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Brownian motion and classical analysis

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BROWNIAN MOTION AND CLASSICAL ANALYSIS

by Jean-Pierre Kahane

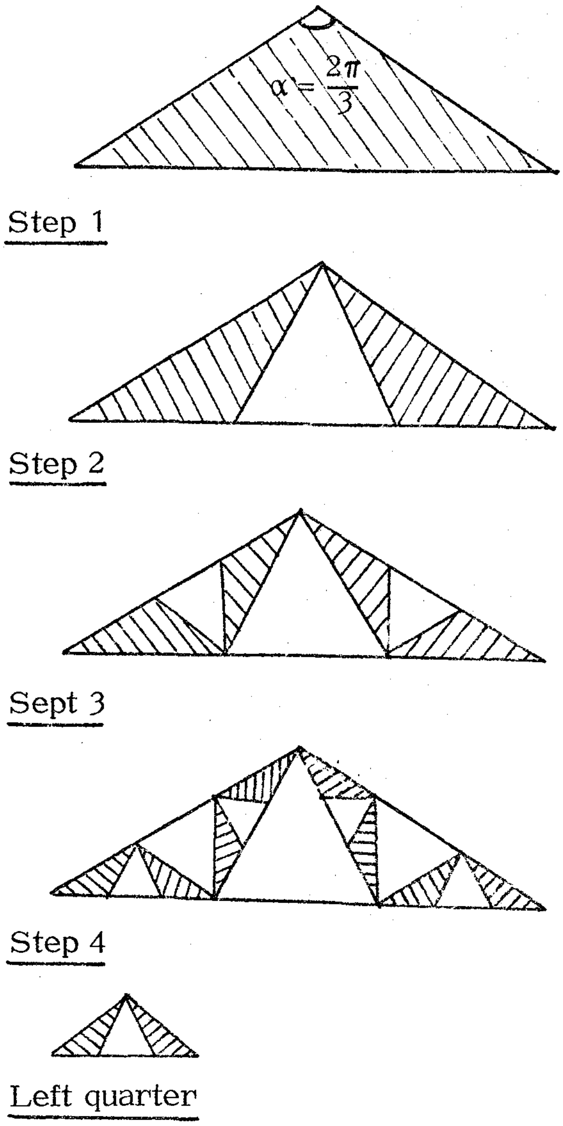
This is an account of an informal talk given at one of the Durham Conferences of the London Mathematical Society, July 1974. I have made no attempt to transform it into an organized paper, with the hope and the excuse that the easier the wirting is, the easier the reading will be.

The subject is very rich. I selected only three topics in classical analysis which are closely related to brownian motion : 1) nowhere differentiable functions 2) totally disconnected perfect sets 3) analytic functions of one complex variable. On each of these topics, the Brownian motion sheds much light, and there are interesting new results.

1. NOWHERE DIFFERENTIABLE FUNCTIONS.

Weierstrass gave his celebrated example of a continuous nowhere differentiable function in 1872. It was regarded as a curiosity at the time, a kind of a mathematical monster. A geometrical construction of a nowhere differentiable simple curve was described by Von Koch in 1904. Maybe it is worthwhile to look at Von Koch's curve, if only because it apparently attracted the attention of Paul Lévy as a child.

I hope the adjacent drawing explains how to obtain Von Koch's curve. It has infinite length, vanishing area, and a positive finite Hausdorff measure in dimension $\frac{\log 4}{\log 3}$ (observe that a dilation of the left quarter by a ratio 3 transforms the left quarter into the whole curve, therefore multiplies its measure by 4). If we increase the angle α from $\frac{2\pi}{3}$ to π, the dimension of the corresponding Von Koch curve decreases from $\frac{\log 4}{\log 3}$ to 1 ; for $\alpha = 1$, it is a straight line. If we decrease the angle α from $\frac{2\pi}{3}$ to $\frac{\pi}{2}$, the dimension increase from $\frac{\log 4}{\log 3}$ to 2 ; for $\alpha = \frac{\pi}{2}$, it is the simplest example of a Peano curve, filling a triangle.



Let us return to Weierstrass's example. It is given by a lacunary trigonometric series

$$(1) \qquad \sum_1^\infty a_n \cos(\lambda_n t + \varphi_n) \qquad \frac{\lambda_{n+1}}{\lambda_n} > q > 1, \quad \sum_1^\infty |a_n| < \infty .$$

Let $f(t)$ be the sum of (1). It is nowhere differentiable as soon as $\overline{\lim}\, |a_n| \lambda_n > 0$ (Hardy). Moreover for $0 < \alpha < 1$ the following conditions are equivalent (Geza Freud)

a) $f(t_o + h) - f(t_o) = 0(|h|^\alpha)$ (t_o fixed, $h \to 0$)

b) $a_n = 0(\lambda_n^{-\alpha})$

c) $f \in \Lambda_\alpha$ (meaning a) for all t_o and uniformly).

The implication a) $\Longrightarrow$ c) means that a Lipschitz condition of order $\alpha < 1$ at one point implies a uniform Lipschitz condition of order α : no point is better than the others. More precisely, if $0 < \overline{\lim}_{n\to\infty} |a_n| \lambda_n^{\alpha} < \infty$, there exist two positive numbers c and C such that

(2) $$c < \overline{\lim}_{h\to 0} \frac{|f(t+h) - f(t)|}{|h|^{\alpha}} < C$$

for all t. For series (1) of a particular type, for example $\sum_1^{\infty} 3^{-\alpha n} \cos 3^n t$, we can even say more : the $\overline{\lim}$ in (2) is a constant almost everywhere (Michel Bruneau).

Like the Von Koch curves, the Weierstrass functions are very regularly irregular.

There is a strong analogy between lacunary series (1) and random Fourier series

(3) $$\sum_0^{\infty} \alpha_n (\xi_n' \cos 2\pi nt + \xi_n'' \sin 2\pi nt)$$

where the ξ_n' and ξ_n'' are independent normal gaussian variables. Since the Brownian motion can be expressed by means of such a series (series of Fourier-Wiener) it can be expected that it behaves somehow like a Weierstrass function. We shall see howfar it is true.

2. BROWN, EINSTEIN, JEAN PERRIN.

The discovery of a very irregular and apparently perpetual motion of very thin particles in a liquid is due to Brown about 1850.

For half a century the Brownian motion was a mystery in physics. The explanation (chocs of atoms) together with an estimate of the dependence of time (role of the quadratic variation) was made by A. Einstein in one of his three famous papers of Annalen der Physik 1905. Is it the one B. Russell had in mind when he advised N. Wiener, then a student in Cambridge, to read Einstein rather than study logic ?

Jean Perrin used Einstein's theory together with other methods to compute the dimensions of atoms, and his little book "Les Atomes" (1912) describes carefully the experiments and results. A long quotation may be useful here. It will allow the reader to practise his French (by the way, Perrin's French is beautiful), to see that physicists can have sharp views on mathematics, and to understand why Wiener himself quoted Jean Perrin very often as a motivation for his work on Brownian motion. Moreover, the drawings provide a good intuition of what the curve of the two-dimensional Brownian motion looks like.

« *Einstein et Smoluckowski ont caractérisé de la même façon l'activité du mouvement brownien. Jusqu'alors on s'était efforcé de définir une « vitesse moyenne d'agitation » en suivant aussi exactement que possible le trajet d'un grain. Les évaluations ainsi obtenues étaient toujours de quelques microns par seconde pour des grains de l'ordre du micron.*

Mais de telles évaluations sont grossièrement fausses. *L'enchevêtrement de la trajectoire est tel que la trajectoire notée est toujours infiniment plus simple et plus courte que la trajectoire réelle. En particulier, quand la durée qui sépare deux pointés d'un même grain décroît, la vitesse moyenne de ce grain pendant cette durée, loin de tendre vers une limite, grandit sans cesse, et varie follement en direction, comme on le voit de façon simple, en notant les positions d'un grain à la chambre claire de minute en minute, puis, par exemple, de 5 en 5 secondes, et mieux encore en les photographiant de vingtaine en vingtaine de secondes, comme ont fait Victor Henri, Comandon ou M. de Broglie, pour cinématographier le mouvement. On voit du même coup que l'on ne peut fixer une tangente en aucun point de la trajectoire, et c'est un cas où il est vraiment*

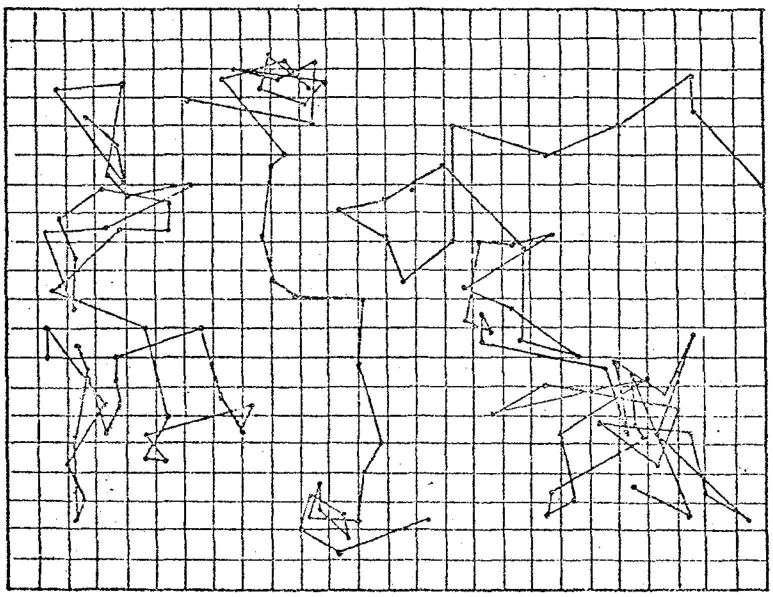


FIG. 1.

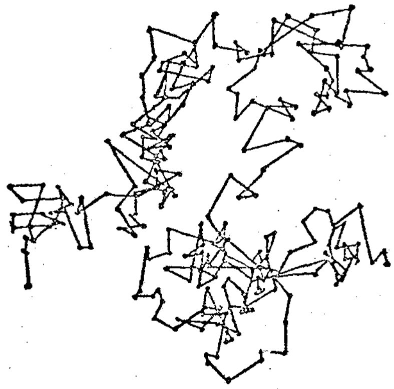


FIG. 2.

naturel de penser à ces fonctions continues sans dérivées que les mathématiciens ont imaginées, et que l'on regardait à tort comme de simples curiosités mathématiques, puisque l'expérience peut les suggérer...

... Si probable que parut la vérification, elle était utile, et d'ailleurs très facile. Il suffit de noter à la « chambre claire » la position d'un même grain à intervalles de temps égaux. A titre d'exemple, j'ai réuni sur la figure ci-contre (1908), où 16 divisions représentent 50 microns, 3 dessins obtenus en traçant les projections horizontales (1).

Incidemment, une telle figure, et même le dessin suivant (2), *où se trouvent reportés à une échelle arbitraire un plus grand nombre de déplacements, ne donnent qu'une idée bien affaiblie du prodigieux enchevêtrement de la trajectoire réelle. Si en effet on faisait des pointés en des intervalles de temps 100 fois plus rapprochés, chaque segment serait remplacé par un contour polygonal relativement aussi compliqué que le dessin entier, et ainsi de suite. On voit comment s'évanouit en de pareils cas la notion de tangente à une trajectoire...*

... On fera des réflexions analogues pour toutes les propriétés qui, à notre échelle, semblent régulièrement continues, telles que la vitesse, la pression, la température. Et nous les verrons devenir de plus en plus irrégulières, à mesure que nous augmenterons le grossissement de l'image toujours imparfaite que nous nous faisons de l'Univers. La densité était nulle en tout point, sauf exceptions ; plus généralement, la fonction qui représente la propriété physique étudiée (mettons que ce soit le potentiel électrique) formera dans toute matière un continum présentant une infinité de points singuliers, et dont les mathématiciens nous permettront de poursuivre l'étude. »

J. PERRIN, *Atomes* (1912 et 1935), pp. 138-139, 142-143 et XXII-XXIII.

3. THE BROWNIAN MOTION.

The Brownian motion on $\mathbb{R}$, the real line, is given by a function of two variables $X(t,\omega)$, where $t \in \mathbb{R}^+$ (t is the time) and $\omega \in \Omega$ (a convenient probability space). Nothing is lost if we take $\Omega = [0,1]$, equipped with the Lebesgue σ-field and the Lebesgue measure. Measurable functions on Ω are called random variables, integration on Ω is denoted by E (expectation) and "almost everywhere on Ω" is written a. s. (almost surely).

A (real, centered) gaussian variable is a random variable X such that $E(e^{\lambda X}) = e^{\frac{1}{2}\lambda^2\tau^2}$ for some $\tau > 0$ and all real λ. It is easy to check that $X \in L^2(\Omega)$ and $\|X\|_2 = \tau$. There exist infinite dimensional closed subspaces of $L^2(\Omega)$ which consist of gaussian random variables ; we denote any such space by $\mathcal{H}$. If X and Y are orthogonal in $\mathcal{H}$, one checks $E(e^{\lambda X+\mu Y}) = E(e^{\lambda X})E(e^{\mu Y})$ $(\lambda, \mu \in \mathbb{R})$, therefore X and Y are independent. Any orthonormal system in $\mathcal{H}$ (that is, any system of independent unitary gaussian variables) will be called a normal system.

Here are two equivalent "definitions" of $X(t,\omega)$.

First definition.

1. For some space $\mathcal{H}$, all random variables $X(t,.)$ $(t \in \mathbb{R}^+)$ belong to $\mathcal{H}$ (definition of a gaussian process)

2. If $0 \leq \alpha < \beta < \gamma < \delta$, the random variables $X(\beta,.) - X(\alpha,.)$ and $X(\delta,.) - X(\gamma,.)$ are independent (definition of a process with independent increments)

3. $E(X^2(t,.)) = t$ (normalization condition).

Second definition. We consider $I = [0,a]$ or $I = \mathbb{R}^+$.

For some space $\mathcal{H}$ and some linear isometric mapping W from $L^2(I)$ into $\mathcal{H}$, $X(t,.) = W(1_{[0,t]})$ $(t \in I)$.

From now on we write $X(t)$ instead of $X(t,\omega)$ or instead of $X(t,.)$.

The second definition suggests the consideration of a basis in $L^2(I)$, and also the extension of the definitions to a <u>complex</u> brownian motion (a complex gaussian variable being $X + iY$ with $\|X\| = \|Y\|$ and $X \perp Y$, $\mathcal{H}$ being a complex Hilbert space consisting of complex gaussian variables, and $L^2(I)$ being also a complex Hilbert space).

The image of a basis $\{e_n\}$ of $L^2(I)$ by W is a normal sequence $\{\xi_n\}$. A simple computation gives

$$X(t) = \sum \xi_n \int_0^t e_n(s)ds$$

and the series converges both in $L^2(\Omega)$ and a. s..

If $I = [0,1]$ the two most useful basis are the Haar system (in the real case) and the trigonometric system (in the complex or in the real case). Using the Haar system

$$X(t) = \xi_0 t + \sum \xi_{jn} \Delta_{jn}(t) \qquad (j = 0,\ 1,\ \ldots\ ;\ n = 0,\ 1,\ \ldots\ 2^j - 1)$$

where $\{\xi_0, \xi_{jn}\}$ is a (real) normal system, and Δ_{jn} is the triangular function of height $2^{-j/2-1}$ and basis $\left[\frac{n}{2^j}, \frac{n+1}{2^j}\right]$. Using the trigonometric system

(4) $$X(t)(\text{complex}) = \xi_0 t + \sum_{n \neq 0} \frac{\xi_n}{2\pi i n} (e^{2\pi i n t} - 1)$$

where $\{\xi_n\}$ $(n \in \mathbf{Z})$ is a complex normal system and

(5) $$X(t)\ (\text{real}) = \xi_0 t + \sum_{n=1}^{\infty} \frac{\sqrt{2}}{2\pi n} (\xi_n'(\cos 2\pi n t - 1) + \xi_n'' \sin 2\pi n t),$$

where $\{\xi_0, \xi_1', \xi_1'', \xi_2', \ldots\}$ is a real normal sequence. (4) and (5) are the Fourier-Wiener series.

4. HOW IRREGULAR IS THE BROWNIAN MOTION ?

Before answering the question, let us point out that we are interested in almost sure properties of the random function $X(t)$. There are at least three questions to be considered : a) is $X(t)$ continuous, and if so what is its modulus of continuity ? b) given t, how wild the oscillation $X(t+h) - X(t)$ can be as $h \to 0$? c) is $X(t)$ nowhere differentiable, and what is the strongest statement in that direction ? In all cases we have to consider an expression of the form

(6) $$\overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{\varphi(|h|)}$$

for some increasing function φ. In a) we are interested in a uniform limit as $0 \le t$, $t+h \le 1$, therefore (6) reads as

$$\overline{\lim_{h\to 0}} \frac{\omega_X(h)}{\varphi(h)}$$

where $\omega_X(h)$ is the modulus of continuity of X on $[0,1]$. In b) and c) we consider (6) when t is fixed ; let us denote it by $Y(t)$ or $Y(t,\omega)$. For each $t > 0$, $Y(t)$ is constant a. s. (zero-one law) and the constant does not depend on t (stationary property), that is

(7) $$\forall t > 0 \quad \text{a. s. } (\omega) \quad Y(t,\omega) = C$$

for some C, $0 \le C \le \infty$. The point of view b) is to look for C when φ is given. As a consequence of (7) and Fubini's theorem

$$\text{a. s. } (\omega) \quad \text{a. s. } (t) \quad Y(t,\omega) = C$$

(which of course does not mean a. s. (ω) $\forall t$ $Y(t,\omega) = C$!). The question c) is to consider events such as

$$\{\forall t \quad Y(t) > 0\}, \quad \{\forall t \quad Y(t) < \infty\};$$

by the zero-one law again, their probabilities are 0 or 1.

Here are the classical answers.

a) $X(t)$ is continuous, its modulus of continuity is $0(\sqrt{h \log \frac{1}{h}})$ (Wiener) and moreover

$$\overline{\lim_{h\to 0}} \frac{\omega_X(h)}{\sqrt{2\, h \log \frac{1}{h}}} = 1 \qquad \text{(Paul Lévy).}$$

b) Given t,

$$\overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{\sqrt{2\,|h| \log \log \frac{1}{|h|}}} = 1$$

(law of the iterated logarithm ; Khintchine, Kolmogorov).

c) For all t

$$\overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{|h|^{1/2+\varepsilon}} = \infty \qquad (\varepsilon > 0 \quad \text{given arbitrarily})$$

(Paley-Wiener-Zygmund).

The nowhere differentiability is provided by c). A stronger result is

c') For all t

$$\overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{|h|^{1/2}} > 0 \qquad \text{(Dvoretsky 1963).}$$

Another formulation of b) is

$$\text{meas}\left\{t \,\middle|\, \overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{\sqrt{2\,|h| \log \log \frac{1}{|h|}}} \neq 1\right\} = 0.$$

Surprizingly, it has not been known for along time whether this set is empty or not.

Actually it is not empty, because there are "rapid points" and "slow points".

The rapidity is gauged by a), nevertheless

$$\dim\left\{t \,\Big|\, \overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{\sqrt{|h| \log \frac{1}{|h|}}} > 0\right\} = 1$$

(S. Orey and S. J. Taylor 1973), where dim is the Hausdorff dimension.

The slowness is controlled by b) ; nevertheless

$$\dim\left\{t \,\Big|\, \overline{\lim_{|h|\to 0}} \frac{|X(t+h) - X(t)|}{|h|^{1/2}} < \infty\right\} = 1$$

(J.-P. Kahane 1974), therefore, Dvoretsky's result is best possible.

If we compare these results with the statements concerning the Weierstrass functions, we see that $X(t)$ behaves grosso modo like $\sum 2^{-n} \cos 4^{-n} t$, but instead of being very regularly irregular, it is rather irregularly irregular. Anyhow the Brownian motion provides a very natural example of a nowhere differentiable function, as Jean Perrin had guessed.

5. TOTALLY DISCONNECTED PERFECT SETS.

In the quotation of Jean Perrin given above the mathematicians are urged to study Cantor sets and to provide thereby good mathematical models for physics. Cantor sets on the line have much to do with Fourier analysis, and part of their most important properties depend on the behaviour at infinity of Fourier transforms of measures carried by these sets. Random Cantor sets are provided in a most easy way by Brownian motion, and many precise results were obtained in the last ten years thereabout. How far they can apply to physics is not clear, but the question is not irrelevant : approximate random perfect sets have been constructed by computers and compared with the distribution of Galaxies and solar systems (B. Mandelbrot). On the other hand (or rather at the other extreme) I learned in Durham that the physicist Holges Bock Nielsen had just defined the dimension of the electron (1.3).

The Hausdorff dimension of a closed set E in R^n can be defined in several ways. One of them (Frostman) is the following. Let $h(t)$ be an increasing function of $t > 0$, either convex or concave. The Hausdorff measure of E with respect to h is strictly positive (maybe infinite) if and only if E carries a positive measure $\mu \not\equiv 0$ such that, for each ball B of diameter $|B|$, $\mu(B) \leq h(|B|)$. The Hausdorff dimension of E is the supremum of the $\alpha \geq 0$ such that $h(t) = t^\alpha$ has this property. With this definition it is easy to check or to compute the Hausdorff dimension of a sphere, a cube, a Van Koch curve or the classical triadic Cantor set.

The Peano curve (described above) enjoys a nice property with respect to the Hausdorff dimension. Let E be a closed set on $[0,1]$, and $x = f(t)$ ($t \in [0,1]$, $x \in \mathbb{R}^2$) the natural parametric equation of the curve. Then $\dim f(E) = 2 \dim E$. This is also a property of the n-dimensional brownian motion $X(t)$ (meaning that the coordinates $X_1(t), \ldots, X_n(t)$ are independent real brownian motions). First, given $E \subset \mathbb{R}^+$, $\dim X(E) = 2 \dim E$ a. s.. Secondly (and this is much more difficult) it is almost sure that $\dim X(E) = 2 \dim E$ <u>for all</u> closed sets $E \subset \mathbb{R}^+$ (R. Kaufman 1969).

An alternative definition of the Hausdorff dimension of $E \subset \mathbb{R}^n$ is the supremum of the $\alpha \in [0,n]$ such that, for some measure $\mu \neq 0$ carried by E,

$$\int_{u \in \mathbb{R}^n,\ |u| \geq 1} |\hat{\mu}(u)|^2 |u|^{\alpha - n}\, du < \infty \tag{8}$$

where $\hat{\mu}(u)$ is the Fourier transform of μ. (8) does not imply that $\hat{\mu}(u)$ tends to 0 as $|u| \to \infty$. Actually this is false when E is the boundary of a cube in $\mathbb{R}^n$

(dim E = n-1) or when E is the triadic Cantor set on the line $(\dim E = \frac{\log 2}{\log 3})$. On the other hand, if

$$\hat{\mu}(u) = 0(|u|^{-\beta/2}) \qquad (|u| \to \infty) \tag{9}$$

and $\alpha < \beta$, then (8) holds. The supremum of the β such that (9) holds is also a kind of dimension of E ; we call it $\mathfrak{F}$-dimension. In any case $\mathfrak{F}\text{-}\dim E \leq \dim E$. If E is a sphere in R^n, $\mathfrak{F}\text{-}\dim E = \dim E = n-1$.

Strikingly no simple construction is known of any closed set E on the line R for which $0 < \mathfrak{F}\text{-}\dim E = \dim E < 1$. Such sets were exhibited by Salem using both probabilistic and number theoretic arguments. Actually this property ($\mathfrak{F}$-dim = dim) looks unusual for nice geometric linear sets, like the triadic Cantor set $(\mathfrak{F}\text{-}\dim E = 0)$, but it is quite standard for random sets. The zero-set of one dimensional brownian motion satisfies $\mathfrak{F}\text{-}\dim X^{-1}(0) = \dim X^{-1}(0) = \frac{1}{2}$ a. s.. Given <u>any</u> closed linear set E, its image by the n-dimensional Brownian motion satisfies $\mathfrak{F}\text{-}\dim X(E) = \dim X(E) = 2 \dim E$, a. s.

More precisely, if h is a concave Hausdorff function and if $E \subset \mathbf{R}^+$ has a strictly positive measure with respect to h, the image set $X(E) \subset \mathbf{R}^n$ carries a. s. a measure μ whose Fourier transform satisfies

$$\hat{\mu}(u) = O(\sqrt{\log |u| h(|u|^{-2}}) \qquad (|u| \to \infty). \tag{10}$$

If $\dim E = 0$, it is not difficult to see that X(E) is linearly independent over the rationals, and (10) shows that it can carry a measure whose Fourier transform tends to z zero at infinity rather fast (the first example of this kind was given by Rudin). If E is still smaller, in the sense that, for some sequence $\varepsilon_n \to 0$, E is covered by

$O(\log \frac{1}{\varepsilon_n})$ intervals of length ε_n, then no transform of E by any function satisfying a Lipschitz condition of order $\alpha > 0$ carries a measure $\mu \neq 0$ with $\lim_{|u| \to \infty} |\hat{\mu}(u)| = 0$. This is sharp because of (10) : if we replace $O(\log \frac{1}{\varepsilon})$ by, say, $O(\log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon})$, we can build E in order to have $\hat{\mu}(u) = o(1)$. Conversely, (10) is sharp : it is not possible to replace $\log |u|$ by a smaller function.

If E is totally disconnected on $\mathbf{R}^+$, and X is the n-dimensional Brownian motion with $\underline{n \geq 2}$, X(E) is also totally disconnected. If $\dim E < \frac{1}{2}$, we have $\dim X(E) < 1$, therefore the result holds also for $n = 1$. A brillant result of Fourier methods in the study of Brownian motion is that, whenever $\dim E > \frac{1}{2}$ and $n = 1$, X(E) contains an interval (R. Kaufman 1974).

6. ANALYTIC FUNCTIONS.

If we transform a Brownian path in the plane (see Perrin's figures) by a conformal mapping, we obtain a Brownian path (with a change of time). This is intuitive, due to the isotropic property of the 2-dimensional Brownian motion X(t), and it is a theorem of Paul Lévy. If we identify $\mathbf{R}^2$ and $\mathbf{C}$, the plane of the complex variable, and if f(z) is an analytic function in a neighbourhood of 0, with $f(0) = 0$ and $f \not\equiv 0$, then $f(X(t)) = X'(T(t))$ where X' is another version of the 2-dimensional Brownian motion, and T(t) is an increasing random function. Actually the proof is not obvious, and it is a motivation for stochastic integrals, since f(X(t)) can be written as $\int_0^t f'(X(s))dX(s)$. For derivation and applications, the reader can consult the book of McKean.

Starting from this theorem, Burgess Davis gave new and exciting proofs of old results. Let us sketch two such proofs (1973).

1. A theorem of E. Stein and G. Weiss. Let E be a Borel set on the circle $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, χ its indicator function ($= 1$ on E and 0 outside), $\tilde{\chi}$ the conjugate (Hilbert transform) of χ with mean value 0. Then the distribution of $\tilde{\chi}$ depends only on $|E|$ (Lebesgue measure of E).

Proof. Let $F(z)$ be the analytic function in $|z| < 1$ whose boundary values are $F(e^{2\pi i\theta}) = \chi(\theta) + i\tilde{\chi}(\theta)$. When $X(t)$ moves in the disc $|z| < 1$, $F(X(t))$ moves in the strip $0 < \operatorname{Re} Z < 1$, starting from $F(0) = |E|$. Let τ be the time when $X(t)$ reaches the circle $|z| = 1$, and let I be any real interval. The event $\operatorname{Im} F(X(\tau)) \in I$ has a probability which depends only on $|E|$ and I, namely, the probability that a Brownian motion starting from $|E|$ hits the boundary of the strip $0 < \operatorname{Re} Z < 1$ at a height $Y \in I$. Writing $X(\tau) = e^{2\pi i\theta}$, we have $\operatorname{Im} F(X(\tau)) = \tilde{\chi}(\theta)$, and the probability of the event $\tilde{\chi}(\theta) \in I$ is the Lebesgue measure of $\tilde{\chi}^{-1}(I)$. Therefore the Lebesgue measure of $\tilde{\chi}^{-1}(I)$ depends only on $|E|$ and I, QED.

2. The Picard theorem. A non constant entire function takes every complex value with at most one exception.

The proof relies on two topological properties of Brownian paths. The first is classical ; the second, though intuitive, is not so easy to prove by combinatorial methods.

The first property is : given any complex number z,

$$\Pr(\exists t > 0 \quad X(t) = z) = 0 \quad , \quad \Pr(\underline{\lim}_{t\to\infty} |X(t) - z| = 0) = 1.$$

In words, the Brownian path never contains a given point (except 0), and it is dense in $\mathbf{C}$.

Now let $F(z)$ be a non zero entire function, such that $F(0) = 0$ and $F(C)$ does not contain two points a et b. Let W be a compact neighbourhood of 0, and $\theta_j \to \infty$ a sequence of stopping times such that $X(\theta_j) \in W$. Then $F(X(\theta_j)) \in V = F(W)$, and we consider both $X(t)$ $(0 \le t \le \theta_j)$ and $F(X(t))$ $(0 \le t \le \theta_j)$ as closed curves. The first, being a closed curve, is homotopic to 0 in $\mathbb{C}$. Therefore the second, being its image by F, is homotopic to 0 in $F(\mathbb{C})$, therefore in $\mathbb{C} \setminus \{a,b\}$. This contradicts the second property for large j. Therefore the assumption is false, which proves Picard's theorem.

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