.4) and (4.2) gives Theorem 4.

The proof of Theorem 8 proceeds in the same manner. Thus

$$\sum_{m\leqslant y_n} \frac{1}{m}\, c_\alpha(x_n/m) \geqslant {\sum_{m\leqslant n^u}}' \frac{1}{m}\, c_\alpha(x_n/m) + O\left(\frac{1}{\log n}\right)$$

and

$$\sum_{\substack{m\leqslant n^u\\ m\nmid x_n}}{}' \frac{1}{m} \ll \sum_{\substack{p,k\\ k\geqslant 2,\ p>n^{1/k}}} \sum_{m\leqslant n^u/p^k} \frac{1}{mp^k} \ll n^{-1/2}\log n.$$

Hence

$$\sum_{m\leqslant y_n} \frac{1}{m}\, c_\alpha(x_n/m) \geqslant {\sum_{m\leqslant n^u}}' \frac{1}{m} + O\left(\frac{1}{\log n}\right). \tag{4.6}$$

By partial integration,

$$\sum_{m\leqslant n^u}{}' \frac{1}{m} = n^{-u}\Psi(n^u,n) + (\log n)\int_0^u n^{-w}\Psi(n^w,n)\,dw. \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) now establishes Theorem 8.

5. Proof of Theorem 6.

Suppose that

$$0 < \beta < 1. \tag{5.1}$$

Let

$$M_\beta = \left[\frac{x}{v} - \beta\right] \tag{5.2}$$

and

$$S(\beta) = \sum_{M_\beta < m \leqslant x-\beta} \sum_{n \leqslant x/(m+\beta)} \frac{1}{n}. \tag{5.3}$$

Then

$$\theta_{x,y}(\alpha)\log y = S(0) - S(\alpha) + (M_0 - M_\alpha) \sum_{n \leqslant y} \frac{1}{n}. \tag{5.4}$$

Let

$$N = [x^{1/2}]. \tag{5.5}$$

Then, by (5.3)

$$S(\beta) = \sum_{M_\beta < m \leqslant N} \left(\log \frac{x}{m+\beta} + \gamma + \underset{\sim}{O}\left(\frac{m}{x}\right)\right) + \sum_{n \leqslant \frac{n}{N+\beta}} \frac{1}{n}\left(\left[\frac{x}{n} - \beta\right] - N\right)$$

providing that $N \geqslant M_\beta$. This also holds when $N < M_\beta$, providing that the convention

$$\sum_{M_\beta < m \leqslant N} = - \sum_{N < m \leqslant M_\beta}$$

is adopted. Hence

$$S(0) - S(\alpha) = \sum_{M_0 < m \leqslant N} \log(1 + \frac{\alpha}{m}) - (M_0 - M_\alpha)\log \frac{x}{M_0 + \alpha} + O(1)$$

$$+ \sum_{n \leqslant \frac{x}{N}} \frac{x}{n^2} - \sum_{n \leqslant \frac{x}{N+\alpha}} (\frac{x}{n^2} - \frac{\alpha}{n}) - \frac{1}{2} \sum_{\frac{x}{N+\alpha} < n \leqslant \frac{x}{N}} \frac{1}{n}$$

$$- \sum_{n \leqslant \frac{x}{N}} \frac{1}{n} B(\frac{x}{n}) + \sum_{n \leqslant \frac{x}{N+\alpha}} \frac{1}{n} B(\frac{x}{n} - \alpha)$$

where $B(u) = \{u\} - \frac{1}{2}$. Therefore, by (5.4),

$$\theta_{x,y}(\alpha)\log y = (M_0 - M_\alpha)\log \frac{y(M_0 + \alpha)}{x} + \alpha\log \frac{N}{M_0} + O(1) \tag{5.6}$$

$$+ \alpha\log \frac{x}{N+\alpha} - T(0) + T(\alpha)$$

where

$$T(\beta) = \sum_{n \leqslant x^{1/2}} \frac{1}{n} B(\frac{x}{n} - \beta). \tag{5.7}$$

By (5.2)

$$\alpha \leqslant 1 + (\alpha - 1)\frac{y}{x} \leqslant \frac{y}{x}(M_0 + \alpha) < 2$$

and

$$\frac{y}{2} < \frac{xN}{M_0(N+\alpha)} \leqslant \frac{x}{[\frac{x}{y}]} \leqslant 2y.$$

Hence, by (5.6),

$$\theta_{x,y}(\alpha)\log y = \alpha\log y + O(1) + T(\alpha) - T(0). \tag{5.8}$$

The proof is completed by observing that a trivial modification of the proof of Satz 3.2.2 of Walfisz [11, p. 98] gives

$$T(\beta) \ll (\log x)^{2/3}.$$

6. Proof of Theorem 10.

6.1. We require a lemma which has some independant interest. Let

$$(6.1) \qquad (x) = \underset{p<x}{} \log p.$$

LEMMA 5. *Let* $N(\sigma,T)$ *denote the number of zeros* $\rho = \beta + i\gamma$ *of the Riemann zeta function with* $\beta \geqslant \sigma$, $|\gamma| \leqslant T$. *Suppose that there are positive constants* B, C (with $C \geqslant 2$) *such that*

$$(6.2) \qquad N(\sigma,T) \ll T^{C(1-\sigma)}(\log T)^B \qquad (T \geqslant 2).$$

Then, whenever $x \geqslant 4$ *and* $x^{\varepsilon - 2/C} < \theta \leqslant 1$, *we have*

$$(6.3) \qquad \int_x^{2x} |\vartheta(u+u\theta) - \vartheta(u) - u\theta|^2 du \ll \theta^2 x^3 \exp\left(-C_1\left(\frac{\log x}{\log\log x}\right)^{1/3}\right),$$

where C_1 *is a suitable positive number depending at most on* ε. *If the Riemann hypothesis is assumed instead, then whenever* $x \geqslant 4$

$$(6.4) \qquad \int_x^{2x} |\vartheta(u+u\theta) - \vartheta(u) - u\theta|^2 du \ll \theta x^2 (\log \tfrac{2}{\theta})^2$$

uniformly in θ *with* $0 < \theta \leqslant 1$.

This is essentially due to Selberg [10]. It differs firstly in that in (6.4) the bound is uniform for θ close to 1 whereas Selberg apparently requires $\theta \ll x^{-\varepsilon}$, and secondly it is slightly weaker when $\theta \leqslant x^{-C_2}$ with $0 < C_1 < 1$ since Selberg obtains

$$(6.5) \qquad \int_0^{\theta^{-1/C_2}} |\vartheta(u+u\theta) - \vartheta(u) - u\theta|^2 u^{-2} du \ll \theta(\log \tfrac{2}{\theta})^2.$$

Moreno [6] has observed (6.3) with $c = 5/2$ and ϑ replaced by ψ (where

$$(6.6) \qquad \psi(x) = \sum_{n \leqslant x} \Lambda(n)$$

and $\Lambda(n)$ is von Mangoldt's function), and given only a weaker result for ϑ. In fact, there are at least two obvious ways of deducing a corresponding result for ϑ.

Proof of Lemma 5. Clearly

$$(6.7) \qquad \int_x^{2x} (\psi(u+\theta u) - \psi(u) - \theta u)^2 du \leqslant$$

$$\leqslant \int_1^2 \left(\int_{xv/2}^{2xv} (\psi(u+\theta u) - \psi(u) - \theta u)^2 du \right) dv.$$

Let $\sum_\rho$ denote summation over all the complex zero of ζ grouped in complex conjugate pairs, that is, $\lim_{T \to \infty} \sum_{|\gamma| \leqslant T}$. Then, by the explicit formula (Ingham [3], Theorem 29), whenever $y \geqslant 2$

$$\sum_{n \leqslant y}{}' \Lambda(n) = y - \sum_\rho \frac{y^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1 - y^{-2})$$

where the dash means that if $y \in \mathbf{Z}$, then $\Lambda(y)$ is to be replaced by $\frac{1}{2} \Lambda(y)$. The sum over the zeros is boundedly convergent (cf. Ingham [3], p. 80). Thus

$$(6.8)\qquad \int_{xv/2}^{2xv} (\psi(u+\theta u) - \psi(u) - \theta u)^2 du \ll$$

$$\ll \int_{xv/2}^{2xv} \Big|\sum_{\rho} \frac{(1+\theta)^{\rho} - 1}{\rho} u^{\rho}\Big|^2 du + \int_{xv/2}^{2xv} \Big(\log\Big(1 - \frac{(1+\theta)^2 - 1}{u^2(1+\theta)^2 - 1}\Big)\Big)^2 du$$

and

$$(6.9)\qquad \int_{xv/2}^{2xv} \Big|\sum_{\rho} \frac{(1+\theta)^{\rho} - 1}{\rho} u^{\rho}\Big|^2 du =$$

$$= \sum_{\rho_1} \sum_{\rho_2} \frac{(1+\theta)^{\rho_1} - 1}{\rho_1} \cdot \frac{(1+\theta)^{\overline{\rho}_2} - 1}{\overline{\rho}_2} \cdot$$

$$\cdot \frac{2^{1+\rho_1+\overline{\rho}_2} - 2^{-1-\rho_1-\overline{\rho}_2}}{1+\rho_1+\overline{\rho}_2} (xv)^{1+\rho_1+\overline{\rho}_2} .$$

Trivially

$$(6.10)\qquad \int_{xv/2}^{2xv} \Big(\log\Big(1 - \frac{(1+\theta)^2 - 1}{u^2(1+\theta)^2 - 1}\Big)\Big)^2 du \ll \theta^2 x^{-3}.$$

By Theorem 25a of Ingham [3],

$$(6.11)\qquad N(0,T+1) - N(0,T) \ll \log T \qquad (T \geqslant 1).$$

Thus, the double sum on the right of (6.9) converges absolutely, and uniformly in v on $[1,2]$. Thus, by (6.6), (6.8), (6.9) and (6.10)

$$(6.12)\qquad \int_{x}^{2x} (\psi(u+\theta u) - \psi(u) - \theta u)^2 du \ll \theta^2 x^{-3} + \sum\nolimits_1$$

where

$$\sum_1 = \sum_{\rho_1}\sum_{\rho_2} \frac{(1+\theta)^{\rho_1}-1}{\rho_1} \cdot \frac{(1+\theta)^{\overline{\rho}_2}-1}{\overline{\rho}_2}$$

$$\cdot \frac{2^{1+\rho_1+\overline{\rho}_2} - 2^{-1-\rho_1-\overline{\rho}_2}}{1+\rho_1+\overline{\rho}_2} \cdot \frac{2^{2+\rho_1+\overline{\rho}_2}-1}{2+\rho_1+\overline{\rho}_2} \cdot x^{1+\rho_1+\overline{\rho}_2} .$$

By the trivial inequality $|z_1 z_2| \ll |z_1|^2 + |z_2|^2$,

$$\sum_1 \ll \sum_{\rho_1}\sum_{\rho_2} x^{1+2\beta_1} \min(\theta^2, \gamma_1^{-2})(1+|\gamma_1-\gamma_2|)^{-2}.$$

Thus, by (6.11),

$$(6.13) \qquad \sum_1 \ll \sum_{\substack{\rho \\ \gamma>0,\ \beta\geq 1/2}} x^{1+2\beta}\min(\theta^2, \gamma^{-2})\log\gamma.$$

If the Riemann hypothesis is assumed, then at once from (6.11) and (6.13),

$$\sum_1 \ll x^2 \sum_{0<\gamma\leq\theta^{-1}} \theta^2 \log\gamma + x^2 \sum_{\gamma>\theta^{-1}} \frac{\log\gamma}{\gamma^2} \ll \theta x^2 (\log \tfrac{2}{\theta})^2.$$

This with (6.12) establishes (6.4) with ϑ replaced by ψ. To deduce the corresponding inequality involving ϑ, observe that for $y \geq 1$, $y \notin \mathbb{Z}$,

$$\psi(y) - \vartheta(y) - y^{1/2} + 1 =$$

$$= \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(-\frac{\zeta'(2s)}{\zeta(2s)} - \frac{1}{2s-1} + \sum_p \frac{\log p}{p^s(p^{2s}-1)}\right)\frac{y^s}{s}ds.$$

Let

$$\Delta(v,\theta) = (\Psi(e^v(1+\theta)) - \Psi(e^v) - \vartheta(e^v(1+\theta)) - \vartheta(e^v) -$$
$$- e^{v/2}(1+\theta)^{1/2} + e^{v/2})e^{-v/2}$$

and

$$F(t) = -\frac{\zeta'(1+2it)}{\zeta(1+2it)} - \frac{1}{2iT} + \sum_p \frac{\log p}{p^{\frac{1}{2}+it}(p^{1+2it} - 1)}.$$

Then, by Plancherel's theorem,

$$\int_0^\infty |\Delta(v,\theta)|^2 dv \ll \int_{-\infty}^\infty \frac{|F(t)|^2}{(1+|t|)^2}|(1+\theta)^{\frac{1}{2}+it} - 1|^2 dt$$

$$\ll \int_{-\infty}^\infty (\log(1+|t|))^2 \min(\theta^2, (1+|t|)^{-2}) dt$$

$$\ll \theta(\log \frac{2}{\theta})^2.$$

This combined with the observation

$$\int_x^{2x} (u^{1/2}(1+\theta)^{1/2} - u^{1/2}) du \ll \theta^2 x^2$$

enables one to deduce (6.4) from the corresponding result with replaced Ψ. Another line of approach is to use the relation

$$\vartheta(x) = \sum_k \mu(k)\Psi(x^{1/k})$$

where μ is the Möbius function, but in the proof of (6.3) this gives rise to complications of detail.

To prove (6.3) note that by (6.11),

$$\sum_{\substack{\rho \\ \gamma > x^4}} x^{1+2\beta} \frac{\log \gamma}{\gamma^2} \ll 1.$$

Thus, by (6.13),

$$(6.14)\qquad \sum_1 \ll 1 + x(\theta^2(\log \tfrac{2}{\theta})\sum_2 + \sum_3)$$

where

$$\sum_2 = \sum_{\substack{\rho \\ 0<\gamma\leqslant\theta^{-1} \\ \beta\geqslant 1/2}} x^{2\beta}$$

and

$$(6.15)\qquad \sum_3 = \sum_{\substack{\rho \\ \theta^{-1}<\gamma\leqslant x^4 \\ \beta\geqslant 1/2}} x^{2\beta}\gamma^{-2}\log \gamma .$$

Hence

$$(6.16)\qquad \sum_2 = xN(\tfrac{1}{2},\theta^{-1}) + 2\int_{1/2}^{1} x^{2u}(\log x)N(u,\theta^{-1})du.$$

By (2), page 226, of Walfisz [11], we have

$$(6.17)\qquad N(\sigma,x^4) = 0 \quad \text{whenever} \quad \sigma \geqslant 1-C_3(\log x)^{-2/3}(\log\log x)^{-1/3}.$$

This with (6.2) and (6.11) gives

$$\sum_2 \ll x\theta^{-1}(\log \tfrac{2}{\theta}) + 2\int_{\frac{1}{2}}^{1-\frac{1}{C}} x^{2u}(\log x)\theta^{-1}(\log \tfrac{2}{\theta})du +$$

$$+ 2\int_{1-\frac{1}{C}}^{1-C_2(\log x)^{-2/3}(\log\log x)^{-1/3}} x^{2u}(\log x)\theta^{-C(1-u)}(\log \tfrac{2}{\theta})^{B}du.$$

It is assumed that $x^2\theta^C \gg x^{C\varepsilon} > 1$. Thus

$$(6.18)\quad \sum\nolimits_2 \ll x^{2-\frac{2}{C}}\theta^{-1}\log\frac{2}{\theta} + x^2(\log x)^{B+1}\exp\Big(\frac{c_2(C\log\frac{1}{\theta} - 2\log x)}{(\log x)^{2/3}(\log\log x)^{1/3}}\Big).$$

The sum $\sum_3$ is estimated in the same way. By (6.16), (6.11), (6.17) and (6.2),

$$\sum\nolimits_3 \ll \sum_{\substack{\rho \\ \theta^{-1}<\gamma<x^4 \\ \beta>1/2}} (x^{2\beta}-x)\Big(\frac{\log\gamma}{\gamma^2} - \frac{4\log x}{x^8}\Big) + \frac{(\log x)^2}{x^2} + \theta x\Big(\log\frac{2}{\theta}\Big)^2$$

$$\ll \theta x\Big(\log\frac{2}{\theta}\Big)^2 + \int_{\frac{1}{2}}^{1}\Big(\int_{\theta^{-1}}^{x^4} x^{2u}(2\log x)N(u,t)\frac{\log t}{t^3}dt\Big)du$$

$$\ll \theta x^{2-\frac{2}{C}}\Big(\log\frac{2}{\theta}\Big)^2 + x^2(\log x)^{B+2}\exp\Big(\frac{c_2(C\log\frac{1}{\theta} - 2\log x)}{(\log x)^{2/3}(\log\log x)^{1/3}}\Big).$$

This, with (6.12), (6.14) and (6.18), gives (6.3) with ϑ replaced by ψ. The deduction for ϑ is the same as in the proof of (6.4).

6.2. It is possible to deduce Theorem 10 directly from Lemma 5, or even from the corresponding result with ϑ replaced by ψ. However, the argument is then somewhat more complicated than with the metho we are going to use. Moreover, the following two lemmas also have some interest of their own.

<u>LEMMA 6.</u> <u>Let</u> h <u>be any real number with</u> $0 \leqslant h \leqslant x$. <u>If</u> (6.2) <u>holds, then</u>

$$(6.19)\qquad \int_x^{2x} (\vartheta(u+h) - \vartheta(u) - h)^2 du \ll h^2 x \exp(-c_1 (\frac{\log x}{\log\log x})^{1/3})$$

<u>whenever</u> $x^{\varepsilon - \frac{2}{c} + 1} < h \leqslant x$ <u>and</u> $x \geqslant 3$. <u>On the Riemann hypothesis,</u>

$$(6.20)\qquad \int_x^{2x} (\vartheta(u+h) - \vartheta(u) - h)^2 du \ll h^2 x (\log \frac{2x}{h})^2$$

<u>uniformly in</u> h.

<u>Proof.</u> It suffices to prove the lemma with $h \leqslant x/6$. Suppose that $2h \leqslant v \leqslant 3h$ and $x \leqslant u \leqslant 2x$, so that $h \leqslant v-h \leqslant 2h$ and $x \leqslant u+h \leqslant 3x$. Then, since

$$(\vartheta(u+h) - \vartheta(u) - h)^2 \ll (\vartheta(u+v) - \vartheta(u) - v)^2 + \\ + (\vartheta(u+v) - \vartheta(u+h) - (v-h))^2,$$

on making the substitution $w = \theta u$ $(h \leqslant w \leqslant 3h)$ and on observing that

$$\frac{h}{3x} \leqslant \frac{h}{u} \leqslant \theta \leqslant \frac{3h}{u} \leqslant \frac{3h}{x} \leqslant \frac{1}{2},$$

one has

$$h\int_x^{2x} (\vartheta(u+h) - \vartheta(u) - h)^2 du \ll \\ \ll \int_x^{3x} (\int_h^{3h} (\vartheta(u+w) - \vartheta(u) - w)^2 dw) du \\ x \int_x^{3x} (\int_{h/3x}^{3h/x} (\vartheta(u+u\theta) - \vartheta(u) - \theta u)^2 d\theta) du.$$

The integrand in the last double integral is continuous on $[x,3x] \times [h/3x, 3h/x]$ except on a subset having zero content. Thus the order of integration can be inverted. Hence

$$(6.21)\qquad \int_x^{2x} (\vartheta(u+h) - \vartheta(u) - h)^2 du \ll$$

$$\ll \frac{x}{h}\int_{h/3x}^{3h/x} \left(\int_x^{3x} (\vartheta(u+\theta u) - \vartheta(u) - \theta u)^2 du\right) d\theta.$$

Using this, (6.20) follows from (6.4). If $x \leqslant 3^{2/\varepsilon}$, then (6.19) is trivial. Thus it can be assumed that

$$h \geqslant 3x^{\frac{\varepsilon}{2} + 1 - \frac{2}{C}}.$$

Combining (6.3) and (6.21) then gives (6.19).

<u>LEMMA 7.</u> <u>Suppose that</u> (6.2) <u>holds. Then</u>

$$(6.22)\qquad \int_x^{2x} \max_{0 \leqslant v \leqslant h} |\vartheta(u+v) - \vartheta(u) - v|^2 du \ll_\varepsilon$$

$$\ll_\varepsilon h^2 x \exp\left(-c_4\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)$$

<u>whenever</u> $x^{\varepsilon - \frac{2}{C} + 1} < h \leqslant x$ <u>and</u> $x \geqslant 3$. <u>Moreover, on the Riemann hypothesis,</u>

$$(6.23)\qquad \int_x^{2x} \max_{0 \leqslant v \leqslant h} |\vartheta(u+v) - \vartheta(u) - v|^2 du \ll hx(\log x)^4$$

<u>whenever</u> $0 \leqslant h \leqslant x$.

This follows from Lemma 6 by a similar argument to that used to deduce Lemma 4 from Lemma 3.

6.3. We now proceed with the proof of Theorem 10. By $\sum_m'$ it is meant that possible terms with $m < [x/y]$ are omitted, $[x/y]$ is only counted when $x < y([x/y] + \alpha)$, and if $[x/y]$ is counted, then $x/[x/y]$ is replaced by y in all the appropriate places. Observe now that, by (1.27),

$$(6.24) \qquad y\vartheta_{x,y}(\alpha) = \sum_{m \leq x^{1/2}}' \sum_{\frac{x}{m+\alpha} < p \leq \frac{x}{m}} \log p + \sum_{p \leq x^{1/2}} \sum_{\substack{m \\ x-\alpha p < mp \leq x}} \log p - \sum_{m \leq x^{1/2}} \sum_{\frac{x}{m+\alpha} < p \leq x^{1/2}} \log p.$$

Clearly the contribution from the second double sum is $O(x^{1/2})$, and from the third is $O(\log x)$. Thus, by (6.24),

$$y\vartheta_{x,y}(\alpha) = \sum_{m \leq x^{1/2}}' \left(\vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right)\right) + O(x^{1/2})$$

so that, by (1.2),

$$(6.25) \qquad y(\vartheta_{x,y}(\alpha) - F(\alpha, x/y)) = \sum_{m \leq x^{1/2}}' \left(\vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right) - \frac{\alpha x}{m(m+\alpha)}\right) + O(x^{1/2}).$$

Suppose that

$$(6.26) \qquad 0 < \delta < 1$$

and

$$0 \leqslant u - \frac{x}{m+\alpha} \leqslant \frac{\delta x\alpha}{m(m+\alpha)}.$$

then

(6.27) $$\vartheta(\tfrac{x}{m}) - \vartheta(\tfrac{x}{m+\alpha}) - \frac{\alpha x}{m(m+\alpha)} =$$

$$= \vartheta(\tfrac{x}{m}) - \vartheta(u) - (\tfrac{x}{m} - u) + O\left((\tfrac{\delta x}{m^2} + 1)\log x\right).$$

Let X be of the form

(6.28) $$X = (1+\delta)^k,$$

where k is a non-negative integer, and suppose that

(6.29) $$X^2 \leqslant x.$$

Then

(6.30) $$\sum_{X<m\leqslant X+\delta X} \left|\vartheta(\tfrac{x}{m}) - \vartheta(\tfrac{x}{m+\alpha}) - \frac{\alpha x}{m(m+\alpha)}\right| \ll$$

$$\ll \frac{X^2}{\delta x}\int_{x/(X+\delta X+1)}^{x/X} \sup_{v\leqslant xX^{-2}} |\vartheta(u+v) - \vartheta(u) - v|\,du$$

$$+ \sum_{X<m\leqslant X+\delta X}{}' (\tfrac{\delta x}{m^2} + 1)\log x.$$

Before proceeding further with the proof consider the consequence of assuming the Riemann hypothesis. By Lemma 7 and (6.30),

$$\sum_{X<m\leqslant X+\delta X}{}' \left|\vartheta(\tfrac{x}{m}) - \vartheta(\tfrac{x}{m+\alpha}) - \frac{\alpha x}{m(m+\alpha)}\right| \ll$$

$$\ll \frac{X^2}{\delta X}\left(\frac{\delta x}{X}\cdot\frac{x}{X^2}\cdot\frac{x}{X}(\log x)^4\right)^{1/2} + \sum_{X<m<X+\delta X}{}' (\tfrac{\delta x}{m^2} + 1)\log x.$$

Thus, summing over those X, given by (6.28), for which (6.29), holds, one finds that

$$\sum_{m \leqslant x^{1/2}}{}' \left| \vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right) - \frac{\alpha x}{m(m+\alpha)} \right| \ll$$

$$\ll x^{1/2}\delta^{-3/2}(\log x)^3 + \delta y \log x + x^{1/2} \log x,$$

which with (6.25) and the choice $\delta = y^{-2/5}x^{1/5}(\log x)^{4/5}$, which is consistent with (6.26), gives

$$\vartheta_{x,y}(\alpha) = F(\alpha, x/y) + O\left(y^{3/5}x^{1/5}(\log x)^{9/5}\right)$$

whenever $y > x^{1/2}(\log x)^2$.

To return to the proof, suppose that (6.2) holds. Then, providing that

$$X \leqslant x^{\frac{2}{C+2} - \varepsilon},$$

one has, by (6.30), the Schwarz inequality and Lemma 7,

$$\sum_{X < m \leqslant X+\delta X}{}' \left| \vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right) - \frac{\alpha x}{m(m+\alpha)} \right| \ll$$

$$\ll \frac{X^2}{\delta x}\left(\frac{\delta x}{X} \cdot \frac{x^2}{X^4} \cdot \frac{x}{X} \exp\left(-c_5\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)\right)^{1/2} +$$

$$+ \sum_{X < m \leqslant X+\delta X} \left(\frac{\delta x}{m^2} + 1\right) \log x.$$

Thus summing over all the numbers X of the form (6.28) for which (6.29) holds gives

$$\sum_{m<x^{1/2}} \left|\vartheta(\frac{x}{m}) - \vartheta(\frac{x}{m+\alpha}) - \frac{\alpha x}{m(m+\alpha)}\right| \ll \sum_{x^{\frac{2}{C+2}-\varepsilon}<m\leq x^{1/2}} (\frac{x}{m^2}+1)\log x$$

$$+ y\delta^{-3/2}\exp(-C_6(\frac{\log x}{\log\log x})^{1/3}) + (\delta y + x^{1/2})\log x.$$

This with the choice $\delta = \exp(-\frac{1}{2}C_6(\frac{\log x}{\log\log x})^{1/3})$ and Huxley's theorem [2] that (6.2) holds with $C = 12/5$ establishes Theorem 10.

7. Proof of Theorem 11.

Define N_j inductively by

$$(7.1)\qquad N_1 = 3 \quad \text{and} \quad N_{j+1} = \Pi p$$

where the product is over all those primes p such that

$$(7.2)\qquad p \leqslant e^{N_j}.$$

Let

$$(7.3)\qquad \mathcal{D}_j = \{n : n | N_j, \quad \log N_j < n \leqslant N_j\}$$

and

$$(7.4)\qquad \mathcal{D} = \bigcup_{j=1}^{\infty} \mathcal{D}_j.$$

Further, let j be large and write

$$(7.5)\qquad y = N_j.$$

Let λ_n be the characteristic function of $\mathcal{D}$ and

$$(7.6)\qquad a_n(y) = \begin{cases} \lambda_n / \sum_{m \leqslant y} \lambda_m & (n \leqslant y) \\ 0 & (n > y). \end{cases}$$

By (7.1), (7.2), (7.3) and (7.4), all the elements of $\mathcal{D}$ are odd. Let $\alpha = 1/2$. Hence, by Theorems 3 and 4 of [9],

$$(7.7)\qquad \max(0,\, x-y^2) \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} \tfrac{1}{m} a_{mn}(y)\Big)^2 \ll$$

$$\ll \int_{x_0}^{x_0+x} (\Phi_{u,y}(\alpha) - \alpha)^2 d\alpha \ll (x+y^2) \sum_{n=1}^{\infty}\Big(\sum_{m=1}^{\infty} \tfrac{1}{m} a_{mn}(y)\Big)^2.$$

By (7.1), (7.2), (7.3), (7.4) and (7.5), it is easily seen that y is squarefree and the elements of

$$\mathcal{D} \cap (\log N_j, N_j]$$

are precisely the divisors of y in the range $(\log N_j, N_j]$. Since λ_n is the characteristic function of $\mathcal{D}$,

$$\sum_{m \leq y} \lambda_m = 2^P + O(\log y) \tag{7.8}$$

where P is the number of prime divisors of y. Also,

$$\sum_{n \leq y} \Big(\sum_{m \leq y/n} \frac{1}{m}\lambda_{mn}\Big)^2 = \sum_{\substack{n \\ \log y < n \leq y}} \Big(\sum_{\substack{m \\ mn \mid y}} \frac{1}{m}\Big)^2 + O((\log y)^3)$$

$$= \sum_{\substack{n \\ n \mid y}} \Big(\sum_{\substack{m \\ mn \mid y}} \frac{1}{m}\Big)^2 + O((\log y)^3)$$

$$= 2^P \prod_{p \mid y} \Big(1 + \frac{1}{p} + \frac{1}{2p^2}\Big) + O((\log y)^3).$$

Hence, by (7.1), (7.2), (7.6) and (7.8),

$$\sum_{n=1}^{\infty} \Big(\sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y)\Big)^2 \asymp \frac{\prod_{p \mid y} (1 + p^{-1} + \frac{1}{2}p^{-2})}{\sum_{n \leq y} \lambda_n}. \tag{7.9}$$

Theorem 11 now follows in a straightforward manner from (7.1), (7.2), (7.5), (7.7) and (7.9).

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B. Saffari

1, Place Corneille
92100 Boulogne Billancourt.

R.C. Vaughan

Imperial College, London
University of Michigan
Ann Arbor.

On the fractionnal parts of x/n and related sequences, III

B. SAFFARI and R.C. VAUGHAN

1. Introduction.

The object of this paper is to investigate the behaviour of $\Phi_{x,y}(\alpha,h)$ (for notation see [2] and [3]) when $h(n) = \frac{1}{\log n}$ $(n > 1)$ and $h(n) = \log n$. In contradistinction to the case $h(n) = 1/n$ it is immediately apparent that the behaviour of $\Phi_{x,y}$ is non-trivial even when y is as large as e^x. For simplicity we only investigate the situation when $\mathcal{A}$ is the Toeplitz transformation formed from the simple Riesz means (R,λ_n) with $\lambda_n = 1$.

Theorems 1 and 2 deal with the case $h(n) = 1/\log n$, whereas Theorem 3 deals with $h(n) = \log n$. While it is well known ([1], Example 2.4, p.8) that the sequence $\log n$ is not uniformly distributed modulo 1, Theorem 3 shows that it is uniformly distributed in the present context.

2. Theorems and proofs.

2.1. Let

$$(2.1)\qquad \Xi_{x,y}(\alpha) = y^{-1} \sum_{2 \leq n \leq y} c_\alpha(x/\log n).$$

THEOREM 1. Suppose that $0 < \alpha < 1$ and $\log y \ll x^{\frac{1}{2}}$. Then

$$\Xi_{x,y}(\alpha) = \alpha + O\left(xy^{-1}(\log x)^{-1}\right) + O\left(x^{-1}\log^2 y\right).$$

COROLLARY 1.1. *Suppose that* $x = o(y \log x)$ *and* $\log y = o(x^{\frac{1}{2}})$ *as* $x \to \infty$. *Then*

$$\Xi_{x,y}(\alpha) \to \alpha \qquad \text{as } x \to \infty .$$

Proof. Clearly by (2.1),

$$y\,\Xi_{x,y}(\alpha) = S(0) - S(\alpha) \tag{2.2}$$

where

$$S(\beta) = \sum_{m=1}^{\infty} \sum_{\substack{2 \leq n \leq y \\ n \leq e^{x/(m+\beta)}}} 1 \tag{2.3}$$

and $0 \leq \beta < 1$. Let

$$M_\beta = \left[\frac{x}{\log y} - \beta\right] \tag{2.4}$$

and

$$T(\beta) = \sum_{M_\beta < m \leq \frac{x}{\log 2} - \beta} \; \sum_{2 \leq n \leq e^{x/(m+\beta)}} 1. \tag{2.5}$$

Then, by (2.3),

$$S(\beta) = T(\beta) + ([y] - 1)M_\beta . \tag{2.6}$$

By (2.5),

$$T(\beta) = \sum_{M_\beta < m \leq H-\beta} e^{x/(m+\beta)} +$$

$$+ \sum_{2 \leq n \leq e^{x/H}} \frac{x}{\log n} - He^{x/H} + O(H) + O(e^{x/H})$$

where H is a real number at our disposal. Hence, by (2.4),

$$(2.7)\qquad T(0) - T(\alpha) = \sum_{M_0<m\leqslant H} e^{x/m} - \sum_{M_\alpha<m\leqslant H-\alpha} e^{x/(m+\alpha)} + \\ + O(H) + O(e^{x/H})$$

whenever $H \geqslant M_0 + 1$. Thus

$$(2.8)\qquad T(0) - T(\alpha) = I(0) - I(\alpha) + O(H) + O(e^{x/H}),$$

where

$$(2.9)\qquad I(\beta) = \int_{M_\beta}^{H-\beta} ([u] - M_\beta)e^{x/(u+\beta)} \frac{xdu}{(u+\beta)^2} .$$

Let $b(u)$ denote the first Bernoulli polynomial modulo one, $b(u) = \{u\} - 1/2$. Then, by (2.9),

$$(2.10)\qquad I(\beta) = \int_{M_\beta + \beta}^{H} (v - M_\beta - \beta - 1/2)e^{x/v} xv^{-2} dv \\ - \int_{M_\beta + \beta}^{H} b(u-\beta)e^{x/v} xv^{-2} dv.$$

The argument now divides into two cases according as $M_0 = M_\alpha$ or $M_0 = M_\alpha + 1$.

<u>The case</u> $M_0 = M_\alpha$. Write M for the common value. Then, by (2.10),

$$I(0) - I(\alpha) = \int_M^{M+\alpha} (v - M - \tfrac{1}{2})e^{x/v} xv^{-2} dv + \alpha \int_{M+\alpha}^{H} e^{x/v} xv^{-2} dv \\ - \int_M^H b(v)e^{x/v} xv^{-2} dv + \int_{M+\alpha}^{H} b(v-\alpha)e^{x/v} xv^{-2} dv.$$

The first integral contributes $\ll e^{x/M} xM^{-2}$, the second is $\alpha(e^{x/(M+\alpha)} - e^{x/H})$ and by partial integration the last two are easily seen to contribute $\ll e^{x/M} xM^{-2}$. Hence, by (2.8),

$$\text{(2.11)} \qquad T(0) - T(\alpha) = \alpha e^{x/(M+\alpha)} + O(H) + O(e^{x/H}) + O(e^{x/M}\, xM^{-2}).$$

Recall that $M = M_0 = [x/\log y]$ and $\log y \ll x^{1/2}$. Thus

$$e^{x/(M+\alpha)} = \exp(\log y + O(x^{-1}\log^2 y)) = y(1 + O(x^{-1}\log^2 y))$$

and $e^{x/M}\, xM^{-2} = O(yx^{-1}\log^2 y)$. Hence, by (2.2), (2.6) and (2.11)

$$y_{x,y}(\alpha) = \alpha y + O(H) + O(e^{x/H}) + O(yx^{-1}\log^2 y).$$

The choice $H = \dfrac{x}{\log(x/\log x)}$ now gives the desired conclusion.

<u>The case</u> $M_0 = M_\alpha + 1$. Write M for M_α. Then, by (2.10),

$$I(0) - I(\alpha) = (\alpha - 1)\int_{M+1}^{H} e^{x/v}\, xv^{-2}\, dv - \int_{M+\alpha}^{M+1} (v - M - \alpha - \tfrac{1}{2}) e^{x/v}\, xv^{-2}\, dv + O(e^{x/(M+\alpha)}\, x(M+\alpha)^{-2}).$$

Now proceeding as in the previous case we obtain

$$T(0) - T(\alpha) = (\alpha-1)y + O(H) + O(e^{x/H}) + O(yx^{-1}\log^2 y).$$

Since $M_0 = M_\alpha + 1$, this with (2.6) and (2.2) and the choice $H = \dfrac{x}{\log(x/\log x)}$ gives the required result once more.

2.2. One might expect that the theorem holds even when y is close to e^x, but this is false. In fact the next theorem indicates that Theorem 1 is essentially best possible, at least as for as the upper bound on y is concerned.

THEOREM 2. Suppose that $0 < \alpha < 1$, $\frac{1}{2} < \theta < 1$ and $y = \exp(x^{\theta})$. Then

$$\limsup_{x \to \infty} \Xi_{x,y}(\alpha) = 1 \quad \text{and} \quad \liminf_{x \to \infty} \Xi_{x,y}(\alpha) = 0.$$

Proof. We begin by following the proof of Theorem 1 as far as (2.7). Suppose that $0 < \beta < 1$,

$$y = \exp(x^{\theta}) \tag{2.12}$$

and

$$H = x. \tag{2.13}$$

Then, by (2.4),

$$\frac{x}{(M_{\beta} + 2)(M_{\beta} + 3)} \gg x^{-1}(\log y)^{2} = x^{2\theta-1}.$$

Thus, by (2.13),

$$\sum_{M_{\beta} + 1 < m \leq H-\beta} e^{x/(m+\beta)} \ll xe^{x/(M_{\beta} + 1 + \beta)} \exp(-C_{1} x^{2\theta-1}).$$

Hence, by (2.7) and (2.13),

$$\begin{aligned} T(0) - T(\alpha) &= \\ &= (e^{x/(M_{0} + 1)} - e^{x/(M_{\alpha} + 1 + \alpha)})(1 + O(x^{-1})) + O(x). \end{aligned} \tag{2.14}$$

To obtain the inferior limit, let N be a large natural number and let

$$x = x_{N} = (N+\alpha)^{\frac{1}{1-\theta}}. \tag{2.15}$$

Then, by (2.4) and (2.12), $M_{0} = M_{\alpha} = N$. Hence, by (2.2), (2.6), (2.12), (2.14) and (2.15),

$$y\,\Xi_{x,y}(\alpha) = e^{x/(N+1)} = o(y)$$

as $N \to \infty$.

For the superior limit, take instead

(2.16) $$x = x_N = N^{\frac{1}{1-\theta}}.$$

Then, by (2.4) and (2.12), $M_\alpha = M_0 - 1 = N-1$, so that, by (2.2), (2.6), (2.12), (2.14) and (2.16),

$$y \Xi_{x,y}(\alpha) \sim - e^{x/(N+\alpha)} + y \sim y$$

as $N \to \infty$.

2.3. The latter part of the paper is devoted to $h(n) = \log n$. It is well known that the sequence $\log n$ is not uniformly distributed modulo 1, and in view of this the next theorem is perhaps rather surprising. However, one can take the view that x being permitted to go to infinity, however slowly by comparison with y, crushes any unruly behaviour of the logarithmic function.

Let

(2.17) $$\Omega_{x,y}(\alpha) = y^{-1} \sum_{n \leqslant y} c_\alpha(x \log n).$$

THEOREM 3. *Suppose that* $0 < \alpha < 1$, $x \geqslant 2$ *and* $y \geqslant 2$. *Then*

$$\Omega_{x,y}(\alpha) = \alpha + O\,(x^{-1} \log x + x^{1/3}\, y^{-2/3}(\log xy)^{2/3}).$$

COROLLARY 3.1. *Suppose that* $x^{1/2}\log x = o(y)$ *as* $x \to \infty$. *Then*

$$\Omega_{x,y}(\alpha) \to \alpha \quad \text{as} \quad x \to \infty .$$

Proof. Let

(2.18) $$M = [y^{2/3}\, x^{-1/3}(\log xy)^{-2/3}] + 1.$$

Then, by Theorem 1 of [2] and (2.17),

(2.19) $$\Omega_{x,y}(\alpha) - \alpha \ll$$

$$\ll y^{-1} + M^{-1} + y^{-1} \sum_{k=1}^{M} k^{-1} \Big|\sum_{n\leq y} e(kx\log n)\Big| .$$

Let

(2.20) $$Y = [y] + \frac{1}{2}$$

and

(2.21) $$T = 4\pi kx.$$

Then, by Lemma 3.12 of Titchmarsh [4],

$$\sum_{n\leq y} e(kx\log n) =$$

$$= \frac{1}{2\pi i} \int_{1+\frac{1}{\log y} - iT}^{1+\frac{1}{\log y} + iT} \zeta(s-2\pi ikx)\frac{Y^s}{s} ds + \underline{O}\left(\left(\frac{y}{T} + 1\right)\log xy\right)$$

where ζ is the Riemann zeta function. By moving the path of integration to the line $\sigma = 1/\log y$, one obtains

$$\sum_{n\leq y} e(kx\log n) =$$

$$= \frac{y^{1+2\pi ikx}}{1+2\pi ikx} + \frac{1}{2\pi i} \int_{\frac{1}{\log y} - iT}^{\frac{1}{\log y} + iT} \zeta(s-2\pi ikx)\frac{Y^s}{s} ds +$$

$$+ \underline{O}\left(\left((kx)^{1/2} + y \log kx\right)T^{-1}\right).$$

Hence, by (2.21),

$$\sum_{n \leqslant y} e(kx\log n) \ll$$

$$\ll (kx)^{1/2} \int_0^T \frac{dt}{t + \frac{1}{\log y}} + (kx)^{-1/2} + \frac{y \log kx}{kx}$$

$$\ll (kx)^{1/2}(\log\log y + \log kx) + y(\log kx)(kx)^{-1}.$$

Thus

$$\sum_{k=1}^{M} k^{-1} \left|\sum_{n \leqslant y} e(kx \log n)\right| \ll (Mx)^{1/2}(\log\log y + \log Mx) + yx^{-1} \log x.$$

Therefore, by (2.18) and (2.19), we have the theorem.

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B. Saffari

1, Place Corneille
92100 Boulogne Billancourt.

R.C. Vaughan

Imperial College, London
University of Michigan,
Ann Arbor.