N° d'ordre :

UNIVERSITÉ PARIS-SUD XI
FACULTÉ DES SCIENCES D'ORSAY

# THÈSE

Présentée pour obtenir

LE GRADE DE DOCTEUR EN SCIENCES
DE L'UNIVERSITÉ PARIS-SUD XI

Spécialité : Mathématiques

par

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# PROCESSUS SLE ET SENSIBILITÉ AUX PERTURBATIONS DE LA PERCOLATION CRITIQUE PLANE

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Soutenue le 5 Décembre 2008 devant la commission d'examen :

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# Remerciements

Je souhaite avant tout remercier Wendelin Werner pour son attention constante, ses conseils précieux et sa disponibilité. Grâce a lui, à l'environnement stimulant qu'il a su créer, aux rencontres et aux voyages qu'il m'a permis de faire, cette thèse aura été à l'opposé d'un long travail solitaire. Je lui suis aussi particulièrement reconnaissant de m'avoir laissé une grande liberté par rapport aux sujets que j'ai pu aborder, tout en restant continuellement à l'écoute. Je garderai un excellent souvenir de ces années de thèse.

Je voudrais aussi adresser une pensée toute particulière à Oded Schramm sans qui cette thèse ne serait pas ce qu'elle est. J'ai appris énormément à son contact au cours des mois que j'ai passés à Seattle et j'espère m'être imprégné de cette intuition qui le caractérisait. Je le remercie pour tout ce qu'il m'a apporté à la fois scientifiquement et personnellement.

Un certain nombre de chercheurs avec qui j'ai longuement discuté ou travaillé ont contribué à ce travail, je tiens à citer en particulier Vincent Beffara, Itai Benjamini (pour son enthousiasme, sa gentillesse, et les "problèmes ouverts" dont il n'est jamais à court !), Greg Lawler (qui m'a accueilli à Cornell et m'a enseigné les processus SLE), Pierre Nolin (avec qui j'ai eu le plaisir de partager mon bureau pendant mes années de thèse), Steffen Rohde (pour ses explications particulièrement intuitives de résultats d'analyse complexe), Vladas Sidoravicius, Stanislav Smirnov, Jeff Steif et Boris Tsirelson (dont la théorie des bruits noirs a beaucoup influencé cette thèse).

J'ai eu la chance de faire mes premiers pas en recherche avec deux "ainés" qui ont eu la patience entre autres de m'apprendre à mettre un article en forme : Gábor Pete avec qui j'ai passé des moments mémorables à Seattle et José Trujillo Ferreras dont l'humour a égayé plus d'une soirée à Ithaca.

Chuck Newman et Jeff Steif ont accepté la lourde tâche d'être rapporteurs de cette thèse et je les en remercie. J'ai pu constater par leurs questions et remarques au cours de discussions récentes qu'ils s'étaient acquitté de ce travail avec coeur. Je tiens également à remercier Itai Benjamini, Raphaël Cerf, Jean-François Le Gall, Yves le Jan et Stanislav Smirnov pour avoir accepté de faire partie de mon Jury. Ils ont tous d'une manière ou d'une autre contribué à ma formation.

Merci à toute l'équipe du DMA où j'ai effectué l'essentiel de cette thèse, en particulier à Amandine, Gilles, Grégory, Marie, Mathilde, Mikaël, Nicolas, Philippe, Sébastien, Thierry, Viviane ainsi que Bénédicte et Zaïna.

Je dois beaucoup à mes amis mathématiciens Elie Aïdekon, Nathanaël Berestycki, Guillaume Chapuy, Mathieu Cossutta, Sylvain Ervedoza, Ori Gurel-Gurevich, Alan Hammond, Mathieu Merle (qui a été le premier à m'expliquer le Mouvement Brownien!), Serban Nacu, Ron Peled, Guillaume Pouchin, Simon Riche (dont l'aide a été précieuse pour la mise en forme de ce mémoire), Léonardo Rolla, Benjamin Schraen; ainsi qu'à tous les autres, non-mathématiciens, qui se reconnaîtront...

Je tiens à remercier ma famille en général qui m'a toujours soutenue, et tout particulièrement ma mère et ma soeur, Claire, à qui je dois tellement.

Enfin je remercie Laure qui m'apporte tant.

*à mon père, à ma mère*

# Contents

# Chapter I

# Introduction (français)

## 1 Contexte et résultats

Dans cette thèse, nous étudions certaines propriétés concernant la percolation critique plane ainsi que les processus SLE. Nous commencerons dans cette partie par introduire ces modèles. Nous motiverons la définition et l'utilité de ces processus SLE à travers l'exemple de la percolation critique. Il existe de nombreux livres ou "surveys" sur le sujet, nous opterons donc dans cette partie pour une présentation concise.

### 1.1 Modèle de la percolation et transition de phase

La percolation est l'un des modèles les plus simples qui possède une transition de phase. Considérons tout d'abord le cas du réseau $\mathbb{Z}^d, d \geq 2$; soit $\mathbb{E}^d$, l'ensemble des arêtes de $\mathbb{Z}^d$. Si $p \in [0,1]$, on définit un sous-graphe aléatoire de $\mathbb{Z}^d$ de la manière suivante : indépendamment pour chaque arête $e \in \mathbb{E}^d$, on garde cette arête avec probabilité $p$ et on la retire avec probabilité $1-p$. De manière équivalente, cela revient à définir une configuration aléatoire $\omega \in \{0,1\}^{\mathbb{E}^d}$ où, indépendamment pour chaque arête $e \in \mathbb{E}^d$, on déclare cette arête ouverte ($\omega(e) = 1$) avec probabilité $p$ ou fermée ($\omega(e) = 0$) avec probabilité $1-p$. On notera $\mathbb{P}_p$ la loi de ce sous-graphe (ou configuration) aléatoire. En théorie de la percolation, on s'intéresse aux propriétés de connectivité à grande échelle (ou échelle macroscopique) de la configuration aléatoire $\omega$. Si $x, y \in \mathbb{Z}^d$ sont deux points, on note $\{x \leftrightarrow y\}$, l'événement où il existe un chemin ouvert dans $\omega$ reliant $x$ et $y$; en particulier $\{x \leftrightarrow y\}$

désigne l'événement où le point $x$ est connecté à l'infini (cela signifie que la composante connexe du point $x$ dans $\omega$ est infinie).

La *transition de phase* peut être décrite de la façon suivante : pour tout $d \geq 2$, il existe une probabilité critique $0 < p_c(\mathbb{Z}^d) < 1$ telle que si $p < p_c(\mathbb{Z})$, alors presque sûrement toutes les composantes connexes sont finies, mais si $p > p_c(\mathbb{Z}^d)$, alors presque sûrement il existe une **unique** composante connexe infinie.

La *fonction densité* $\theta_{\mathbb{Z}^d}(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$ fournit des informations importantes conçernant les propriétés à grande échelle de la configuration aléatoire $\omega$. Elle correspond à la densité d'occupation (en moyenne sur l'espace $\mathbb{Z}^d$) de la composante connexe infinie. La transition de phase signifie en terme de fonction densité que $\theta_{\mathbb{Z}^d}(p) = 0$ si $p < p_c(\mathbb{Z}^d)$, alors que $\theta_{\mathbb{Z}^d}(p) > 0$ si $p > p_c(\mathbb{Z}^d)$. Que ce passe-t-il exactement au point de transition $p_c(\mathbb{Z}^d)$ ? Est ce qu'il existe presque sûrement une composante connexe infinie à $p = p_c(\mathbb{Z}^d)$ ou non ? Il se trouve que c'est une question difficile en général. La "continuité" de la transition de phase (caractéristique des transitions dites de second-ordre) est connue pour $d = 2$ ainsi qu'en grande dimension ($d \geq 19$), mais par exemple c'est un problème ouvert de savoir si $\theta_{\mathbb{Z}^3}(p_c(\mathbb{Z}^3))$ est égal à zéro ou non. Pour plus de détails sur la percolation dans $\mathbb{Z}^d$, on renvoie le lecteur vers [Gri99]. Nous nous concentrerons désormais sur la percolation plane, en particulier au niveau du point critique.

## 1.2 Percolation planaire, invariance conforme et processus SLE

La théorie de la percolation critique plane a connu de rapides progrès au cours des dix dernières années, en particulier grâce à la preuve de Smirnov de l'invariance conforme de la percolation sur réseau triangulaire, ainsi que la découverte par Schramm des processus SLE. Il est conjecturé que la limite d'échelle de la percolation critique sur $\mathbb{Z}^2$ est également invariante conforme. L'invariance conforme supposée de ces systèmes a permis aux physiciens théoriciens de prédire, a l'aide de la *théorie des champs conformes*, de nombreuses probabilités asymptotiques pour la percolation critique. Par exemple ils ont pu prédire les valeurs des exposants critiques de la percolation qui décrivent en quelque sorte les propriétés fractales des grandes composantes connexes etc..

Même si l'invariance conforme de la percolation sur $\mathbb{Z}^2$ n'est à ce jour pas

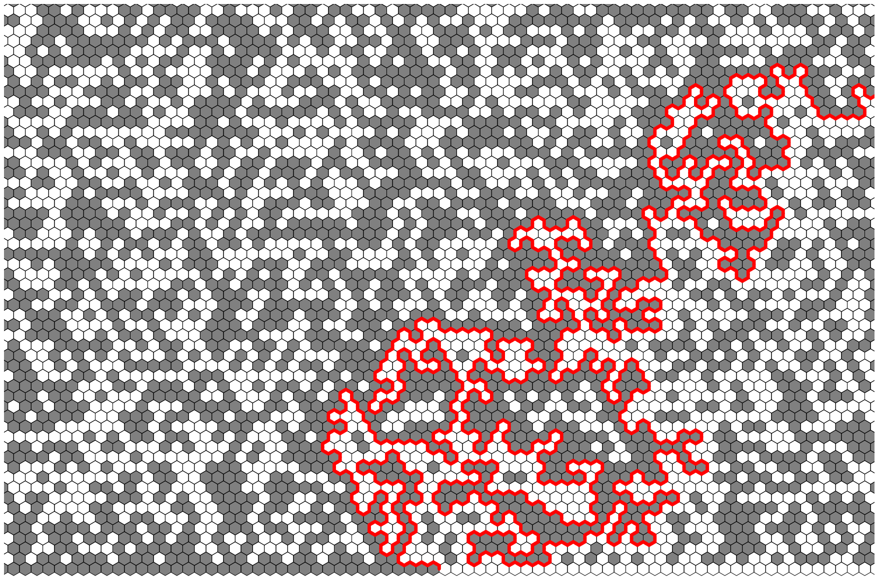


Figure 1.1: Le processus d'exploration dans le demi-plan supérieur.

démontrée, Stanislav Smirnov a prouvé dans [Smi01] qu'elle avait lieu (du moins asymptotiquement) pour le réseau triangulaire $\mathbb{T}$. Plus précisément, il a prouvé qu'une grande famille d'événements de type "croisement" sont asymptotiquement invariants par transformation conforme. Une conséquence de cette preuve est l'obtention de la formule de Cardy pour la probabilité asymptotique de traverser un rectangle.

Nous introduisons donc à présent ce modèle de percolation par site sur le réseau triangulaire. Il est défini de façon similaire : pour tout $p \in [0,1]$, indépendamment pour chaque site $x$ dans le réseau triangulaire $\mathbb{T}$, on déclare le site ouvert (représenté en noir sur les images) avec probabilité $p$ et fermé (blanc) avec probabilité $1-p$. Comme sur $\mathbb{Z}^2$, il y a une probabilité critique $p_c(\mathbb{T})$, telle que si $p \le p_c(\mathbb{T})$ alors presque sûrement toutes les composantes connexes de sites ouverts sont finies, alors que pour $p > p_c(\mathbb{T})$, il existe presque sûrement une unique composante connexe infinie (de sites ouverts). Un célèbre théorème dû à Harry Kesten affirme que $p_c(\mathbb{T}) = p_c(\mathbb{Z}^2) = \frac{1}{2}$. Le graphe triangulaire est intimement relié à son graphe dual, le graphe hexagonal. C'est commode (esthétiquement du moins) de représenter les configurations de percolation par site sur $\mathbb{T}$ à l'aide du graphe hexagonal, voir figure 1.1

Avant la preuve de Smirnov (en 2001), Oded Schramm avait identifié en 1999 quelles devraient être, en supposant que l'invariance conforme a effectivement lieu, les courbes qui décrivent le bord des composantes connexes "macroscopiques" à la limite continue. Ça l'a conduit à définir les fameux processus SLE, où SLE signifie *Stochastic-Loewner-Evolution* ou *Schramm-Loewner-Evolution*. Plutôt que de considérer tous les bords des composantes connexes en même temps, Schramm a eu l'idée d'en considérer un en particulier : le *processus d'exploration* dans le demi-plan $\overline{\mathbb{H}}$ (voir figure 1.1 dans le cas du réseau triangulaire), qui se trouve entre les composantes connexes ouvertes attachées à la demi-droite $\mathbb{R}_-$ et les composantes connexes fermées attachées à la demi-droite $\mathbb{R}_+$. Le processus d'exploration peut être réalisé de manière inductive en découvrant le statut des sites un par un.

Charles Loewner a élaboré dans les années vingt une façon de représenter des courbes dans le plan afin de résoudre la conjecture de Bieberbach sur la croissance des coefficients des fonctions univalentes. Sa théorie lui a permis de contrôler la taille du troisième coefficient (les deux premiers coefficients peuvent être contrôlés a l'aide de techniques usuelles en analyse complexe). Il se trouve que bien des années plus tard, la preuve par De Branges de la conjecture de Bieberbach (1985) elle aussi repose sur les chaînes de Loewner. Appliqué à notre cadre de la percolation, on peut considérer le processus d'exploration ci-dessus comme une courbe simple $\gamma : [0, \infty) \to \overline{\mathbb{H}}$, avec une paramétrisation quelconque. Pour tout $t \geq 0$, $H_t := \mathbb{H} \backslash \gamma[0, t]$ est un domaine simplement connexe, par conséquent en utilisant le théorème de représentation conforme de Riemann, il existe une application conforme $g_t$ de $H_t$ vers $\mathbb{H}$. Il y a trois degrés de liberté pour le choix de $g_t$; on fixe donc $g_t(\infty) = \infty$ et $g_t(z) = z + o(1)$, quand $z$ tend vers l'infini. Il est facile de vérifier que cela détermine de façon unique l'application conforme $g_t$. Maintenant si l'on développe $g_t$ au voisinage de l'infini on trouve

$$g_t(z) = z + \frac{a_t}{z} + O(\frac{1}{z^2}),$$

où $t \mapsto a_t$ est une fonction (réelle) croissante. Si on reparamétrise la courbe $\gamma$ de telle façon que $a_t = 2t$ (ce que l'on peut toujours faire), alors le Théorème de Loewner affirme que les applications conformes $(g_t)_{t \geq 0}$ vérifient l'équation différentielle suivante

$$\begin{cases} g_0(z) & = & z & \forall z \in \mathbb{H}\,, \\ \frac{\partial}{\partial t} g_t(z) & = & \frac{2}{g_t(z) - \beta(t)} & \text{if } t < T(z)\,, \end{cases}$$

où $t \mapsto \beta(t)$ est appelée la *fonction directrice* de la courbe $\gamma$, et $T(z)$ est le "temps d'explosion", c.a.d. le temps à partir duquel, lorsque l'on suit la trajectoire $t \mapsto g_t(z)$, l'équation différentielle n'est plus définie (a posteriori, la courbe $\gamma(0,t]$ est l'ensemble des points $z$ tels que $T(z) \le t$). Ainsi la courbe $\gamma$ est déterminée par sa fonction directrice $\beta$: en effet, pour reconstruire $\gamma$ à partir de $t \mapsto \beta(t)$, il suffit de résoudre l'équation différentielle ci-dessus.

Considérons à présent le processus d'exploration sur un réseau triangulaire de très petite maille ("mesh") $\epsilon\mathbb{T}$. Cela correspond à une certaine courbe aléatoire $\gamma^\epsilon : [0,\infty] \to \overline{\mathbb{H}}$, que l'on peut paramétriser de telle façon que la famille des applications conformes $(g_t)$ qui lui est associée vérifie la normalisation ci-dessus ($a_t = 2t$). Ce processus d'exploration $\gamma^\epsilon$ est donc "dirigé" par un certain processus aléatoire (réel) $\beta^\epsilon(t)$. Supposons que l'on arrête l'exploration à un certain temps $t > 0$ (c.a.d on a découvert les sites un par un jusqu'à l'obtention de la courbe $\gamma^\epsilon[0,t]$). L'observation cruciale est que ce qu'il reste à découvrir dans $\mathbb{H}\backslash\gamma^\epsilon$ suit toujours la loi de la percolation critique i.i.d. En particulier, si on suppose que l'invariance conforme a lieu à la limite continue, alors on peut "renvoyer" dans le demi-plan $\mathbb{H}$, la configuration de percolation qu'il reste à découvrir dans $\mathbb{H} \setminus \gamma^\epsilon$, ceci grâce à l'application conforme $g_t$. Heuristiquement, l'invariance conforme affirme que si la maille $\epsilon$ du réseau est petite, alors le processus d'exploration dans le réseau "déformé" (image par $g_t$ du réseau $\epsilon\mathbb{T}$) "ressemble" beaucoup au processus d'exploration dans le réseau d'origine (lui aussi de très petite maille). Autrement dit l'image $g_t(\gamma^\epsilon((t,\infty]))$ est proche en loi du processus d'exploration $\gamma^\epsilon$.

Il n'est pas difficile de vérifier que cela se traduit de la façon suivante en termes de fonction directrice : quand la maille $\epsilon$ tend vers 0, pour tout $t > 0$, la loi de $(\beta^\epsilon(t+u))_{u>0}$ est indépendante de la loi de $\beta^\epsilon([0,t])$ et a même loi que $(\beta^\epsilon(t))_{t>0}$. Puisque la continuité de la fonction directrice est préservée à la limite continue, alors par le Théorème de Levy, le processus limite (quand $\epsilon$ tend vers 0) $\beta$ est nécessairement un mouvement Brownien $\sqrt{\kappa}B_t + \mu t$. Seulement, par symétrie de notre processus d'exploration (par rapport à $z \to -\bar{z}$), il est clair que $\beta(t)$ et $-\beta(t)$ ont même loi, ce qui force le drift $\mu$ de s'annuler. C'est précisément ce par quoi on désigne les processus $\mathrm{SLE}_\kappa$ : ce sont les évolutions de Loewner aléatoires "conduites" ou "dirigées" par $\sqrt{\kappa}B_t$, où $B_t$ est un mouvement Brownien standard.

Notons que l'on a un peu "triché" ici puisque nous avons expliqué la théorie de Loewner dans le cas des courbes simples du demi-plan $\mathbb{H}$, mais il se trouve que la limite continue du processus d'exploration se trouve être une loi supportée sur des courbes avec de nombreuses auto- intersections. En fait, la théorie de Loewner s'adapte au cas plus général des familles croissantes de compacts qui satisfont à une certaine condition de "croissance locale" (condition qui est satisfaite dans le cas de la percolation).

Il est en aucun cas évident (pour un paramètre $\kappa$ quelconque) de démontrer que la construction décrite ci-dessus génère effectivement une courbe aléatoire à partir d'un mouvement Brownien $\beta = \sqrt{\kappa} B_t$. Cela a été prouvé par Rohde et Schramm dans [RS05]. Dans cet article, ils prouvent que dans le demi-plan supérieur, la courbe SLE avec paramètre $\kappa$ existe presque sûrement et est continue. De plus ils montrent également que cette courbe est simple seulement si $\kappa \leq 4$, alors qu'elle a des points doubles et touche le bord du demi-plan dès que $\kappa > 4$. Notons aussi qu'un processus SLE dans un domaine simplement connexe quelconque est défini comme étant l'image du SLE dans le demi-plan par une application conforme.

En résumé, en combinant la preuve par Smirnov de l'invariance conforme sur réseau triangulaire avec la description par Schramm des processus d'exploration, on obtient que ce processus d'exploration de la percolation a une limite continue quand la maille du réseau tend vers zéro. Cette limite continue est donnée par un processus $\mathrm{SLE}_\kappa$ pour un certain paramètre $\kappa > 0$. Une fois que l'on a déterminé la limite continue comme étant un processus SLE, il n'est pas très difficile de voir que $\kappa$ doit être égal à 6; l'une des raisons étant que le $\mathrm{SLE}_6$ est la seule courbe SLE dont la loi de croissance est locale (comme c'est déjà le cas au niveau discret). Pour prendre un exemple extrême, si l'on considère le $\mathrm{SLE}_0$, cela correspond à une géodésique pour la métrique de Poincaré; si on perturbe le domaine dans lequel on définit le $\mathrm{SLE}_0$, cela affecte la métrique de Poincaré et du coup affecte la courbe; cette influence du domaine ne se ressent pas dans le cas du $\mathrm{SLE}_6$ (a part bien sûr quand le $\mathrm{SLE}_6$ touche le bord).

Une fois que l'invariance conforme de la percolation est démontrée, il reste encore certains arguments non triviaux avant de pouvoir en déduire la convergence du processus d'exploration discret vers le $\mathrm{SLE}_6$. La première preuve détaillée se trouve dans Camia et Newman [CN07]. Une autre approche a été exposée dans [Smi06] (dont les détails peuvent être trouvés dans [Wer07]).

En général, de nombreux modèles planaires issus de la physique statistique sont conjecturés être invariants conformes au niveau de la transition de phase (c.a.d au point critique). Cela a été prouvé dans un certain nombre de cas, dont les modèles suivants.

- Les modèles LERW (Loop Erased Random Walk) et UST (Uniform Spanning Tree) ont des limites continues qui correspondent respectivement au $SLE_2$ et au $SLE_8$ (et sont donc invariants conforme), voir [LSW04a].
- La frontière du Mouvement Brownien plan correspond au $SLE_{8/3}$, [LSW01b] (ce qui a impliqué en particulier la conjecture de Mandelbrot).
- Les lignes de niveau du Champ libre Gaussien discret convergent vers le $SLE_4$, [SchShe06].
- Smirnov a prouvé récemment l'invariance conforme pour le modèle d'Ising ($SLE_3$) ainsi que son modèle de FK-percolation correspondant ($SLE_{16/3}$); voir [Smi06, Smi07].

## 1.3 Exposants critiques

La convergence des interfaces de percolation vers le $SLE_6$ sur le réseau triangulaire permet de prouver l'existence de certains exposants critiques et de calculer leur valeur. Nous donnerons deux exemples : l'exposant à un bras et l'exposant à quatre bras. Pour tout $R > 1$, soit $A_R^1$ l'événement où le site 0 est connecté à distance $R$ par un chemin de sites ouverts (ou chemin ouvert). On note également $A_R^4$ l'événement où il y a quatre "bras" (ou chemins) de statut (ou couleur) alternés qui partent du site 0 (de couleur quelconque) et qui vont jusqu'à distance $R$ de l'origine (autrement dit, on peut trouver quatre chemins, deux fermés, deux ouverts qui vont de 0 jusqu'à distance $R$ et les chemins fermés se trouvent entre les chemins ouverts). La figure 1.3 représente deux configurations de percolation satisfaisant respectivement les événements $A_R^1$ et $A_R^4$.

Il a été prouvé dans [LSW02] que la probabilité de l'événement à un bras décroît de la façon suivante

$$\mathbb{P}\big[A_R^1\big] := \alpha_1(R) = R^{-\frac{5}{48}+o(1)}\,,$$

où $\frac{5}{48}$ est ce que l'on appelle un *exposant critique*.

Pour l'événement à quatre bras, Smirnov et Werner ont prouvé dans [SW01] que sa probabilité décroît comme

$$\mathbb{P}\big[A_R^4\big] := \alpha_4(R) = R^{-\frac{5}{4}+o(1)}\,.$$

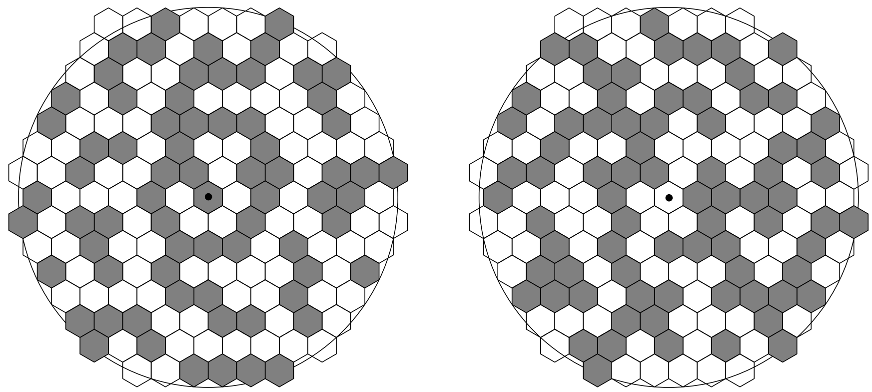


Figure 1.2: La configuration à gauche satisfait l'événement à un bras, celle de droite satisfait l'événement à quatre bras

L'événement à quatre bras se révélera être d'importance capitale dans tous les résultats qui concernent la percolation dans cette thèse. En effet supposons que l'événement à quatre bras est satisfait pour un certain site $x \in \mathbb{T}$ jusqu'à une distance $R$. Cela signifie que l'information provenant du seul site $x$ est importante pour ce qui concerne les connections à grande échelle dans la boule $B(x,R)$. En changeant le statut de $x$, on affecte radicalement "l'image" que l'on voit dans $B(x,R)$. Un tel point sera appelé **point pivot** jusqu'à distance $R$.

En utilisant les relations de "scaling" dues à Kesten [Kes87], la détermination de ces deux exposants critiques implique (voir[Wer07, Nol07]) le comportement suivant pour la fonction densité $\theta(p)$ sur le réseau triangulaire au voisinage de $p_c = 1/2$

$$\theta(p) = (p - 1/2)^{5/36+o(1)},$$

quand $p \to 1/2+$. Cela fait partie de la description de la percolation *presque-critique*.

Comme nous l'avons mentionné ci-dessus, les exposants critiques fournissent des informations sur les propriétés fractales de la percolation à la limite continue. Par exemple si l'on considère l'exposant à un bras, cela signifie qu'en moyenne on trouve $R^{91/48+o(1)}$ sites dans le carré $[-R,R]^2$ qui appartiennent à une composante connexe de diamètre plus grand que $R$. Puisque il y a seulement un nombre "fini" de tels composantes macroscopiques, cela signifie que à la limite continue, les composantes connexes de percolation sont des compacts aléatoires dont la dimension fractale est p.s. $\frac{91}{48}$ (ce qui peut être prouvé rigoureusement).

Une des difficultés au niveau discret provient du fait que les probabilités ci-dessus ne sont connues qu'au niveau de l'exposant (c.a.d. ce sont des équivalents logarithmiques). Par exemple on ne sait pas si $\alpha_1(R)/R^{-5/48}$ reste bornée ou pas.

On a défini ces événements pour la percolation critique sur réseau triangulaire, mais on peut les définir de la même façon sur $\mathbb{Z}^2$; par exemple nous utiliserons souvent la probabilité $\alpha_4(R)$ dans le contexte du graphe $\mathbb{Z}^2$. Un certain nombre de propriétés sont connues sur les probabilités de ces événements rares; par exemple on sait qu'il existe des constantes $1 < \alpha < \beta < 2$, telles que lorsque $R$ est suffisamment grand

$$R^{-\beta} < \alpha_4(R) < R^{-\alpha}.$$

Toutefois, ne serait-ce que l'existence des exposants critiques pour $\mathbb{Z}^2$ demeure encore ouverte à ce jour.

## 1.4 Aperçu des résultats

La partie principale de cette thèse est constituée de quatre chapitres indépendants :

- L'aire moyenne de la boucle Brownienne planaire. Dans ce premier chapitre, on montre que l'aire moyenne comprise à l'intérieur d'une boucle Brownienne planaire de temps un est égale à $\frac{\pi}{5}$. Afin de déterminer cette aire moyenne, on utilise de façon essentielle le $\mathrm{SLE}_{8/3}$ qui a la propriété de décrire le "bord du mouvement Brownien". C'est un exemple de problème où il semble que l'on doit utiliser les processus SLE afin de déterminer des quantités concernant le mouvement Brownien qui semblent hors de portée des techniques standards de calcul stochastique. Cette valeur de $\frac{\pi}{5}$ a des conséquences sur les propriétés fractales du modèle des "soupes Browniennes" (ou Brownian Loop Soups) introduites dans [LW04].

- Dans le second chapitre, on démontre un analogue du théorème de Makarov (concernant la mesure harmonique) pour les processus $\mathrm{SLE}_\kappa$. Autrement dit, on étudie en quelque sorte quelle est la "taille" possible de l'ensemble $\partial D \cap \gamma$ pour un SLE dans un domaine quelconque $D$. On montre également que pour tout $\kappa \in [0, 8)$, les courbes $\mathrm{SLE}_\kappa$ dans un domaine (simplement connexe) quelconque sont continues. Ce résultat était connu pour $\kappa \leq 4$ mais ne l'était pas pour $4 < \kappa < 8$ où les $\mathrm{SLE}_\kappa$ touchent le bord du domaine; hors le bord d'un domaine simplement connexe quelconque peut être "sauvage".

- Le spectre de Fourier de la percolation critique. Dans ce troisième chapitre, on obtient des résultats optimaux sur la sensibilité au bruit de la percolation (que ce soit dans $\mathbb{T}$ ou dans $\mathbb{Z}^2$). Diverses applications de ces résultats sont déduites pour le modèle de la percolation dynamique. Ce dernier modèle correspond à une configuration de percolation qui évolue au cours du temps et où le statut de chaque site est indépendamment mis à jour à taux un (c.a.d. après des temps exponentiels de paramètres un). On montre en particulier que si $Exc$ est l'ensemble aléatoire des temps exceptionnels pour la percolation dynamique (à $p_c = 1/2$) sur réseau triangulaire où l'origine percole, alors $Exc$ a p.s. dimension $31/36$. On montre également l'existence de tels temps exceptionnels dans le cas de la percolation dynamique sur $\mathbb{Z}^2$.

- Limite continue de la percolation presque-critique et de la percolation dynamique. Ce dernier chapitre fait parti d'un projet en cours où nous comptons démontrer que ces modèles de percolation presque-critique et de percolation dynamique, une fois renormalisés convenablement, ont une limite continue (unique). On ne présente pas la preuve complète dans cette thèse, mais nous incluons deux théorèmes (intéressants en soi, indépendamment du plus large projet) qui constitueront des étapes clés dans la preuve ultérieure de la limite continue.

Ces chapitres sont tous reliés aux objets bidimensionnels invariants par transformation conforme. Les deux premiers chapitres utilisent et étudient les processus SLE. Les deux derniers ne relèvent pas directement des techniques type SLE, mais elles utilisent des résultats (par exemple les exposants critiques) qui proviennent du SLE. Soulignons que même si les chapitres 3 et 4 sont tous les deux reliés à la percolation dynamique, ils sont en fait complètement indépendants l'un de l'autre, et se focalisent sur des perspectives assez différentes.

Le reste de l'introduction est organisé de la façon suivante : tout d'abord nous décrivons les deux premiers chapitres. Ces résultats peuvent être énoncés sans nécessiter de connaissances supplémentaires. Mais avant de décrire le contenu des deux derniers chapitres, nous avons choisi de présenter, afin de donner une image plus claire des résultats, une introduction détaillée des objets mathématiques (comme le spectre de Fourier) qui sont abondamment utilisés dans le chapitre 3.

## 2 Aire moyenne de la boucle Brownienne planaire

Notre premier résultat, en collaboration avec *José Trujillo Ferreras*, concerne l'aire moyenne comprise dans une boucle Brownienne planaire de temps un. Plus précisément, soit $B_t, 0 \leq t \leq 1$ une boucle Brownienne (un mouvement Brownien dans $\mathbb{C}$ conditionné à $B_0 = B_1$). On considère le compact obtenu en remplissant tous les trous de la boucle Brownienne, c.a.d. le complémentaire de l'unique composante non-bornée de $\mathbb{C} \setminus B[0,1]$. Appelons $\mathcal{A}$ l'aire de ce compact aléatoire; dans [GT06], nous prouvons le théorème suivant

**Théorème 2.1.**

$$\mathbb{E}[\mathcal{A}] = \frac{\pi}{5}$$

Les autres moments de la variable aléatoire $\mathcal{A}$ sont pour le moment inconnus. Ce travail était motivé par les "soupes Browniennes" (Brownian Loop Soups) introduites dans [LW04]; voir aussi [Wer03, Wer05b] pour les liens avec les $CLEs$ (Conformal Loop Ensembles) qui sont les candidats naturels pour la limite continue de systèmes supposés être invariants conformes (comme Ising, Potts, etc..).

Plus précisément, une Soupe Brownienne d'intensité $c > 0$ dans un domaine simplement connexe $\Omega \neq \mathbb{C}$, est un nuage de Poisson de boucles Browniennes (enracinées et restreintes à rester dans $\Omega$) d'intensité $c\mu^{\mathrm{loop}}$, où la mesure infinie $\mu^{\mathrm{loop}}$ est définie par

$$\mu^{\mathrm{loop}} := \int_{\mathbb{C}} \int_0^\infty \frac{dt}{2\pi t^2} \mu^\sharp(z,z,t) dt dA(z).$$

Ici $\mu^\sharp(z,z,t)$ correspond à la mesure de probabilité sur les boucles Browniennes de temps $t$ enracinées en $z$. Pour une telle soupe Brownienne d'intensité $c > 0$, on considère le complémentaire (dans $\Omega$) de toutes les boucles "remplies" de la soupe. Comme il est expliqué dans [Wer05b], cet ensemble aléatoire dans $\Omega$ a la même "structure" que la percolation fractale de Mandelbrot. Par analogie avec le cas de la percolation fractale, si l'on veut évaluer

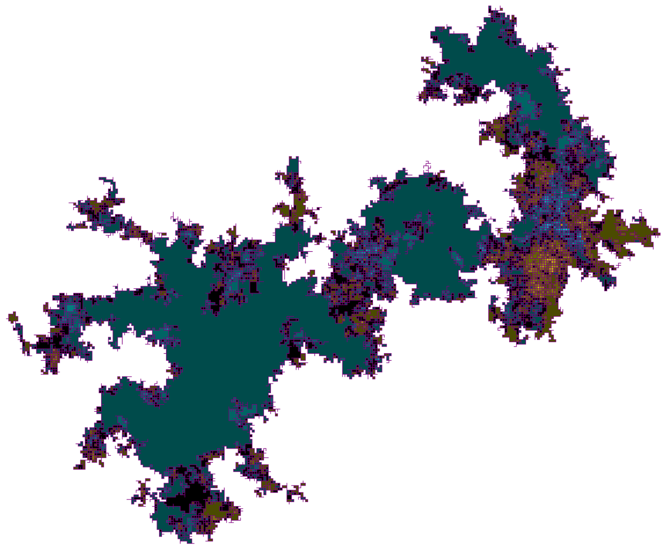


Figure 2.1: Différents indices dans une marche aléatoire de 50000 pas, les régions en noir correspondent aux régions d'indice zéro.

la dimension de Hausdorff du complémentaire de la soupe Brownienne (c.a.d l'ensemble des points qui ne sont entourés par aucune boucle), la quantité que l'on a besoin de connaître est le premier moment de la taille des boucles (à un certain niveau fixé, par exemple $t = 1$). Cette quantité est précisément celle que l'on détermine dans le théorème 2.1. En utilisant ce résultat, on peut montrer (voir [Tha06]) que cette dimension est p.s. égale à $2 - \frac{c}{5}$, où $c$ est l'intensité de la soupe Brownienne (en particulier, quand l'intensité dépasse 10, p.s. tous les points dans $\mathbb{C}$ sont entourés par au moins une boucle).

La preuve du théorème 2.1 repose sur les processus $\mathrm{SLE}_\kappa$, et plus précisément sur le processus $\mathrm{SLE}_{8/3}$, qui décrit (du moins "localement") le bord des boucles Browniennes (voir [LSW01b]). Une approche naturelle pour prouver le théorème 2.1 sans utiliser de processus SLE aurait été d'utiliser la formule de Yor ([Yor80]) donnant la loi de l'indice d'une boucle Brownienne. Soit $B_t, 0 \leq t \leq 1$ une boucle Brownienne; on définit pour tout $n \in \mathbb{Z} \setminus 0$, $\Omega_n$ comme étant l'ouvert aléatoire du plan correspondant à tous les points de $\mathbb{C}$ dont l'indice est $n$ par rapport à la boucle $B([0, 1])$. Appelons $\mathcal{W}_n$ l'aire de

$\Omega_n$, c.a.d.

$$\mathcal{W}_n = \int_{\mathbb{C}} 1_{n_z=n} dA(z),$$

où $n_z$ est l'indice de $z$ par rapport à $B([0,1])$. La formule de Yor donne la loi de l'indice $n_z$ en fonction de la position $z$. En intégrant cette loi sur le plan complexe $\mathbb{C}$, on trouve que pour tout $n \in \mathbb{Z} \setminus 0$,

$$\mathbb{E}\big[\mathcal{W}_n\big] = \int_{\mathbb{C}} \mathbb{P}\big[n_z = n\big] dA(z) = \frac{1}{2\pi n^2}.$$

Ce résultat avait déjà été obtenu dans la littérature physique ([CDO90]) à l'aide des méthodes de Gaz de Coulomb. Puisque un point $z$ d'indice $n_z \neq 0$ est nécessairement à l'intérieur de la boucle Brownienne remplie, on en déduit que $\sum_{n\neq 0} \mathcal{W}_n \leq \mathcal{A}$. Les points qui restent à comptabiliser sont les points d'indice zéro qui se trouvent à l'intérieur de la boucle Brownienne. Appelons $\mathcal{W}_0$ l'aire de l'ensemble des points d'indice zéro à l'intérieur de la boucle. Même si la formule de Yor donne la probabilité qu'un point $z$ soit d'indice $n_z = 0$, on ne peut pas "voir" si le point est à l'intérieur où à l'extérieur de la boucle Brownienne. (par exemple, un point distant de l'origine aura forte probabilité d'être d'indice nul). Puisque la preuve de la formule de Yor est basée sur des techniques de martingales qui suivent l'évolution de l'angle vu depuis le point $z$, il n'y a aucune chance d'adapter cette preuve en ajoutant de l'information géométrique du type intérieur/extérieur. C'est pourquoi il semble que les processus SLE sont ici nécessaires. Dans [CDO90], Comtet, Desbois et Ouvry (qui ont calculé les aires moyennes $\mathbb{E}\big[\mathcal{W}_n\big]$ pour $n \neq 0$ à l'aide de gaz de Coulomb) ont posé la question de déterminer quelle est l'aire moyenne des points d'indice zéro à l'intérieur de la boucle (ce que l'on a appelé $\mathbb{E}\big[\mathcal{W}_0\big]$). En combinant les résultats ci-dessus, on obtient

**Théorème 2.2.**

$$\mathbb{E}\big[\mathcal{W}_0\big] = \frac{\pi}{30}.$$

La figure 2 représente les différentes régions $\Omega_n$ colorées différemment selon leur indice $n$. Notons que si l'on voulait évaluer les moments supérieurs de $\mathcal{A}$, par exemple le second moment, l'un des ingrédients nécessaires serait de connaître la "two-point function" pour la courbe $\mathrm{SLE}_{8/3}$, c.a.d si l'on se donne deux points $z_1, z_2 \in \mathbb{H}$ : quelle est la probabilité que la courbe $\mathrm{SLE}_{8/3}$ passe à leur droite; ce qui est connu comme étant une question difficile.

# 3 Analogue du théorème de Makarov pour les processus $\mathrm{SLE}_\kappa$, et continuité des courbes SLE dans un domaine quelconque

Ce chapitre est en collaboration avec *Steffen Rohde* et *Oded Schramm*.

Le théorème de Makarov sur le support de la mesure harmonique affirme que pour n'importe quel domaine simplement connexe $\Omega \subsetneq \mathbb{C}$, il existe un ensemble $E \subset \partial\Omega$ de dimension de Hausdorff un tel que pour tout $z \in \Omega$, presque sûrement un mouvement Brownien qui part de $z$ va sortir du domaine $\Omega$ en un point de $E \subset \partial\Omega$. On considère ici la situation analogue pour les processus $\mathrm{SLE}_\kappa$. Par exemple dans le cas de $\kappa = 6$, cela peut être décrit de la façon suivante. Soit $\Omega \subsetneq \mathbb{C}$ un domaine simplement connexe et soit $z \in \Omega$. Plutôt que de démarrer un mouvement Brownien en $z$, on peut imaginer "envoyer" un cluster de "percolation continue" (c.a.d à la limite continue) en $z$; par exemple en conditionnant par l'événement de probabilité 0 que $z$ soit connecté au bord $\partial\Omega$ (il est possible de donner un sens à ce conditionnement dégénéré, voir par exemple [Kes86]). Puisque le cluster de percolation va rencontrer le bord à de nombreux endroits, on ne s'attend pas à trouver un ensemble $E$ de dimension un qui va presque sûrement "absorber" tous les points du bord qui seront connectés à $z$. Est ce que le bord entier est nécessaire pour absorber les clusters ? Nous allons montrer qu'il existe une constante absolue $1 < d < 2$ telle que pour tout domaine simplement connexe $\Omega$, il existe un ensemble $E \subset \partial\Omega$ de dimension de Hausdorff plus petite que $d$ qui presque sûrement absorbe sur le bord tous les clusters macroscopiques de percolation dans $\Omega$. Voir figure 3.1 pour une illustration.

Dans le cas général des processus $\mathrm{SLE}_\kappa$, on lance une courbe $\mathrm{SLE}_\kappa$ dans un domaine $\Omega$ (par exemple un $\mathrm{SLE}_\kappa$ radial depuis un point $z \in \Omega$ jusqu'à un prime-end de $\Omega$) et on se demande à quel point la courbe $\mathrm{SLE}_\kappa$ s'engouffre dans les fjords de $\Omega$. On prouve le résultat suivant

**Théorème 3.1.** *Soit $\Omega \subsetneq \mathbb{C}$ un domaine simplement connexe, soient $a, b$ deux prime-ends de $G$, soit $z_0 \in \Omega$ et $\kappa \in (4, 8)$. Alors il existe un ensemble Borelien $E \subset \partial\Omega$ tel que le $\mathrm{SLE}_\kappa$ chordal dans $\Omega$ de $a$ vers $b$ ainsi que le $\mathrm{SLE}_\kappa$ radial dans $\Omega$ de $a$ vers $z_0$ presque sûrement vérifient*

$$\gamma(0, \infty) \cap \partial\Omega \subset E,$$

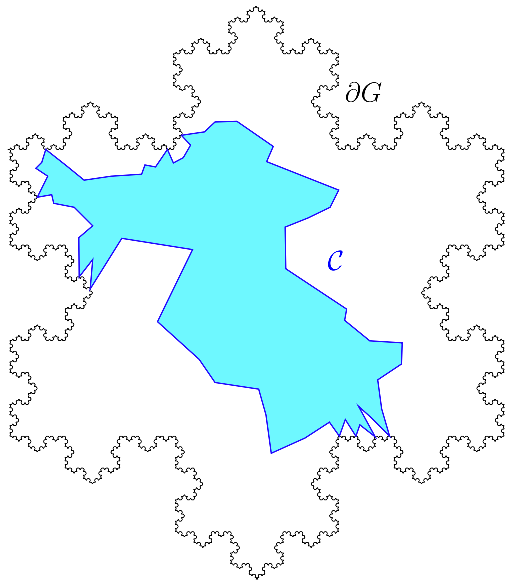



Figure 3.1: Une vue schématique d'un cluster de percolation $\mathcal{C}$ (ou bien "l'enveloppe" d'un $\mathrm{SLE}_6$) à l'intérieur d'un domaine fractal $\Omega$; la courbe bleue représente le bord extérieur du cluster.

*et*

$$\dim E \leq d(\kappa) < 2\,,$$

*où* $d(\kappa)$ *est une constante qui ne dépend que de* $\kappa$*.*

On montre également que le théorème ne peut pas être vérifié pour $d(\kappa) = 1$. De plus on obtient certaines estimées explicites sur la dimension $d(\kappa)$; en particulier on obtient que $\lim_{\kappa\to 4} d(\kappa) = 1$.

Les techniques utilisées pour démontrer ce résultat nous permettent de répondre à une question reliée qui concerne les processus $\mathrm{SLE}_\kappa$ : les courbes $\mathrm{SLE}_\kappa$ sont-elles continues dans n'importe quel domaine ? Plus précisément, soit $\Omega \subsetneq \mathbb{C}$ un domaine quelconque et soient $a, b$ deux prime-ends de $\Omega$. Soit $f : \mathbb{H} \to \Omega$ une application conforme qui envoie 0 sur le prime-end $a$ et $\infty$ sur le prime-end $b$. Le $\mathrm{SLE}_\kappa$ dans $\Omega$ est défini comme étant l'image par $f$ du $\mathrm{SLE}_\kappa$ dans $\mathbb{H}$. Sans restrictions sur le domaine $\Omega$, on ne peut pas prolonger $f$ par continuité sur $\overline{\mathbb{H}}$. Vu que pour $\kappa > 4$, le $\mathrm{SLE}_\kappa$ dans $\mathbb{H}$ touche le bord sur un ensemble de type Cantor, pour prouver que son image dans $\Omega$ est encore une courbe continue, on doit montrer qu'en quelque sorte les courbes SLE dans $\mathbb{H}$ évitent les points du bord où l'application conforme $f$ "explose" (c'est une image naïve car il existe des domaines pour lesquels l'application conforme $f : \mathbb{H} \to \Omega$ ne peut être prolongée nulle part sur le bord). On montre le théorème suivant

**Théorème 3.2.** *Soit* $\Omega \subsetneq \mathbb{C}$ *un domaine simplement connexe, soient* $a, b$ *deux prime-ends de* $\Omega$*, soit* $z_0 \in \Omega$ *et* $\kappa \in [0, 8)$*. Alors le* $\mathrm{SLE}_\kappa$ *chordal de* $a$ *vers* $b$ *dans* $\Omega$ *et le* $\mathrm{SLE}_\kappa$ *radial de* $a$ *vers* $z_0$ *dans* $\Omega$ *sont presque sûrement des courbes continues sur* $(0, \infty)$*.*

Bien évidemment, ce résultat était déjà connu pour $0 \leq \kappa \leq 4$, paramètres pour lesquels les $\mathrm{SLE}_\kappa$ sont des courbes simples qui ne rencontrent pas le bord.

Ces résultats qui concernent les propriétés générales des processus $\mathrm{SLE}_\kappa$ ont été en partie motivés par la situation suivante. Schramm et Smirnov montrent dans [SS] que la limite continue de la percolation peut être vue comme un bruit noir bidimensionnel au sens de Tsirelson (voir [Tsi04]). Etre un bruit signifie que si $A$ et $B$ sont deux ensembles ouverts lisses, alors toute l'information sur les connections de la percolation continue dans $A$ ($\mathcal{F}_A$) plus toute l'information sur les connections de la percolation continue dans $B$ ($\mathcal{F}_B$) suffisent à reconstruire toutes les connections dans $\overline{A \cup B}$. Cela veut dire que la filtration du processus de percolation (à la limite continue) est "factorisable". Il se trouve ([Tsi04]) que les bruits noirs se factorisent moins bien que les bruits blancs. Dans ce contexte particulier de la percolation, on peut illustrer cette baisse de factorisabilité en se demandant quelle est la situation pour des ouverts $A$ et $B$ de régularité quelconque. Si on désirait "recoller" l'information provenant de $\mathcal{F}_A$ et $\mathcal{F}_B$ "cluster par cluster", on aurait besoin de savoir à quel point les clusters de percolation pénètrent dans les fjords du domaine $A$ (et $B$), ce qui est relié au théorème 3.1 ci-dessus. Plus précisément il y a un résultat sur la mesure harmonique dû à Bishop, Carleson, Garnett et Jones ([BCGJ89], voir aussi [Roh91]) qui affirme qu'il existe des courbes $\gamma$ pour lesquelles la mesure harmonique vue d'un côté et la mesure harmonique vue de l'autre côté sont des mesures singulières. Par analogie, les mêmes techniques utilisées pour les théorèmes ci-dessus impliquent que pour tout $\kappa \in (4, 8)$, il existe un certain domaine $\Omega = \Omega(\kappa)$ et un ensemble $E \subset \partial\Omega$ tels que si $\gamma_1$ et $\gamma_2$ sont respectivement des $\mathrm{SLE}_\kappa$ conduits à l'intérieur et à l'extérieur de $\Omega$, alors p.s. $\gamma_1(0, \infty) \cap \partial\Omega \subset E$ tandis que $\gamma_2(0, \infty) \cap \partial\Omega \subset E^c$. Appliqué au cas de $\kappa = 6$, cela signifie qu'il existe des domaines $\Omega$ pour lesquels (à la limite continue) les clusters à l'intérieur sont invisibles pour les clusters extérieurs.

# 4 Le spectre de Fourier de la percolation critique

Avant d'expliquer nos résultats dans le contexte de la percolation, nous présentons ci-dessous un petit "survey" sur la sensibilité au bruit des fonctions Booléennes, et nous donnons quelque prérequis sur la percolation dynamique.

## 4.1 Sensibilité au bruit des fonctions Booléennes

Commençons pas un exemple. Imaginons que l'on s'intéresse à la sensibilité du résultat d'une élection par rapport au faible taux d'erreurs dans le comptage des votes (autrement dit, dû au faible niveau de "bruit"). Pour simplifier supposons qu'il y a seulement deux candidats (+1 et -1) et que chaque personne participant au scrutin fait son choix de façon indépendante et uniforme. Un mode de scrutin correspond à une certaine fonction Booléenne $f$ de $\{-1,1\}^n$ vers $\{-1,1\}$, où $n$ est le nombre de personnes. On peut supposer de plus que le mode de scrutin est équilibré dans le sens où il ne favorise pas tel ou tel candidat (cela se traduit par $\mathbb{E}\big[f\big] = 0$). Le faible taux de bruit (ou d'erreurs) peut être modélisé comme suit : supposons que indépendamment pour chaque bulletin, une erreur se produit avec probabilité $\epsilon$, où $\epsilon \in (0,1)$ est une constante fixée. Cela veut dire que indépendamment pour chaque personne, avec probabilité $\epsilon$ le vote est mal pris en compte (+1 devient -1 et vice-versa). La sensibilité au bruit du mode de scrutin $f$ correspond ici à la probabilité que le résultat de l'élection soit affecté par les erreurs. Par exemple un scrutin à la majorité absolue sera moins sensible au bruit qu'un scrutin à plusieurs niveaux (comme c'est le cas aux états-unis).

Plus formellement, nous considérerons des fonctions Booléennes $f$ de $\{-1,1\}^n$ vers $\{-1,1\}$ (souvent, les fonctions Booléennes vont plutôt de $\{0,1\}^n$ vers $\{0,1\}$ mais pour des raisons de symétrie il s'avérera être plus pratique de les considérer de $\{-1,1\}^n$ vers $\{-1,1\}$ et plus généralement de $\{-1,1\}^n$ dans $\mathbb{R}$). Les propriétés des fonctions Booléennes sont étudiées de façon approfondie en informatique ainsi que dans d'autres domaines (voir [KS06] par exemple).

Comme nous l'avons motivé ci-dessus, pour une fonction Booléenne fixée $f$ de $n$ bits, nous serons principalement intéressés par la sensibilité de la fonction $f$ quand les données sont affectées par du "bruit". En informatique, on poserait la question de la manière suivante : est ce que la fonction $f$ est robuste aux erreurs (dans la transmission des données par exemple) ? Plus précisément, soit $f : \{-1,1\}^n \to \{-1,1\}$. Supposons que l'hypercube $\{-1,1\}^n$ est muni de la mesure de probabilité uniforme. La théorie peut être facilement étendue aux mesures produit sur $\{-1,1\}^n$, mais nous nous restreindrons à ce cas (qui est déjà très riche). Pour une configuration aléatoire $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$, soit $y = (y_1, \ldots, y_n)$ une perturbation aléatoire de $x$, où indépendamment pour chaque bit $i \in \{1, \ldots, n\}$, avec probabilité $\epsilon$, $y_i = -x_i$ et avec probabilité $1-\epsilon$, $y_i = x_i$. Ici $\epsilon$ est une petite constante qui correspond au niveau de bruit. La fonction Booléenne $f$ sera dite sensible au bruit si pour une majeur partie des configurations $x$, connaissant les données initiales $x$, il est difficile de prédire ce que sera $f(y)$. Plus quantitativement cela peut être mesuré par la quantité suivante :

$$N(f, \epsilon) := \mathrm{var}\left[\mathbb{E}\left[f(y_1, \ldots, y_n) \;\middle|\; x_1, \ldots, x_n\right]\right]. \tag{4.1}$$

On s'intéressera au cas asymptotique où le nombre de bits $n$ tend vers l'$\infty$.

*Définition* 4.1. Soit $(n_m)_{m\in\mathbb{N}}$ une suite croissante dans $\mathbb{N}$. Une suite de fonctions Booléennes $f_m : \{-1,1\}^{n_m} \to \{-1,1\}$ sera dite **asymptotiquement sensible au bruit** (ou juste sensible au bruit) si pour tout $\epsilon > 0$,

$$\lim_{m\to\infty} N(f_m, \epsilon) = 0. \tag{4.2}$$

Cela peut être paraphrasé en disant que asymptotiquement, la donnée initiale $(x_1, \ldots, x_{n_m})$ ne donne presque aucune information sur le résultat $f(y_1, \ldots, y_{n_m})$.

La situation opposée correspond à la **stabilité au bruit**. Une suite de fonctions Booléennes $f_m : \{-1,1\}^{n_m} \to \{-1,1\}$ sera dite (asymptotiquement) stable au bruit si

$$\sup_{m\geq 0} \mathbb{P}\left[f(x_1, \ldots, x_{n_m}) \neq f(y_1, \ldots, y_{n_m})\right] \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Bien évidemment, la sensibilité au bruit et la stabilité au bruit sont des cas extrêmes; il y a de nombreux exemples qui se trouvent entre les deux. On trouve la même situation dans la théorie des bruits de Tsirelson où les bruits noirs et les bruits blancs sont les cas extrêmes.

Dans certains contextes, d'autres façons de mesurer la sensibilité au bruit peuvent sembler plus naturelles, mais dans la plupart des cas, notre mesure de la sensibilité $N(f,\epsilon)$ contrôle les autres critères. Par exemple, il est immédiat par Cauchy-Schwarz de vérifier que pour $f:\{-1,1\}^n \to \mathbb{R}$, on contrôle la corrélation

$$\left|\mathbb{E}\big[f(x)f(y)\big] - \mathbb{E}\big[f\big]^2\right| \leq \sqrt{N(f,\epsilon)}\sqrt{\mathrm{var}(f)},$$

ce qui peut être traduit dans le cas d'une fonction Booléenne équilibrée ($\mathbb{E}\big[f\big]=0$) à valeurs dans $\{-1,1\}$ par

$$\left|\mathbb{P}\big[f(x)\neq f(y)\big] - \frac{1}{2}\right| \leq \frac{1}{2}\sqrt{N(f,\epsilon)}.$$

Cette dernière expression est ce que l'on considérerait dans le cas du résultat d'un mode de scrutin.

Il se trouve que l'analyse de Fourier discrète fournit des outils très utiles pour l'étude de la sensibilité au bruit.

## 4.2 Analyse de Fourier des fonctions Booléennes et application à la sensibilité au bruit

Commençons par une analogie avec l'analyse de Fourier classique. Imaginons qu'une certaine fonction $f$ dans $L^2(\mathbb{R}/\mathbb{Z})$ nous soit donnée. On choisit une point $x$ au hasard, uniformément sur le cercle. Soit $y$ une perturbation de $x$ (c.a.d. $x$ plus un petit bruit), par exemple $y = x + \mathcal{N}(0,\epsilon^2)$ pour une petite valeur $\epsilon > 0$. On aimerait prédire la valeur de $f(y)$ sachant $x$. Par exemple si $f(x) = \sin(\pi 2^{100} x)$, il est clair que si le bruit $\epsilon$ vaut $10^{-3}$, la sensibilité sera très forte. En général on peut quantifier la sensibilité de $f$ par

$$N(f,\epsilon) = \mathrm{var}\left[\mathbb{E}\big[f(y)|x\big]\right]. \tag{4.3}$$

On sait que les coefficients de Fourier de $f$ donnent des informations sur la "régularité" de $f$. Si le spectre de $f$ est concentré sur les petites fréquences, $f$ sera très régulière et peu sensible au bruit, alors que si $f$ a beaucoup de hautes fréquences, le résultat $f(y)$ sera moins prévisible. On peut facilement calculer $N(f,\epsilon)$ à l'aide de la décomposition en série de Fourier $f(x) =$

$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2i\pi nx}$, en effet

$$\begin{aligned}
N(f,\epsilon) &= \mathbb{E}\big[[\mathbb{E}\big[f(y)\bigm| x\big] - \mathbb{E}\big[f\big]]^2\big] \\
&= \int_0^1 \left(\sum_n \hat{f}(n)\mathbb{E}\big[e^{2i\pi ny}|x\big] - \hat{f}(0)\right)^2 dx \\
&= \int_0^1 \left(\sum_{n\neq 0} \hat{f}(n)e^{2i\pi nx}\mathbb{E}\big[e^{2i\pi n\mathcal{N}(0,\epsilon^2)}\big]\right)^2 dx \\
&= \int_0^1 \left(\sum_{n\neq 0} \hat{f}(n)e^{2i\pi nx}e^{-2\pi^2 n^2\epsilon^2}\right)^2 dx \\
&= \sum_{n\neq 0} |\hat{f}(n)|^2 e^{-4\pi^2 n^2\epsilon^2} \ \text{ since } \hat{f}(n) = \overline{\hat{f}(-n)}\ .
\end{aligned}$$

Ainsi on peut voir sur cette formule que les hautes fréquences favorisent la sensibilité au bruit.

On aimerait suivre la même approche pour l'étude des fonctions Booléennes. Il existe une riche théorie de l'analyse de Fourier sur l'hypercube $\{-1,1\}^n$. Considérons le cas plus général de l'espace $L^2(\{-1,1\}^n)$ des fonctions réelles de $n$ bits dans $\mathbb{R}$, muni du produit scalaire :

$$\begin{aligned}
\langle f, g\rangle &= \sum_{x_1,\ldots,x_n} 2^{-n} f(x_1,\ldots,x_n)g(x_1,\ldots,x_n) \\
&= \mathbb{E}\big[fg\big],
\end{aligned}$$

avec la mesure de probabilité uniforme sur l'hypercube. Pour tout $S \subset \{1,2\ldots,n\}$, soit $\chi_S$ la fonction sur $\{-1,1\}^n$ définie par

$$\chi_S(x) := \prod_{i\in S} x_i\,, \tag{4.4}$$

pour tout $x = (x_1,\ldots,x_n)$. Il est immédiat de voir que cet ensemble de $2^n$ fonctions forme une base orthonormale de $L^2(\{-1,+1\}^n)$. Ainsi toute fonction $f$ peut être décomposée comme

$$f = \sum_{S\subset\{1,\ldots,n\}} \hat{f}(S)\,\chi_S,$$

où $\hat{f}(S)$ sont les coefficients de Fourier de $f$. Ils sont parfois appelés les coefficients de **Fourier-Walsh** de $f$. Il vérifient

$$\hat{f}(S) = \langle f, \chi_S \rangle = \mathbb{E}\big[f \chi_S\big].$$

Notons que $\hat{f}(\emptyset)$ correspond à la moyenne $\mathbb{E}\big[f\big]$.

Bien sûr on pourrait trouver d'autres bases orthonormales pour $L^2(\{-1,1\}^n)$, mais il y a de nombreuses situations où ce choix particulier de fonctions $(\chi_S)_S$ apparaît naturellement. Tout d'abord il y a une théorie approfondie de l'analyse de Fourier sur les groupes, théorie qui est particulièrement simple et élégante pour les groupes Abeliens (ce qui inclue notre cas de l'hypercube $\{-1,1\}^n$, mais aussi $\mathbb{R}/\mathbb{Z}$, $\mathbb{R}$ etc..). Pour les groupes Abeliens, l'ensemble $\hat{G}$ des caractères de $G$ (c.a.d. le groupe des morphismes de $G$ dans $\mathbb{C}^*$) se trouve être le bon point de vue pour faire de l'analyse harmonique sur $G$. Dans notre situation où $G = \{-1,1\}^n$, les caractères sont précisément les fonctions $\chi_S$ indexées par $S \subset \{1,\ldots,n\}$ car $\chi_S(x \cdot y) = \chi_S(x)\chi_S(y)$.

Ces fonctions apparaissent aussi naturellement si l'on considère la marche aléatoire simple sur l'hypercube (équipé de la structure de graphe de Hamming), car ce sont les fonctions propres du noyau de la chaleur sur $\{-1,1\}^n$.

Enfin, cette base $(\chi_S)$ est particulièrement bien adaptée à notre étude de la sensibilité au bruit. En effet, de la même façon que pour les fonctions sur le cercle $\mathbb{R}/\mathbb{Z}$, on obtient que si $f$ est une fonction de $L^2(\{-1,1\}^n)$, alors

$$\begin{aligned} N(f,\epsilon) &= \mathbb{E}\big[[\mathbb{E}\big[f(y) \mid x\big] - \mathbb{E}\big[f\big]]^2\big] \\ &= \mathbb{E}\big[[\sum_{S \subset \{1,\ldots,n\}} \hat{f}(S)\mathbb{E}\big[\chi_S(y) \mid x\big] - \hat{f}(\emptyset)]^2\big]. \end{aligned}$$

Il est facile de vérifier que $\mathbb{E}\big[\chi_S(y) \mid x\big] = \prod_{i\in S} \mathbb{E}\big[y_i \mid x_i\big] = (1-2\epsilon)^{|S|}$ par indépendance des bits. Ainsi en utilisant le fait que si $S_1 \neq S_2$, $\chi_{S_1}$ et $\chi_{S_2}$ sont orthogonales on obtient

$$N(f,\epsilon) = \sum_{S \subset \{1,\ldots,n\},\, S \neq \emptyset} \hat{f}(S)^2 (1-2\epsilon)^{2|S|}. \tag{4.5}$$

Par conséquent, dans le contexte des fonctions Booléennes, les "hautes fréquences" correspondent aux sous-ensembles $S$ de $\{1,\ldots,n\}$ de grand cardinal. La formule de Parseval implique que

$$\sum_S \hat{f}(S)^2 = \|f\|_2^2.$$

Pour toute fonction Booléenne $f : \{-1,1\}^n \to \mathbb{R}$, on définit sa **mesure spectrale** sur l'ensemble des parties de $\{1,\ldots,n\}$ par

$$\mathbb{Q}_f\big[\mathscr{S} = S\big] = \mathbb{Q}\big[\mathscr{S} = S\big] := \hat{f}(S)^2,$$

où le sous-ensemble "aléatoire" $\mathscr{S}$ ($\mathbb{Q}$ n'est pas forcément une mesure de probabilité ici) sera appelé le **spectre aléatoire de Fourier** de $f$. En particulier, si $f$ est à valeurs dans $\{-1,1\}$, alors $\|f\|_2 = 1$, et on obtient une **mesure de probabilité spectrale**,

$$\mathbb{P}_f\big[\mathscr{S} = S\big] = \mathbb{P}\big[\mathscr{S} = S\big] := \hat{f}(S)^2.$$

Notons qu'il y a un léger abus de notation ici étant donné que $\mathbb{Q}$ et $\mathbb{P}$ ne sont pas définis ici sur le même espace de probabilité que $x \in \{-1,1\}^n$, donc formellement on aurait dû utiliser d'autres notations.

Pour toute fonction Booléenne $f$ (à valeurs dans $\mathbb{R}$), on peut réécrire sa sensibilité $N(f,\epsilon)$ en terme de sa mesure spectrale de la façon suivante

$$\begin{aligned} N(f,\epsilon) &= \sum_{S\subset\{1,\ldots,n\},\, S\neq\emptyset} \hat{f}(S)^2(1-2\epsilon)^{2|S|} \\ &= \sum_{k=1}^{n} \mathbb{Q}\big[|\mathscr{S}| = k\big](1-2\epsilon)^{2k}. \end{aligned}$$

Pour une fonction Booléenne de norme $L^2$ égale à un, cela correspond à

$$N(f,\epsilon) = \sum_{k=1}^{n} \mathbb{P}\big[|\mathscr{S}| = k\big](1-2\epsilon)^{2k} = \mathbb{E}\big[(1-2\epsilon)^{2|\mathscr{S}|}\big],$$

où $\mathbb{E}$ correspond ici à l'espérance par rapport au spectre aléatoire $\mathscr{S}$. Par conséquent cela montre qu'une suite de fonctions Booléennes $(f_m)$ (à valeurs dans $\{-1,1\}$) sera asymptotiquement sensible au bruit si et seulement si les mesures spectrales $(\mathbb{P}_{f_m})$ sont supportées sur des ensembles de plus en plus grands et qu'il ne reste pas de masse sur les "fréquences finies" (à part éventuellement $\emptyset$). Plus précisément

**Proposition 4.2.** *Une suite de fonctions $f_m$ de $\{-1,1\}^{n_m} \to \{-1,1\}$ est asymptotiquement sensible au bruit si et seulement si pour tout $N > 0$,*

$$\lim_{m\to\infty} \mathbb{P}_{f_m}\big[0 < |\mathscr{S}| < N\big] = 0.$$

Ainsi la distribution de la taille du spectre aléatoire $\mathscr{S}$ réunit toute l'information nécessaire à l'étude de la sensibilité au bruit de $f$. On pourrait donc être tenté de restreindre notre étude seulement à la taille de $\mathscr{S}$, mais il s'avérera être utile dans le chapitre V de considérer $\mathscr{S}$ "géométriquement".

## 4.3 Quelque exemples simples de fonctions Booléennes

- Commençons par un exemple simple relié à la situation des modes de scrutin, décrite précédemment. Pour tout entier impair $n \geq 1$, on définit la fonction Majorité $\mathrm{MAJ}_n$ sur l'hypercube $\{-1,1\}^n$ (toujours muni de la mesure uniforme) de la façon suivante : pour tout $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$ soit

$$\mathrm{MAJ}_n(x) = \mathrm{sign}(\sum_i x_i).$$

  Pour tout niveau de bruit $\epsilon > 0$, si on se représente $x_1 + \ldots + x_n$ comme une marche aléatoire simple sur $\mathbb{Z}$ de $n$ pas, $y = (y_1, \ldots, y_n)$ sera une version $\epsilon$-bruitée de la marche aléatoire $x$; en particulier pour $n$ grand, $\frac{1}{\sqrt{n}}(x_1, \ldots, x_n)$ et $\frac{1}{\sqrt{n}}(y_1, \ldots, y_n)$ sont approximativement proches à $\sqrt{\epsilon}$ près. Ainsi si $x$ est tel que $|x_1 + \ldots + x_n| > 100\sqrt{\epsilon}$ (ce qui se produit avec grande probabilité si $\epsilon$ est petit), on peut prédire $f(y)$ avec une bonne précision. Par conséquent la fonction Majorité est (asymptotiquement) stable au bruit.

  On peut en fait calculer exactement dans cette situation la distribution de la taille du spectre aléatoire. Regardons le premier niveau ($|\mathscr{S}| = 1$) de la distribution de Fourier. Pour tout bit $i \in \{1, \ldots, n\}$, $\mathbb{P}\big[\mathscr{S} = \{i\}\big] = \mathbb{E}\big[\mathrm{sign}(x_1 + \ldots + x_n)x_i\big]^2$. La seule contribution à l'espérance provient des configurations $x$ telles que $\sum x_i = \pm 1$; l'ensemble de ces configurations (asymptotiquement) a probabilité $\frac{2}{\sqrt{2\pi n}}$. Cela donne $\mathbb{P}\big[\mathscr{S} = \{i\}\big] = \frac{2}{\pi n} + o(\frac{1}{n})$, d'où $\mathbb{P}\big[|\mathscr{S}| = 1\big] = \frac{2}{\pi} + o(1)$. On observe donc que asymptotiquement, une fraction positive reste concentrée sur le niveau un des coefficients de Fourier; il en est de même pour tous les niveaux impairs $k \geq 1$ et de plus la masse ne se propage pas à l'infini (quand $n$ tend vers l'infini). La fonction Majorité est en quelque sorte, sous des hypothèses raisonnables, la fonction Booléenne la plus stable.

- La fonction Parité $\mathrm{PAR}_n$ : soit $n \geq 1$, considérons la fonction qui retourne 1 si parmi les $n$ bits on trouve un nombre pair de $-1$; $-1$ sinon. La fonction Parité peut être écrite pour tout $x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$ comme

$$\mathrm{PAR}_n(x) = \prod_{i=1}^{n} x_i = \chi_{\{1,2,\ldots,n\}}(x).$$

  En particulier dans cet exemple, la mesure spectrale est concentrée sur le singleton $\delta_{\{1,\ldots,n\}}$. C'est la fonction Booléenne la plus sensible au bruit (plus haute fréquence) que l'on peut trouver sur l'hypercube.

- On se tourne à présent vers les fonctions Booléennes qui nous intéresseront tout particulièrement dans la suite (chapitre V) à savoir les événements de croisement (radial ou de type Gauche-droite dans un rectangle) en percolation critique (2d). Par exemple, si on considère la percolation sur $\mathbb{Z}^2$ à $p_c = 1/2$; pour le rectangle $n \times (n+1)$, on peut considérer la fonction Booléenne $f_n$ sur les arêtes de ce rectangle, qui retourne 1 si il y a un croisement de Gauche à Droite, -1 sinon. Par dualité cette fonction est équilibrée : $\mathbb{E}\big[f_n\big] = 0$. On aimerait comprendre quelle est la sensibilité au bruit de la percolation; ou plus précisément comment ses connections, clusters etc.. sont affectées quand la configuration est bruitée. Si on voulait calculer les coefficients de Fourier de $f_{10}$, puisqu'il y a environ 200 bits concernés, on aurait besoin de calculer environ $2^{200}$ termes. Il n'existe pas à ce jour de manière de calculer ces coefficients de Fourier. On ne sait pas non plus "simuler" une réalisation de $\mathscr{S}$ de façon raisonnable (on peut comparer par exemple avec la situation des $SAW$ où il est difficile de compter le nombre de chemins-auto-évitants, mais au moins il est possible en utilisant l'algorithme pivot de les simuler). On a calculé ci-dessous la décomposition de Fourier-Walsh de $f_1$ (il y aurait déjà $2^{13}$ termes pour $f_2$).

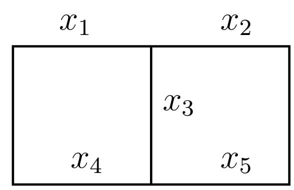



Figure 4.1: Variables pour la fonction $f_1$.

$$\begin{aligned} f_1(x_1,\dots,x_5) &= \frac{1}{2^5}(12\chi_1+12\chi_2+4\chi_3+12\chi_4+12\chi_5)\\ &+\frac{1}{2^5}(-8\chi_{1,2}+8\chi_{1,4}+8\chi_{2,5}-8\chi_{4,5})\\ &+\frac{1}{2^5}\left\{\begin{array}{l} -4\chi_{1,2,3}-4\chi_{1,2,4}-4\chi_{1,3,4}+4\chi_{2,3,4}-4\chi_{1,2,5}\\ +4\chi_{1,3,5}-4\chi_{2,3,5}-4\chi_{1,4,5}-4\chi_{2,4,5}-4\chi_{3,4,5}\end{array}\right.\\ &+\frac{4}{2^5}\chi_{1,2,3,4,5}\end{aligned}$$

Ce qui donne $\mathbb{P}\big[|\mathscr{S}|=1\big]=\frac{592}{2^{10}}\approx 0.58$, $\mathbb{P}\big[|\mathscr{S}|=2\big]=\frac{252}{2^{10}}=1/4$, $\mathbb{P}\big[|\mathscr{S}|=3\big]=\frac{160}{2^{10}}\approx 0.156$ et $\mathbb{P}\big[|\mathscr{S}|=5\big]=\frac{16}{2^{10}}\approx 0.016$.

## 4.4 Résultats obtenus précédemment sur la sensibilité au bruit de la percolation

L'étude de ce problème remonte à l'article séminal [BKS99]. Benjamini, Kalai et Schramm ont prouvé que l'événement de croisement dans le rectangle $n\times(n+1)$ est en effet (asymptotiquement) sensible au bruit. Ils prouvent le théorème suivant

**Théorème 4.3.** *Si $f_n$ correspond à la fonction indicatrice (dans $\{-1,1\}$) du croisement gauche-droite dans le rectangle $n\times(n+1)$ , alors pour tout $N>0$,*

$$\lim_{n\to\infty}\mathbb{P}_{f_n}\big[0<|\mathscr{S}_{f_n}|<N\big]=0. \tag{4.6}$$

L'ingrédient principal de leur preuve est l'utilisation de la propriété d'hypercontractivité pour le "noise Operator" (l'analogue du semi-groupe de Ornstein-Uhlenbeck qui associe à toute fonction Booléenne $f$ son espérance conditionnelle $\mathbb{E}\big[f(y)\bigm| x\big]$). Dans cet article, les auteurs soulèvent la question de savoir à quel point la percolation est-elle sensible au bruit. Plus précisément, plutôt que de fixer le niveau de bruit à une valeur fixe $\epsilon > 0$, le niveau du bruit peut décroître vers 0 avec la taille du système. On considère donc une suite $(\epsilon_n)$ tendant vers 0 et on regarde $N(f_n, \epsilon_n)$. Benjamini, Kalai et Schramm ont posé la question suivante

*Question* 4.4. Est ce que $N(f_n, n^{-\beta})$ tend vers 0 pour un certain exposant $\beta > 0$ ?

C'est équivalent au problème de savoir si $\mathbb{P}_{f_n}\big[0 < |\mathscr{S}| < n^\beta\big]$ converge vers zéro ou non (on s'intéresse à la vitesse à laquelle la masse de la mesure spectrale se propage vers l'infini). Dans l'article, [BKS99] les techniques d'hypercontractivité permettaient de montrer que la percolation est au moins $\epsilon_n = \frac{c}{\log n}$-sensible au bruit, pour une certaine constante $c > 0$.

Cette question a été résolue par Schramm et Steif dans [SS05]. Ils ont prouvé le Théorème

**Théorème 4.5.** *Il existe une exposant $\gamma > 0$ tel que*

$$\lim_{n\to\infty} \mathbb{P}_{f_n}\big[0 < |\mathscr{S}_{f_n}| < n^\gamma\big] = 0.$$

C'est équivalent au fait que la sensibilité $N(f_n, n^{-\gamma})$ tend vers 0 lorsque la taille du rectangle tend vers l'infini. Dans le cas de la percolation critique sur réseau triangulaire, en se basant sur la connaissance des exposant critiques, ils obtiennent des estimées quantitatives de la sensibilité des événements de croisement. Si $g_n$ est la fonction indicatrice (dans $\{-1, 1\}$) de l'événement de croisement de gauche à droite dans un domaine approximant le carré de coté $n$ (on pourrait aussi choisir une forme plus adaptée au réseau triangulaire, par exemple un losange de coté $n$) ils montrent

**Théorème 4.6.** *Pour tout $\gamma < 1/8$,*

$$\lim_{n\to\infty} \mathbb{P}_{g_n}\big[0 < |\mathscr{S}_{g_n}| < n^\gamma\big] = 0.$$

En fait ils obtiennent des résultats plus fins sur les mesures spectrales, puisqu'ils parviennent à contrôler la queue (inférieure) de distribution du spectre, qui se trouve être la quantité clé pour l'étude de la percolation dynamique.

Leur preuve utilise des techniques très différentes de celles utilisées dans [BKS99]. Ils ont observé en particulier le phénomène suivant : si une fonction Booléenne peut être évaluée par un algorithme aléatoire (randomized algorithm) de telle façon que chaque bit est lu avec faible probabilité, alors la fonction est sensible au bruit (et en quelque sorte sa sensibilité dépend de "l'efficacité" de l'algorithme). Prenons le cas de la fonction Majorité (que nous avons vu être stable), il est clair que si l'on veut déterminer le résultat de l'élection en regardant les votes un par un (en suivant un certain algorithme), dans tous les cas, on devra relever au moins $n/2$ suffrages; ainsi il n'existe pas d'algorithme pour lequel chaque bit a une faible probabilité d'être lu. Leurs techniques sont valides pour toute fonction Booléenne et ne restreignent donc pas au cas de la percolation.

Nous concluons cette sous-partie en montrant que si le bruit décroît trop vite quand la taille du système tend vers l'infini, alors le bruit peut ne plus avoir aucun effet sur les propriétés de connection de la percolation. Plus précisément considérons $f_n$ : l'indicatrice ($\pm 1$) du croisement de gauche à droite dans le rectangle $n \times (n+1)$ dans $\mathbb{Z}^2$. Soit $\omega$ une configuration i.i.d sur l'ensemble $E_n$ des arêtes du rectangle $n \times (n+1)$; pour $\epsilon_n > 0$ soit $\omega^{\epsilon_n}$ une configuration $\epsilon_n$-bruitée de $\omega$. Cela signifie qu'il y a $N \sim \mathcal{B}(|E_n|, \epsilon_n)$ arêtes au hasard qui ont changé de statut. Pour produire $\omega^{\epsilon_n}$ en partant de $\omega$ on peut procéder comme suit : indépendamment pour chaque arête $e \in E_n$, soit $u_e$ une variable aléatoire uniforme sur l'intervalle unité. Si $u_e < \epsilon_n$ on change le statut de $e$ dans la configuration $\omega$, sinon on garde le même statut pour $e$. Cet aléas en plus nous procure un ordre sur l'ensemble $S = \{e_1, \ldots, e_N\}$ des arêtes qui changent de statut entre $\omega$ et $\omega^{\epsilon_n}$. Soit $\omega_0 = \omega$, on définit par induction pour $0 \leq i < N$, la configuration $\omega_{i+1}$ comme étant la configuration $\omega_i$ dont le statut de l'arête $e_{i+1}$ a changé. En particulier on a $\omega_N = \omega^{\epsilon_n}$. Remarquons que pour tout $0 \leq i \leq N$, $\omega_i$ suit la loi de la percolation i.i.d. sur $E_n$ et la $i^{\text{me}}$ arête $e_i$ est distribuée uniformément sur $E_n$. Connaissant le nombre total de changement $N$, on obtient que

$$\begin{aligned} \mathbb{P}\big[f_n(\omega) \neq f_n(\omega^{\epsilon_n}) \bigm| N\big] &\leq \sum_{0 \leq i < N} \mathbb{P}\big[f_n(\omega_i) \neq f_n(\omega_{i+1})\big] \\ &= \sum_{0 \leq i < N} \mathbb{P}\big[e_i \text{ is pivotal for } \omega_i\big]. \end{aligned}$$

Mais puisque pour tout $0 \leq i \leq N$, $\omega_i$ suit la loi de percolation i.i.d et vu que $e_i$ est distribué uniformément sur le rectangle, toutes ces probabilités sont égales et il est facile de voir qu'elles dont de l'ordre de $\alpha_4(n)$; il y a certes ici des "effets de bord", mais c'est un calcul classique de vérifier que ces contributions venant du bord (ou des coins) sont négligeables. On obtient donc que $\mathbb{P}\big[f_n(\omega) \neq f_n(\omega^{\epsilon_n})\big] \leq O(1)\mathbb{E}\big[N\big]\alpha_4(n) = O(1)\epsilon_n n^2 \alpha_4(n)$. Par conséquent si le niveau de bruit $(\epsilon_n)$ satisfait asymptotiquement $\epsilon_n \ll \frac{1}{n^2\alpha_4(n)}$, alors les événements de croisement sont stable.

La conjecture naturelle était qu'il y a une transition "brusque" entre sensibilité et stabilité, dans le sens où, dès que l'on commence à toucher de nombreux points pivots, alors toute l'information devrait disparaître à la limite. En d'autres termes, si $\epsilon_n \gg \frac{1}{n^2\alpha_4(n)}$, alors les événements de croisement devraient être sensibles au bruit. La résolution de cette conjecture que nous décrirons plus bas est l'une des principales contributions de cette thèse. Sur le réseau triangulaire, on a vu que $\alpha_4(n) = n^{-5/4+o(1)}$, en particulier le seuil de sensibilité au bruit se trouve au voisinage de $\epsilon_n = n^{-3/4+o(1)}$. On peut comparer ce seuil avec le théorème ci-dessus de [SS05] qui montre que sur le réseau triangulaire, les croisements de percolation sont au moins $n^{-1/8+o(1)}$-sensibles au bruit.

## 4.5 Autres utilisations du spectre de Fourier dans des contextes proches

Avant d'expliquer plus en détail comment l'étude de la sensibilité au bruit de la percolation nous permet de mieux appréhender la percolation dynamique, nous mentionnons brièvement deux autres contextes où des techniques similaires se sont avérées être conséquentes.

- Dans [BKS03], il est prouvé que les longueurs des géodésiques en percolation de premier passage ont des fluctuations (en variance) majorée par $O(n/\log(n))$, et diffèrent ainsi des fluctuations gaussiennes. La conjecture (toujours ouverte) est que l'écart-type est en $n^{1/3}$. Avant cet article, Kesten ([Kes93]) avait montré que les fluctuations (en variance) sont en $O(n)$, ce qui n'excluait pas à priori un comportement Gaussien. Remarquons que les techniques de "sensibilité" au bruit sont utilisées ici dans un but différent, à savoir comprendre les fluctuations autour d'une forme asymptotique déterministe.

- Dans [FK96], il est montré que toute fonction Booléenne d'un graphe aléatoire $G(n,p), 0 \leq p \leq 1$ (dont on oublie le label des points) admet nécessairement un "sharp threshold" autour d'une certaine probabilité critique $p_c = p_c(n)$. Cela signifie que pour tout événement monotone $\mathcal{A}$, si on considère la fonction $f_n : p \mapsto \mathbb{P}_{n,p}\big[\mathcal{A}\big]$, alors $f_n$ a asymptotiquement une forme en "cut-off". En d'autre termes, n'importe quel événement monotone apparaît "tout d'un coup" quand on augmente la valeur de $p$. La preuve de ce résultat utilise entre autre le fait que l'influence totale (où l'énergie) d'un événement monotone quelconque est nécessairement grande (ce qui à nouveau provient de l'hypercontractivité). Ce fait (uniforme en quelque sorte par rapport à $p \in (0,1)$) combiné avec le lemme de Russo implique leur résultat.

## 4.6 Percolation dynamique

La percolation dynamique consiste en un dynamique naturelle sur l'espace des configurations de percolation; plus précisément c'est un processus de Markov sur l'ensemble de ces configurations. Le modèle est défini de façon élémentaire comme suit : pour tout graphe $G = (V, E)$, on part d'une configuration initiale $\omega_0$, qui suit la loi $\mathbb{P}_p$ (pour un certain $p \in [0,1]$) où chaque arête est ouverte avec probabilité $p$, et on laisse évoluer le statut de chaque arête $e \in E$ selon un processus de Poisson de taux un : indépendamment pour chaque arête, à taux un le statut de l'arête (ouvert ou fermé) est retiré au hasard : ouvert avec probabilité $p$, fermé avec probabilité $1 - p$. Par conséquent, la percolation dynamique $(\omega_t)$ est une dynamique où à chaque temps fixé $t_0$, la configuration $\omega_{t_0}$ suit la loi de percolation i.i.d. $\mathbb{P}_p$; en d'autres termes $\mathbb{P}_p$ est la loi de probabilité invariante pour ce processus de Markov. Ce modèle a été introduit par Häggström, Peres et Steif dans [HPS97]. Les principales questions que l'on rencontre sont du type suivant : est ce qu'une propriété qui est vérifiée presque sûrement pour la percolation statique $(\mathbb{P}_t)$ sera également vérifiée pour tous les temps $t$ de la dynamique ? Si la réponse se trouve être négative, alors il existe au long de cette dynamique des **temps exceptionnels** où la propriété cesse d'être vérifiée. Puisque la propriété est supposée être vérifiée presque sûrement pour la percolation statique, l'ensemble de ces temps exceptionnels est nécessairement de mesure de Lebesgue nulle.

Dans [HPS97], les auteurs considèrent le cas général des graphes $G$, infinis, connexes et localement finis. Soit $p_c = p_c(G)$ la probabilité critique. Appelons $\mathcal{C}$, l'événement qu'il existe une composante connexe infinie. Ils montrent que à part peut-être au point critique $p_c$, il n'y a pas de temps exceptionnels pour l'événement $\mathcal{C}$. Plus précisément, ils montrent que si $p > p_c$ alors presque sûrement (par rapport à la mesure de probabilité régissant le processus de Markov) l'événement $\neg\mathcal{C}$ est vérifié pour tous les temps de la dynamique. Ainsi l'étude de la percolation dynamique s'est focalisée depuis sur le comportement de la percolation dynamique au niveau du point critique. Toujours dans [HPS97], les auteurs ont soulevé cette question pour la percolation dans $\mathbb{Z}^d, d \geq 2$ au point critique $p_c(\mathbb{Z}^d)$. En utilisant des résultats obtenus par Hara et Slade sur la percolation en grande dimension ($d \geq 19$), et en particulier le fait que la densité du cluster infini $\theta_{\mathbb{Z}^d}(p)$ a une dérivée finie en $p_c$ (c.a.d. $\theta_{\mathbb{Z}^d}(p) = \mathbb{P}_p\big[0 \leftrightarrow \infty\big] = O(p - p_c(\mathbb{Z}^d))$), ils ont montré que à $p = p_c$, il n'y a pas de temps exceptionnels où un cluster infini apparaît. En dimension deux, la situation est différente car lorsque l'on fait croître $p$ et que l'on passe la valeur critique $1/2$, le cluster infini apparaît en quelque sorte plus subitement ($\frac{d}{dp}\big|_{p_c}\theta_{\mathbb{Z}^2}(p) = \infty$). La question de l'existence des temps exceptionnels pour les clusters infinis dans $\mathbb{Z}^2$ est restée ouverte (maintenant résolue, voir plus bas), mais dans [SS05], Schramm et Steif ont apporté une contribution décisive en montrant que de tels temps exceptionnels existent pour la percolation sur réseau triangulaire (à $p_c = 1/2$). Ils ont prouvé le Théorème suivant.

**Théorème 4.7.** *Presque sûrement, l'ensemble des temps exceptionnels* $t \in [0, 1]$ *tels que la percolation dynamique critique sur réseau triangulaire a une composante connexe infinie est non vide.*

*De plus, la dimension de Hausdorff de cet ensemble de temps exceptionnels est presque sûrement une constante dans* $[1/6, 31/36]$.

Ils ont conjecturés que la dimension de ces temps exceptionnels est presque sûrement $31/36$.

La percolation dynamique est intimement reliée à la sensibilité au bruit de la percolation. En effet, pour la percolation dynamique sur réseau triangulaire, la configuration $\omega_{t+s}$ au temps $t+s$ est une configuration $\epsilon$-bruitée de $\omega_t$ avec $\epsilon = \frac{1}{2}(1-\exp(-s))$; ici le facteur $1/2$ provient du fait que l'on a définit la percolation dynamique en retirant au hasard le statut de chaque site à taux un, plutôt que de changer la statut de chaque site à taux un (la première définition étant plus commode pour les graphes où $p_c \neq 1/2$). Comme c'est souvent le cas, c'est beaucoup plus facile de contrôler la borne supérieure pour la dimension de Hausdorff de l'ensemble des temps exceptionnels. D'un autre coté, si pour un certain événement (de probabilité “statique” 0), on veut prouver qu'il existe des temps exceptionnels, alors cela passe généralement par la détermination d'une borne inférieure strictement positive pour la dimension de l'ensemble de ces temps exceptionnels (ce qui est en général la partie difficile).

Nous allons expliquer d'où vient la borne supérieure égale à 31/36 dans le cas du réseau triangulaire. Appelons $\mathcal{E}$ l'ensemble (aléatoire) des temps exceptionnels $t \in [0,1]$ où il y a une composante connexe infinie dans $\omega_t$, on veut montrer que p.s. $\dim_H(\mathcal{E}) \leq \frac{31}{36}$. Pour tout site $x$ dans le réseau triangulaire $\mathbb{T}$, appelons $\mathcal{I}_x$, l'événement qu'il existe un chemin ouvert de $x$ vers l'infini, et soit $\mathcal{E}_x$ l'ensemble des temps exceptionnels $t \in [0,1]$ tels que $x \overset{\omega_t}{\longleftrightarrow} \infty$. Par définition, $\mathcal{E} = \cup_{x\in\mathbb{T}} \mathcal{E}_x$. Par dénombrabilité, il suffit de montrer que l'ensemble $\mathcal{E}_0$ des temps exceptionnels où l'origine est connectée à l'infini est p.s. de dimension plus petite que $\frac{31}{36}$. Pour tout entier $n \geq 1$, on partitionne l'intervalle unité $[0,1)$ en $n$ intervalles $I_k = [\frac{k}{n}, \frac{k+1}{n})$, $0 \leq k < n$. Pour tout $0 \leq k < n$, on veut majorer la probabilité que $\mathcal{E}_0 \cap I_k \neq \emptyset$. Pour cela remarquons que $\omega_{k/n}$ suit la loi de la percolation critique ($p = 1/2$); on définit maintenant la configuration $\tilde{\omega_k}$ comme étant l'ensemble des sites ouverts de $\omega_{k/n}$ plus tous les sites qui ont changé de statut (au moins une fois) de fermé vers ouvert pendant l'intervalle $I_k$. Ainsi par définition, pour tout $t \in I_k$, la configuration $\omega_t$ est dominée par $\tilde{\omega_k}$. Mais il est facile de voir que $\tilde{\omega_k}$ suit précisément la loi de la percolation i.i.d avec paramètre $p = \frac{1}{2} + \frac{1}{4}(1 - e^{-1/n}) \leq \frac{1}{2} + \frac{1}{4n}$. Par conséquent, la probabilité qu'il y ait un temps $t \in I_k$ pour lequel 0 soit connecté à l'infini est dominé par la probabilité que 0 soit connecté à l'infini pour $\tilde{\omega_k}$. Cette probabilité est donnée par la fonction de densité $\theta(\frac{1}{2} + \frac{1}{4n})$. Maintenant, grâce à la connaissance des exposants critiques dans le cas du réseau triangulaire, on sait que $\theta(p) = (p - 1/2)^{5/36+o(1)}$, quand $p \to p_c = 1/2$ (voir par exemple [Wer07]). En particulier pour tout $\alpha > 0$ et $n$ assez grand, on obtient que pour tout $0 \leq k \leq n$, $\mathbb{P}\big[I_k \cap \mathcal{E}_0\big] \leq (\frac{1}{n})^{5/36-\alpha}$. Cela implique que pour $n$ assez grand, le nombre moyen d'intervalles de largeur $1/n$ nécessaires pour recouvrir $\mathcal{E}_0$ est majoré par $n^{31/36-\alpha}$ ce qui (en prenant $n \to \infty$ et $\alpha \to 0$) prouve que p.s. $\dim(\mathcal{E}) = \dim(\mathcal{E}_0) \leq \frac{31}{36}$. Voir [SS05] pour plus de détails.

D'un autre coté, ne serait-ce que la preuve de l'existence des temps exceptionnels se trouve être une tâche bien plus difficile. En effet, on a besoin pour cela de comprendre les corrélations entre les configurations $\omega_t$ et $\omega_{t+s}$. En d'autres termes, les estimées de type "premier moment" (comme ci-dessus) sont suffisantes pour les bornes supérieures, mais pour les bornes inférieures on a recourt au moins à un contrôle du type "second moment" (d'où le besoin de regarder les corrélations).

Heuristiquement, si la configuration percolation $\omega_t$ change très vite au cours du temps $t$, alors elle aura plus de chance de créer des chemins infinis à certains temps exceptionnels. Autrement dit, si la percolation se trouve être très sensible au bruit, alors les propriétés de connection décorréleront vite ce qui facilitera l'émergence de clusters infinis.

Plus mathématiquement, pour tout rayon $R > 1$, on introduit $Q_R$, l'ensemble des temps où 0 est connecté à distance $R$:

$$Q_R := \{t \in [0,1] : 0 \overset{\omega_t}{\longleftrightarrow} R\}.$$

Prouver l'existence des temps exceptionnels revient à montrer qu'avec probabilité strictement positive $\cap_{R>0} Q_R \neq \emptyset$. Même si les ensembles $Q_R$ ne sont pas fermés, avec quelque techniques supplémentaires (voir [SS05]), il suffit de montrer qu'il existe une constante $c > 0$ telle que $\inf_{R>1} \mathbb{P}\big[Q_R \neq \emptyset\big] > c$. Cela peut être établi en introduisant le montant de temps $X_R$ où 0 est connecté à distance $R$ : plus précisément on définit

$$X_R := \int_0^1 1_{0 \overset{\omega_t}{\longleftrightarrow} R}\, dt.$$

Or par Cauchy-Schwarz,

$$\mathbb{P}\big[Q_R \neq \emptyset\big] = \mathbb{P}\big[X_R > 0\big] \geq \frac{\mathbb{E}\big[X_R\big]^2}{\mathbb{E}\big[X_R^2\big]},$$

(c'est ce que l'on appelle la méthode du second moment); il reste donc à prouver qu'il existe une constante $C > 0$ telle que pour tout $R > 1$, $\mathbb{E}\big[X_R^2\big] < C\mathbb{E}\big[X_R\big]^2$. Remarquons que le second moment peut être écrit

$$\begin{aligned} \mathbb{E}\big[X_R^2\big] &= \iint_{\substack{0 \leq s \leq 1 \\ 0 \leq t \leq 1}} \mathbb{P}\big[0 \overset{\omega_s}{\longleftrightarrow} R,\, 0 \overset{\omega_s}{\longleftrightarrow} R\big]\, ds dt \\ &\leq 2 \int_0^1 \mathbb{P}\big[0 \overset{\omega_0}{\longleftrightarrow} R, 0 \overset{\omega_t}{\longleftrightarrow} R\big]\, dt\,. \end{aligned}$$

Maintenant, soit $f_R = f_R(\omega)$ la fonction indicatrice de l'événement $\{0 \overset{\omega}{\longleftrightarrow} R\}$. $f_R$ peut être vue comme une fonction Booléenne des bits du disque de rayon $R$ à valeurs dans $\{0, 1\}$. On peut donc calculer la corrélation de la façon suivante

$$\begin{aligned}
\mathbb{P}\big[0 \overset{\omega_0}{\longleftrightarrow} R,\, 0 \overset{\omega_t}{\longleftrightarrow} R\big] &= \mathbb{E}\big[f_R(\omega_0) f_R(\omega_t)\big] \\
&= \mathbb{E}\Big[\Big(\sum_{S\subset B(0,R)} \hat{f}_R(S)\chi_S(\omega_0)\Big)\Big(\sum_{S\subset B(0,R)} \hat{f}_R(S)\chi_S(\omega_t)\Big)\Big] \\
&= \mathbb{E}\big[f_R\big]^2 + \sum_{\emptyset\not\subset S\subset B(0,R)} \hat{f}_R(S)^2 \exp(-t|S|) \\
&= \mathbb{E}\big[f_R\big]^2 + \sum_{k\geq 1} \mathbb{Q}\big[|\mathscr{S}| = k\big] e^{-kt}, \qquad (4.7)
\end{aligned}$$

où $\mathbb{Q}$ est la mesure spectrale de $f_R$ (ce n'est pas une mesure de probabilité puisque $\|f_R\|_2 < 1$). En intégrant le long de l'intervalle unité, cela donne

$$\mathbb{E}\big[X_R^2\big] \leq 2\,\mathbb{E}\big[X_R\big]^2 + 2\sum_{k\geq 1} \frac{\mathbb{Q}\big[|\mathscr{S}| = k\big]}{k}.$$

Par conséquent, afin d'obtenir le second moment désiré, on doit contrôler la queue de distribution inférieure de la taille du spectre aléatoire $\mathscr{S}$. Cela a été concrétisé dans [SS05], ce qui leur a permis de montrer l'existence des temps exceptionnels sur le réseau triangulaire. Comme nous l'avons mentionné ci-dessus, leur contrôle de la queue de distribution inférieure leur a permis d'obtenir la borne inférieure de $1/6$ pour la dimension de Hausdorff de l'ensemble des temps exceptionnels. Afin d'atteindre la borne supérieure de $31/36$, des estimées optimales sur la queue de distribution (inférieure) du spectre sont nécessaires, ce qui constitue une partie des résultats que nous décrivons dans la sous-partie qui suit.

## 4.7 Notre contribution à la sensibilité au bruit et à la percolation dynamique

Ces résultats sont en collaboration avec *Gábor Pete* et *Oded Schramm.*

Les énoncés qui suivent ont lieu à la fois pour le réseaux triangulaires et $\mathbb{Z}^2$ (et ne se basent pas sur les SLE). Pour tout $n \geq 1$, $f_n$ correspondra à l'indicatrice du croisement de gauche à droite dans le rectangle $n \times (n+1)$ dans le cas de $\mathbb{Z}^2$, et dans un domaine approximant le carré de coté $n$ dans le cas du réseau triangulaire. $\alpha_4(n)$ désignera la probabilité de l'événement à quatre bras de l'origine jusqu'à distance $n$ sur le réseau en considération.

Nous avons vu ci-dessus que si $\epsilon_n n^2 \alpha_4(n)$ tend vers 0, alors les événements de croisement sont stables. Cela signifie que si $y^n$ est une configuration $\epsilon_n$-bruitée de $x^n$, alors $\mathbb{P}\big[f_n(y^n) \neq f_n(x^n)\big]$ tend vers zéro. On montre que la transition de la stabilité vers la sensibilité est “sharp” :

**Théorème 4.8.** *Si le niveau de bruit satisfait* $\epsilon_n n^2 \alpha_4(n) \to \infty$, *alors*

$$\lim_{n\to\infty} N(f_n, \epsilon_n) = 0.$$

*En terme de corrélations cela veut dire que si* $y^n$ *est une configuration* $\epsilon_n$*-bruitée de* $x^n$, *alors on a*

$$\mathbb{E}\big[f_n(y^n) f_n(x^n)\big] - \mathbb{E}\big[f_n\big]^2 \underset{n\to\infty}{\longrightarrow} 0.$$

Ce théorème est démontré en prouvant que toute la “masse spectrale” est concentrée autour de $n^2\alpha_4(n)$; c.a.d. que pour toute fonction $\delta(n)$ tendant vers 0 (arbitrairement vite), nous avons

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| < \delta(n)\, n^2 \alpha_4(n)\big] \underset{n\to\infty}{\longrightarrow} 0.$$

En fait, nous obtenons des résultats plus fins sur la mesure spectrale, en particulier sur sa queue de distribution inférieure avec le théorème suivant

**Théorème 4.9.** *Le spectre aléatoire* $\mathscr{S}_{f_n}$ *de* $f_n$ *vérifie*

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| < r^2 \alpha_4(r)\big] \asymp \Big(\frac{n}{r}\Big)^2 \alpha_4(r,n)^2,$$

*pour tout* $r \in [1, n]$ *et où* $\asymp$ *correspond à l'équivalence à constantes multiplicatives près.*

Nous montrons également un théorème analogue pour la queue de distribution inférieure de la mesure spectrale de l'événement radial (à un bras). C'est réellement ce contrôle de l'événement radial que l'on applique ensuite à la percolation dynamique. On note que dans le théorème ci-dessus, notre contrôle de la queue inférieure de distribution est optimal (à constantes près).

Dans le cas du réseau triangulaire, en utilisant la connaissance des exposants critiques, on peut réécrire le théorème ci-dessus sous forme de concentration autour de la moyenne de la façon suivante :

**Proposition 4.10.** *Pour tout* $\lambda \in (0,1]$*, on a*

$$\limsup_{n\to\infty} \mathbb{P}\big[0 < |\mathscr{S}_{f_n}| \leq \lambda\, \mathbb{E}\big[|\mathscr{S}_{f_n}|\big]\big] \asymp \lambda^{2/3},$$

*où les constantes impliquées dans* $\asymp$ *sont des constantes absolues.*

Remarquons que si l'on avait suivi l'approche développée dans [SS05] pour obtenir ces contrôles optimaux, on aurait eu besoin de trouver un algorithme pour évaluer $f_n$ avec un "**revealment**" $\delta = \delta(n)$ (le revealment d'un algorithme aléatoire étant le maximum sur l'ensemble des bits de la probabilité qu'un bit soit "demandé" par l'algorithme). Dans notre contexte des événements de croisement, le théorème général qu'ils prouvent, reliant algorithme et sensibilité, énonce que pour tous $k \geq 1$,

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| \leq k\big] \leq \delta(n) k^2. \tag{4.8}$$

Il est clair que dans le but d'évaluer $f_n$, n'importe quel algorithme aura besoin d'utiliser au moins $n$ sites (en fait, avec grande probabilité, au moins $n^\beta$ sites seront nécessaires car les "plus court chemins" en percolation critique ont une structure fractale). En particulier puisqu'il y a $O(1)n^2$ bits impliqués, le revealment est nécessairement supérieur à $c/n$ pour une certaine constante $c > 0$. Ainsi, en utilisant 4.8 et un algorithme optimisé au maximum, on peut espérer prouver au mieux que pour tout $\phi(n) = o(\sqrt{n})$, $\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| < \phi(n)\big]$ tend vers zéro (c.a.d. que la masse spectrale se propage au moins à vitesse $\sqrt{n}$). Mais puisque l'on voulait prouver que la masse spectrale se propage à vitesse $n^2 \alpha_4(n) = n^{3/4+o(1)}$, on a du recourir à une approche complètement différente.

Notre stratégie se concentre plus sur la géométrie spatiale des spectres aléatoires $\mathscr{S}_{f_n}$. Cela nous permet plus de liberté : par exemple on peut sortir du cadre classique de la sensibilité au bruit en bruitant seulement une partie des bits (on peut penser à un mode de scrutin dont le comptage des voies est plus sûr dans une région que dans une autre). On peut par exemple montrer dans le cas de la percolation sur $\mathbb{Z}^2$, que si l'on bruite seulement les arêtes verticales (avec un bruit de niveau $\epsilon > 0$), alors l'événement de croisement est asymptotiquement sensible au bruit. Cette situation peut être poussée à son extrême (avec $\epsilon = 1$), où l'on change (ou retire) l'état d'un ensemble fixé de bits. Cela répond à une conjecture apparue dans [BKS99]. Les techniques précédentes ne permettaient pas d'aborder ce type de sensibilité avec contraintes.

Comme nous l'avons vu, on peut penser à $\mathscr{S}_{f_n}$ comme à un sous-ensemble aléatoire du rectangle $n \times (n+1)$. Gil Kalai a suggéré d'étudier l'existence d'une limite continue pour la loi des spectres aléatoires $\frac{1}{n}\mathscr{S}_{f_n}$. En combinant la théorie des bruits de Tsirelson avec la preuve par Schramm et Smirnov que la limite continue de la percolation peut être vue comme un bruit ([SS]), on déduit que $\frac{1}{n}\mathscr{S}_{f_n}$ a en effet une limite continue. On prouve le théorème suivant

**Théorème 4.11.** *Dans le contexte de la percolation sur réseau triangulaire, la limite en loi de $\frac{1}{n}\mathscr{S}_{f_n}$ existe. C'est p.s. un ensemble de Cantor aléatoire de dimension 3/4.*

On note que $\mathscr{S}_{f_n}$ a beaucoup de propriétés en commun avec l'ensemble aléatoire $\mathcal{P}_n$ des points pivots pour le croisement gauche-droite (par exemple ils ont asymptotiquement la même dimension). Cependant nous voudrions souligner ces deux ensembles aléatoires sont très différents (cela se vérifie par exemple dans le domaine des grandes déviations); nous pensons en fait qu'ils deviennent asymptotiquement singuliers.

Passons désormais à la description des résultats que l'on a pu obtenir concernant la percolation dynamique en utilisant notre contrôle optimal du Spectre. Tout d'abord, nos résultats sur la concentration du spectre de Fourier dans le cas de $\mathbb{Z}^2$ nous permettent de démontrer le théorème suivant.

**Théorème 4.12.** *P.s. il existe des temps exceptionnels où la percolation dynamique critique sur $\mathbb{Z}^2$ a des composantes connexes infinies, et la dimension de Hausdorff de l'ensemble de ces temps est p.s. strictement positive.*

Soulignons ici qu'il manquait peu de chose dans [SS05] pour parvenir à ce résultat et que "rétrospectivement" leurs techniques (et estimées sur le spectre) auraient été suffisantes pour montrer l'existence des temps exceptionnels sur $\mathbb{Z}^2$.

Dans le cas du réseau triangulaire, on prouve le théorème suivant

**Théorème 4.13.** *Dans le contexte de la percolation dynamique critique sur réseau triangulaire, on a les valeurs (presque sûres) suivantes pour les dimensions de Hausdorff.*

1. *L'ensemble des temps où il y a un cluster infini a p.s. dimension de Hausdorff* 31/36*.*

2. *L'ensemble des temps où il y a un cluster infini restreint au demi-plan* $\mathbb{H}$ *a p.s. dimension de Hausdorff* 5/9*.*

3. *L'ensemble des temps où un cluster infini ouvert et un cluster infini fermé coexistent a p.s. dimension de Hausdorff au moins* 1/9 *(la dimension conjecturée étant* 2/3*).*

Les bornes supérieures étaient connues depuis [SS05], mais ne serait-ce que l'existence n'était prouvé que pour le premier point (avec comme borne inférieure 1/6). Notons que dans le troisième point, notre borne inférieure ne correspond pas à la borne supérieure. Ceci est dû à la perte de monotonicité de l'événement en considération, et la monotonicité était utilisée de façon cruciale dans notre manière de contrôler le Spectre.

# 5 Limite continue des percolations presque-critique et dynamique

On insiste tout d'abord sur le fait que en dépit de ce que le titre semble suggérer, ce chapitre n'est en aucune manière la continuation du chapitre précédent et peut être vu comme un projet tout à fait indépendant. Nous énoncerons toutefois un résultat à la fin qui relie les deux chapitres ensemble.

Ce chapitre concerne un projet en cours en collaboration avec *Gábor Pete* et *Oded Schramm*, dont le but est de démontrer que la percolation presque-critique et la percolation dynamique, une fois proprement renormalisées, ont toutes deux une limite continue. Même si nous ne présentons pas de preuve complète dans cette thèse de l'existence (et unicité) de ces limites continues, on énonce et prouve deux résultats d'intérêts indépendants qui constitueront des étapes clés dans la preuve ultérieure de la limite continue. Dans cette introduction, nous présenterons le projet global et décrirons ces deux résultats.

## 5.1 Modèle et prérequis

Pour simplifier, on restreindra l'étude au cas de la percolation par site sur réseau triangulaire (les résultats pour $\mathbb{Z}^2$ sont de toute façon partiels). Commençons pas introduire le modèle de la percolation presque-critique. On explique souvent la transition de phase en percolation " en faisant croître le niveau $p$". Cela correspond à définir un couplage naturel sur les configurations de percolation $\omega_p$ pour tous les niveaux $p \in [0,1]$ en même temps. Une manière de procéder est de tirer indépendamment pour chaque site $x$ du réseau triangulaire $\mathbb{T}$ une variable aléatoire uniforme $u_x$ sur l'intervalle unité. Pour tout $p \in [0,1]$, soit $\omega_p$ la configuration correspondant à l'ensemble des points $x \in \mathbb{T}$ tels que $u_x \leq p$. A présent, presque sûrement (par rapport à la loi du couplage), lorsque l'on fait croître $p$, un (unique) cluster infini apparaît à partir du moment où $p$ dépasse $p_c = 1/2$. On aimerait à l'aide de la compréhension actuelle de la percolation critique sur $\mathbb{T}$, comprendre comment le cluster infini émerge tout à coup dès que $p > 1/2$. En d'autres termes, on aimerait décrire la "naissance" du cluster infini. Si l'on veut utiliser l'invariance conforme (et donc le $\mathrm{SLE}_6$), une idée naturelle est de considérer la limite continue du couplage entier $(\omega_p)_{0\leq p\leq 1}$ quand la maille du réseau triangulaire tend vers 0. On considère donc par exemple la suite des couplages $(\omega_p^n)$ sur les réseaux renormalisés $\frac{1}{n}\mathbb{T}$. Le problème auquel on est confronté avec cette approche est que pour tout niveau fixé $p < 1/2$, les probabilités de connection décroissent exponentiellement vite (c.a.d. qu'il existe des constantes $C_1, C_2 > 0$ qui dépendent de $p$, telles que la probabilité que 0 soit connecté à distance $n$ pour $\omega_p$ est plus petite que $C_1 \exp(-C_2 n)$). Cela implique en particulier que si l'on observe une configuration sous-critique $\omega_p^n$ (sur $\frac{1}{n}\mathbb{T}$) dans la "fenêtre" $[0,1]^2$, alors la plus grande composante connexe dans $\omega_p^n$ sera de diamètre $O(1)\frac{\log(n)}{n}$. Quand $n$ tend vers l'infini on obtient donc une limite triviale. Le même phénomène se produit dans la situation opposée du régime sur-critique $p > 1/2$.

Par conséquent, si l'on veut garder une transition significative du régime sous-critique vers le régime surcritique, tout en renormalisant le réseau, on doit aussi veiller à "ralentir" notre façon de croître le niveau $p$ (de façon à rendre cette transition moins "brutale"). Ainsi nous renormaliserons notre couplage de la façon suivante : pour tout $n \geq 1$, considérons le couplage $(\hat{\omega}_\lambda^n)_{\lambda\in\mathbb{R}}$ où la configuration $\hat{\omega}_\lambda^n$ correspond à $\omega_p^n$ avec $p = 1/2 + \lambda\delta(n)$. Ici $\delta(n)$ est la vitesse à laquelle on ralentit l'augmentation de $p$; pour le moment ce sera juste une fonction qui tend vers zéro. Remarquons que l'on définit ainsi un couplage pour toutes les valeurs réelles de $\lambda \in \mathbb{R}$ (si $\lambda$ est tel que $p = 1/2 + \lambda\delta(n) < 0$, alors on définit $\hat{\omega}^n_\lambda$ comme étant la configuration vide, de même pour les grandes valeurs de $\lambda$, $\hat{\omega}^n_\lambda$ sera la configuration entière). en utilisant des idées semblables à celles du chapitre précédent, il est possible de voir que si $\delta(n)$ décroît trop vite, alors on obtient à la limite un couplage où toutes les configurations $\hat{\omega}_\lambda$ coïncident avec la configuration critique $\hat{\omega}_0$. Cela se produit si et seulement si $\lim_{n\to\infty} \delta(n)n^2\alpha_4(n) = 0$ (ce qui nous rappelle curieusement la transition de sensibilité...). A l'inverse, si l'augmentation de $p$ n'est pas suffisamment ralentie (c.a.d. si $\delta(n)$ ne décroît pas assez vite vers 0) alors on obtient un couplage trivial à la limite, le même que celui déjà obtenu avec le couplage $\omega^n_p$. Cela se produit si et seulement si $\lim_{n\to\infty} \delta(n)\, n^2\alpha_4(n) = \infty$. Voir [NW08] pour plus de détails. Cette question de la "fenêtre d'échelle" ou "scaling window" a été en fait beaucoup étudiée précédemment dans des contextes variés; dans le cas du modèle de percolation, de nombreuses idées proviennent de Kesten [Kes87].

Pour récapituler la discussion ci-dessus, si l'on veut garder un couplage non-trivial à la limite continue, on est obligé de choisir une vitesse $\delta(n)$ qui satisfait $\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$.

Dans [NW08], les auteurs considèrent les interfaces de percolation (par exemple l'interface standard dans $\mathbb{H}$ qui part de 0) sur le réseau triangulaire renormalisés $\frac{1}{n}\mathbb{T}$ au paramètre $p_n = 1/2 + \delta(n)$, où $\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$. Soit $\gamma^n$ l'interface standard sur $\frac{1}{n}\mathbb{T}$ à $p = p_n$ partant de 0 et restant dans $\mathbb{H}$ jusqu'à ce qu'elle sorte du disque de rayon un. Ils ont montré que $(P_n)_{n\geq 1}$, la famille des lois qui gouvernent ces interfaces renormalisées $\gamma^n$ est tendue (pour la topologie induite par une métrique bien choisie sur l'espace des interfaces). En particulier, il existe des limites continues suivant des sous-suites (subsequential scaling limits) $\gamma^{n_k} \overset{\text{loi}}{\to} \gamma$ qui convergent vers des lois supportées sur les interfaces continues dans $\mathbb{H} \cap \mathbb{D}$. Leur Théorème principal énonce que toute subsequential scaling limit est singulière par rapport à la $\mathrm{SLE}_6$ mesure sur les interfaces. Cela signifie que les "images" (du moins les interfaces) que l'on voit en régime presque-critique ("off-critical") sont différentes des images que l'on voit en régime critique.

Cela implique que quelque chose "d'intéressant" se produit à la limite $n \to \infty$ dans le seul cas restant ($\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$) pour nos couplages renormalisés $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$.

Nous définirons dans le chapitre VI une topologie naturelle $\mathcal{T}$ sur l'espace $\mathscr{H}$ de toutes les configurations de percolation. Pour pouvoir travailler avec le même espace $\mathscr{H}$, à la fois aux niveaux discret et continu, on associera à chaque configuration de percolation $\omega$ l'ensemble des "tubes" qui ont un croisement de gauche à droite pour $\omega$. Ainsi en quelque sorte, un élément de $\mathscr{H}$ consiste en un ensemble de tubes, voir chapitre VI pour une définition plus précise. Il est démontré dans [SS] que l'espace topologique $(\mathscr{H}, \mathcal{T})$ est compact. Le processus qui nous intéresse, à savoir le couplage des configurations $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$, peut être vu comme une mesure de probabilité sur les processus càdlàg $\mathbb{R} \to \mathscr{H}$ (càdlàg provient de notre choix de considérer un site $x$ ouvert si et seulement si $u_x \leq p$ plutôt que $u_x < p$). On munit cet espace de trajectoire avec la topologie de la convergence uniforme sur les compacts, soit $\widehat{\mathcal{T}}$ cette topologie.

L'un des principaux Théorèmes de notre projet en cours énonce que dans le régime digne d'intérêt $\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$, il y a une **unique** (à changement d'échelle près) subsequential scaling limit pour le couplage $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$. Plus précisément si $\delta(n) := \frac{1}{n^2\alpha_4(n)}$, on prévoit d'établir le théorème suivant

**Théorème 5.1.** *Les couplages renormalisés $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$ vus comme des processus cádlág dans $\mathscr{H}$ ont une limite continue quand $n$ tend vers l'infini. Ils convergent en loi, pour la topologie de la convergence uniforme sur les compacts ($\widehat{\mathcal{T}}$) vers un couplage de percolations continues $(\hat{\omega}_\lambda)_{\lambda\in\mathbb{R}}$.*

*En particulier, pour tout niveau fixé $\lambda \neq 0$, les configurations de percolation presque-critique $\omega^n$ sur $\frac{1}{n}\mathbb{T}$ à $p_n = 1/2 + \frac{\lambda}{n^2\alpha_4(n)}$ ont une limite continue.*

Pour la percolation dynamique, si l'on souhaite renormaliser le réseau afin d'obtenir une limite continue de la percolation dynamique, pour la même raison, on doit ralentir la dynamique en temps. En effet, si on renormalisait le réseau tout en gardant le même taux pour les horloges de Poisson sur les sites, du fait de la sensibilité de la percolation, on obtiendrait une limite continue où à tout temps $t \in \mathbb{R}$, on verrait une copie de percolation continue complètement indépendante du reste. En suivant la même discussion que pour la percolation presque-critique, si on veut garder une limite non-triviale, lorsque l'on renormalise l'espace par $n$, on doit aussi ralentir le temps par $\delta(n) \asymp 1/(n^2\alpha_4(n))$. Plus précisément pour tout $n \geq 1$, soit $(\omega^n_t)_{t\geq 0}$ une percolation dynamique sur $\frac{1}{n}\mathbb{T}$, où chaque site $x \in \mathbb{T}$ est mis à jour selon une horloge de Poisson de taux $q_n := 1/(n^2\alpha_4(n)) = n^{-3/4+o(1)}$. De même que pour la percolation presque-critique, on prévoit de montrer :

**Théorème 5.2.** *Les percolations dynamiques renormalisées $(\omega^n_t)_{t\geq 0}$ vues comme des processus cádlág aléatoires sur $\mathscr{H}$ ont une limite continue quand* *n tend vers l'infini. Ils convergent en loi, pour la topologie de la convergence uniforme sur les compacts ($\widehat{\mathcal{T}}$) vers une percolation dynamique continue* $(\omega_t)_{t\geq 0}$.

Expliquons brièvement comment on prévoit de démontrer l'existence de ces limites continues. On suit en partie un programme proposé par Camia, Fontes et Newman dans [CFN06]. L'idée est de construire la limite continue du couplage entier des percolations presque-critiques $(\hat{\omega}_\lambda)_{\lambda\in\mathbb{R}}$ ainsi que la limite continue de la percolation dynamique $(\omega_t)_{t\in\mathbb{R}}$, seulement à partir de la configuration critique "initialle" $\hat{\omega}_0 = \omega_0$. Détaillons ce programme dans le cas de la limite continue de la percolation presque-critique. Afin de réaliser $\hat{\omega}_\lambda$ (pour un certain niveau $\lambda > 0$) à l'aide de $\hat{\omega}_0$, de nombreux "sites" devront changer de statut de façon aléatoire en passant de l'état fermé à l'état ouvert. Seulement, puisque l'on est déjà à la limite continue, il n'y a plus à proprement parler de "sites". Néanmoins certains sites restent visibles : l'ensemble $\mathcal{P}$ de tous les points pivots. Dans [CFN06], les auteurs expliquent qu'il devrait être suffisant en principe de suivre l'état de ces points pivots afin de suivre (quand $\lambda$ varie) la configuration $\hat{\omega}_\lambda$. Notons que l'on suit ici le statut des points qui étaient initialement (pour la configuration $\hat{\omega}_0$) pivots; il se pourrait que la configuration $\hat{\omega}_\lambda$ "bouge" de telle façon que l'ensemble de ses points pivots n'est pas préservé; cette partie du programme nécessite donc une preuve. Mais même si l'on admet qu'il est suffisant (et que cela veut dire quelque chose) de suivre l'état des points pivots initiaux, on se trouve confronté à une difficulté : si $\mathcal{P}$ est l'ensemble de tous les points pivots initiaux (c.a.d. pour $\hat{\omega}_0$), alors une infinité (dénombrable) de points dans $\mathcal{P}$ vont passer de l'état fermé à l'état ouvert entre les configurations $\hat{\omega}_0$ et $\hat{\omega}_\lambda$ ! Il est donc difficile à priori de reconstruire la configuration $\hat{\omega}_\lambda$ à partir de la configuration $\hat{\omega}_0$ plus cette quantité "infinie" d'information additionnelle.

## 5.2 Résultats prouvés dans le chapitre VI

C'est la raison pour laquelle on introduit un "cut-off" : plutôt que de considérer tous les points pivots en même temps, on considère seulement les points pivots dont le statut compte au moins jusqu'à une distance $\epsilon$, pour un certain $\epsilon > 0$. Un point $x$ sera appelé $\epsilon$-important si l'événement à quatre bras est satisfait dans $B(x,\epsilon)$. Pour tout $\epsilon > 0$, soit $\mathcal{P}_\epsilon$ l'ensemble des points pivots qui sont initialement (pour $\hat{\omega}_0$) au moins $\epsilon$-importants.

Nous montrerons dans le chapitre VI, que si l'on veut prédire avec une bonne précision le “résultat” $\hat{\omega}_\lambda$, alors il est en effet suffisant de suivre le statut des points $\epsilon$-importants $\mathcal{P}_\epsilon$, le cut-off $\epsilon$ étant choisi arbitrairement petit en fonction du degré de précision que l'on souhaite. Pour montrer ce résultat, on aurait besoin d'exclure des scénarios où des “cascades d'importance” se produisent; c.a.d. des dynamiques (en $\lambda$), où certains points initialement de très faible importance sont “promus” à une importance bien plus élevée (le long de la dynamique en $\lambda$) et subissent ensuite un changement de statut (une mise à jour). En effet puisque l'on ne suit pas le devenir des points initialement peu importants, si un certain nombre d'entre eux deviennent importants et changent d'état, alors on prédirait très mal le résultat $\hat{\omega}_\lambda$. Cette propriété de “non-cascade” est l'objet de notre premier théorème dans le chapitre VI.

Une fois que l'on sait qu'il est suffisant de suivre l'état des points $\epsilon$-importants, on doit maintenant trouver un moyen de réaliser (tirer au hasard) quels points parmi $\mathcal{P}_\epsilon$ vont effectivement changer de statut (dans tout compact du plan, seul un nombre fini de points dans $\mathcal{P}_\epsilon$ changeront d'état, c'est la raison pour laquelle on a introduit un cut-off). Comme il est expliqué dans [CFN06], cet ensemble aléatoire de points devrait correspondre à un certain nuage de Poisson sur l'ensemble $\mathcal{P}_\epsilon$, pour une certaine mesure, qui au niveau discret correspondrait simplement à la mesure de comptage sur l'ensemble $\mathcal{P}_\epsilon^n$ des points $\epsilon$-importants (renormalisé par $n^2\alpha_4(n)$). Par conséquent pour tout $\epsilon > 0$, si on se donne une percolation continue critique $\omega$, on a besoin de définir une mesure Borelienne $\mu^\epsilon = \mu^\epsilon(\omega)$ qui soit l'analogue continu de la mesure de comptage sur $\mathcal{P}_\epsilon$. Plus précisément pour tout $n \geq 1$, soit $\mu_n^\epsilon$ la mesure de comptage sur l'ensemble $\mathcal{P}_\epsilon^n$ des points $\epsilon$-importants de $\omega^n$ renormalisé par $n^2\alpha_4(n)$; c.a.d. $\mu_n^\epsilon$ est définit par

$$\mu_n^\epsilon = \mu_n^\epsilon(\omega^n) = \frac{1}{n^2\alpha_4(n)} \sum_{x \in \frac{1}{n}\mathbb{T} \text{ is } \epsilon\text{-important}} \delta_x \,.$$

Le résultat suivant est le second théorème principal du chapitre VI.

**Théorème 5.3.** *Quand la maille $1/n$ tend vers 0, le couple de variables aléatoires $(\omega^n, \mu_n^\epsilon)$ converge en loi vers $(\omega, \mu^\epsilon)$, où $\omega$ est la limite continue de la percolation continue critique, et la mesure Borelienne $\mu^\epsilon = \mu^\epsilon(\omega)$ est une fonction mesurable de la percolation continue $\omega$.*

Notre preuve peut s'appliquer à d'autres objets aléatoire que l'on retrouve en percolation, par exemple on peut montrer que la mesure de comptage sur le processus d'exploration (voir figure 1.1) proprement renormalisé converge en loi vers une paramétrisation naturelle de la courbe $\mathrm{SLE}_6$. La question de construire une paramétrisation "fidèle" pour les courbes $\mathrm{SLE}_\kappa$ est une question naturelle; elle a été adressée récemment par Lawler et Sheffield dans [LS]. Dans notre cas, nous obtenons des paramétrisations naturelles seulement pour le $\mathrm{SLE}_{8/3}$ et le $\mathrm{SLE}_6$, mais avec l'avantage que ces paramétrisations sont issues du modèle discret (en tant que limites continues des mesures de comptage).

## 5.3 Perspectives

On remarque que ce programme conduit non seulement à la preuve des limites continues pour la percolation presque-critique et la percolation dynamique mais qu'en plus, d'une certaine façon il donne une description des couplages limites. Par exemple, cette stratégie implique que la limite continue de la percolation dynamique $(\omega_t)_{t\geq 0}$ est un processus de Markov sur l'espace des configurations $\mathscr{H}$; cette propriété est évidente au niveau discret mais elle est loin de l'être un fois passé à la limite continue.

Par ailleurs, cette approche nous permet d'utiliser le modèle de percolation presque-critique (ou dynamique) de façon très flexible, par exemple en rendant le taux de mise à jour dépendant de la position dans l'espace. Dans cette direction, on montre dans le chapitre VI que la mesure de comptage asymptotique définie ci-dessus a des propriétés de "covariance conforme". Ce résultat entraîne une structure de covariance conforme pour la percolation presque-critique. Plus précisément, on prévoit de montrer que si $\hat{\omega}_\lambda$ est une percolation presque-critique au niveau $\lambda$ dans un domaine $\Omega$, alors si $f : \Omega \to \widetilde{\Omega}$ est une application conforme, $f(\hat{\omega}_\lambda)$ est une configuration presque-critique "généralisée" dont le niveau $\tilde{\lambda}$ dépend de la position de la façon suivante : pour tout $z \in \Omega$, $\tilde{\lambda}(f(z)) = |f'(z)|^{-3/4}\lambda$.
Dans ce travail en cours, nous prévoyons d'appliquer ces résultats aux modèles ci-dessous qui sont connus pour être reliés à la percolation presque-critique:

- Nous montrerons que l'arbre couvrant minimal (Minimum Spanning Tree) défini sur le réseau triangulaire, a une limite continue, est invariant par rotations (ce modèle ne devrait pas à priori être invariant conforme). Nous décrirons certaines de ses propriétés asymptotiques.

- Le front de la percolation en gradient a une limite continue.

- Le processus “invasion percolation” a une limite continue.

On aimerait conclure par la remarque suivante. Comme nous l’avons mentionné plus haut, les chapitres V et VI sont largement indépendants l’un de l’autre. Toutefois, en combinant les résultats du premier avec le second, on en déduit que pour le modèle de la percolation dynamique à la limite continue, lorsque $t$ tend vers l’infini, $\omega_t$ devient de plus en plus indépendant de $\omega_0$. Autrement dit, au fur et à mesure que le temps passe, on oublie la configuration initiale.

En considérant la percolation dynamique (à la limite continue) comme un processus dans l’espace $\mathscr{H}$, on peut en fait montrer en utilisant les techniques du chapitre V le théorème suivant.

**Théorème 5.4.** *Le processus $t \mapsto \omega_t$ est* **ergodique** *dans l’espace des configurations $\mathscr{H}$.*

# Chapter II

# Introduction

## 1 Context and results

In this thesis, we will be focusing on properties of critical planar percolation as well as SLE processes. Let us first introduce these models in the present section. We will motivate the definition and need of SLE processes through the example of critical percolation. Since there are numerous good surveys and books on the subject (among which [Wer04, Law05, Sch07]), we opt for a concise presentation.

### 1.1 Model of percolation and phase transition

Percolation is one of the simplest models that exhibit a phase transition. Let us first consider the case of $\mathbb{Z}^d$, $d \geq 2$; let $\mathbb{E}^d$ denote the set of edges of $\mathbb{Z}^d$. For any $p \in [0,1]$ we define a random subgraph of $\mathbb{Z}^d$ as follows: independently for each edge $e \in \mathbb{E}^d$, we keep this edge with probability $p$ and remove it with probability $1-p$. Equivalently, this corresponds to defining a random configuration $\omega \in \{0,1\}^{\mathbb{E}^d}$ where, independently for each edge $e \in \mathbb{E}^d$, we declare the edge to be open ($\omega(e) = 1$) with probability $p$ or closed ($\omega(e) = 0$) with probability $1-p$. The law of the so-defined random subgraph (or configuration) is denoted by $\mathbb{P}_p$. In percolation theory, one is interested in large scale connectivity properties of the random configuration $\omega$. If $x, y \in \mathbb{Z}^d$ are two vertices, $\{x \leftrightarrow y\}$ denotes the event that there is an open path between $x$ and $y$ within the configuration $\omega$, and $\{x \leftrightarrow \infty\}$ is the event that the point $x$ is connected to infinity (this means that the connected component attached to $x$ is infinite).

The *phase transition* can be described as follows: For any $d \geq 2$, there is a critical probability $0 < p_c(\mathbb{Z}^d) < 1$ such that if $p < p_c(\mathbb{Z}^d)$, then with probability one all connected components are finite (open) clusters, while if $p > p_c(\mathbb{Z}^d)$ then with probability one, there is a **unique** infinite cluster.

The *density function* $\theta_{\mathbb{Z}^d}(p) := \mathbb{P}_p(0 \leftrightarrow \infty)$ encodes important properties of the large scale connectivities of the random configuration $\omega$. It corresponds to the density (averaged over the space $\mathbb{Z}^d$) of the infinite cluster. The phase transition means for the density function $\theta_{\mathbb{Z}^d}$ that $\theta_{\mathbb{Z}^d}(p) = 0$ if $p < p_c(\mathbb{Z}^d)$, while $\theta_{\mathbb{Z}^d}(p) > 0$ once $p > p_c(\mathbb{Z}^d)$. What exactly is happening at the threshold point $p_c(\mathbb{Z}^d)$ ? Is there almost surely an infinite cluster at $p_c$ or not ? This turns out to be a hard question in general. The "continuity" of the phase transition (characteristic of second-order phase transitions) is known in $d = 2$ as well as in high dimensions, but it is for instance not known whether $\theta_{\mathbb{Z}^3}(p_c(\mathbb{Z}^3))$ is equal to 0 or not. We refer the reader to [Gri99] for more details on percolation in $\mathbb{Z}^d$. We will now focus on percolation for planar graphs, especially at the critical point.

## 1.2 Planar percolation, conformal invariance and SLE processes

The theory of critical planar percolation has undergone rapid growth over the last ten years, especially thanks to Smirnov's proof of conformal invariance for critical percolation on the triangular lattice as well as the discovery of the SLE processes by Schramm. It is believed that the scaling limit of $\mathbb{Z}^2$ percolation at criticality is conformally invariant. This belief has lead theoretical physicists to predict, using *conformal field theory*, many asymptotic properties of critical percolation. For instance they were able to predict critical exponents of percolation which in some sense describe the fractal properties of large clusters and so on.

Even though conformal invariance of $\mathbb{Z}^2$ percolation has not been proved yet, Stanislav Smirnov proved in [Smi01] that conformal invariance holds (asymptotically) for critical site percolation on the triangular lattice $\mathbb{T}$. More precisely, he proved that a large family of crossing events are asymptotically conformally invariant. This in particular implied Cardy's formula for the asymptotic probability of crossing a rectangle.

Let us then introduce this variant model of site percolation on the triangular grid. It is defined similarly as in the case of $\mathbb{Z}^2$: for any $p \in [0, 1]$,

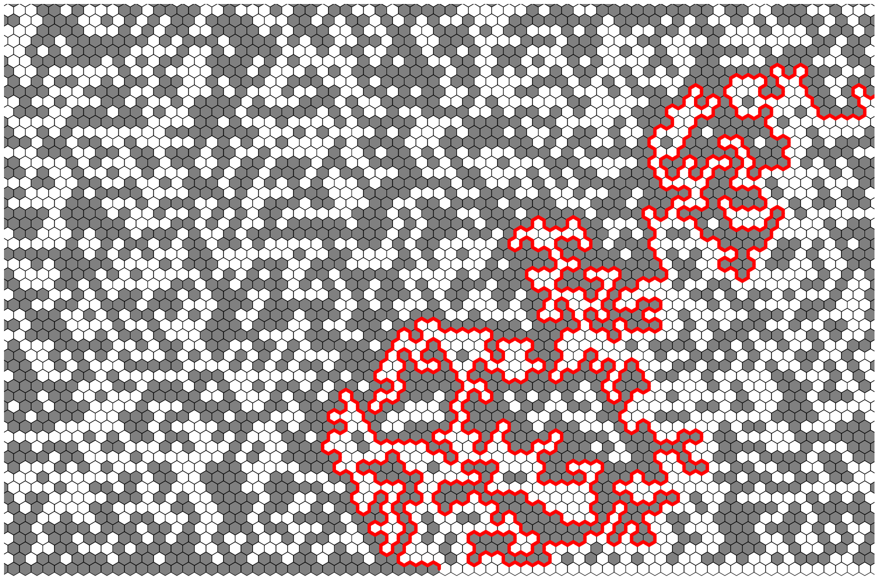


Figure 1.1: The percolation interface in the upper half plane.

independently for each site $x$ in the triangular grid $\mathbb{T}$, declare the site to be open (represented in black in the pictures) with probability $p$ and closed (white) with probability $1-p$. As in $\mathbb{Z}^2$ there is a critical probability $p_c(\mathbb{T})$, so that if $p \le p_c(\mathbb{T})$ then almost surely all clusters of open sites are finite, while if $p > p_c(\mathbb{T})$, there is with probability one a unique infinite open cluster. It is a well known theorem by Harry Kesten that $p_c(\mathbb{T}) = p_c(\mathbb{Z}^2) = \frac{1}{2}$. The triangular graph is intimately related to its dual graph, the hexagonal (or honeycomb) lattice. It is convenient ("esthetically" at least) to draw configurations of triangular site percolation on the honeycomb lattice, see figure 1.1.

Prior to Smirnov's proof (in 2001), Oded Schramm identified in 1999 what should be, assuming conformal invariance holds, the curves describing the boundaries of the ("macroscopic") clusters at the scaling limit. This led him to define the so-called SLE processes, where SLE stands for *Stochastic-Loewner-Evolution* or *Schramm-Loewner-Evolution*. Instead of considering all boundary curves at once, Schramm had the idea to consider one in particular: the *exploration path* in the upper-half plane $\overline{\mathbb{H}}$ (see figure 1.1 in the case of the triangular grid), which lies between the open clusters attached to the negative half-line and the closed clusters attached to the positive half-line. The exploration process can be sampled inductively by discovering the status of the sites one at a time.

Charles Loewner back in the twenties elaborated a way to represent curves (or rather slits) in the plane in order to solve the Bieberbach conjecture on the growth of the coefficients of univalent functions. His theory enabled him to control the size of the third coefficient (the first two coefficients being controlled using "standard" complex analysis techniques). One should point out that De Branges's proof of the Bieberbach conjecture (1985) also relies on Loewner evolutions. Applied to our percolation setting, one can consider the above exploration path as a simple curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$, with some arbitrary parametrization. For any $t \geq 0$, $H_t := \mathbb{H} \setminus \gamma[0, t]$ is a simply connected domain, therefore using the Riemann mapping Theorem, there is some conformal map $g_t$ from $H_t$ to $\mathbb{H}$. There are three (real) degrees of freedom in our choice of $g_t$; let us then fix $g_t(\infty) = \infty$ and $g_t(z) = z + o(1)$, when $z$ goes to infinity. This is easy to check that it uniquely defines the conformal map $g_t$. Now if one expands $g_t$ near infinity, one ends up with

$$g_t(z) = z + \frac{a_t}{z} + O(\frac{1}{z^2}),$$

where $t \mapsto a_t$ is some (real) increasing function. If one reparametrizes the curve $\gamma$ so that $a_t = 2t$ (which we can always do), then Loewner Theorem states that the conformal maps $(g_t)_{t\geq 0}$ satisfy the following ODE

$$\begin{cases} g_0(z) & = & z & \forall z \in \mathbb{H}\,, \\ \frac{\partial}{\partial t} g_t(z) & = & \frac{2}{g_t(z) - \beta(t)} & \text{if } t < T(z)\,, \end{cases}$$

where $t \mapsto \beta(t)$ is called the *driving function* of the curve $\gamma$, and $T(z)$ is the "explosion time", i.e. the time at which while we are following the trajectory $t \mapsto g_t(z)$, the ODE is no longer defined (a posteriori, the curve $\gamma(0, t]$ is the set of points $z$ for which $T(z) \leq t$). So in a way, the driving function $\beta(t)$ encodes the half-plane curve $\gamma$: in order to reconstruct $\gamma$ from $t \mapsto \beta(t)$, one just needs to solve the above ODE.

Imagine now that we consider the exploration path of triangular site percolation on a very small-mesh grid $\epsilon\mathbb{T}$. This corresponds to some random curve $\gamma^\epsilon : [0, \infty] \to \overline{\mathbb{H}}$, that we may parametrize so that the associated conformal maps $g_t$ satisfy the above normalization ($a_t = 2t$). Our exploration process $\gamma^\epsilon$ is thus "driven" by some random process $\beta^\epsilon(t)$. Now suppose we stop our exploration at some time $t > 0$ (this means we discovered the sites one by one, until we obtain the path $\gamma^\epsilon[0,t]$). The crucial observation is that what remains to be discovered in $\mathbb{H} \setminus \gamma^\epsilon$ still has the law of i.i.d critical percolation. In particular if one assumes conformal invariance, one can conformally map the percolation configuration in $\mathbb{H} \setminus \gamma^\epsilon[0,t]$ back to the upper half plane $\mathbb{H}$ using $g_t$. Roughly speaking conformal invariance says that if the mesh $\epsilon$ is small then the exploration process in the "distorded" lattice looks very similar to the exploration process in the orginal lattice (with small mesh as well). This means here that the image $g_t(\gamma^\epsilon((t,\infty]))$ is close in law to the original exploration process $\gamma^\epsilon$.

It is not hard to check that it can be rephrased as follows in terms of the driving functions: when the mesh $\epsilon$ goes to zero, for any $t > 0$, the law of $(\beta^\epsilon(t+u))_{u>0}$ is independent of $\beta^\epsilon([0,t])$ and has same law as $(\beta^\epsilon(t))_{t>0}$. Since the driving function remains continuous at the limit, by Levy's theorem, the limiting process ($\epsilon$ going to zero) $\beta$ necessarily is a Brownian motion $\sqrt{\kappa}B_t + \mu t$. Now, by symmetry of our percolation process (under $z \to -\bar{z}$), it is obvious that $\beta(t)$ and $-\beta(t)$ have the same law, which forces the drift $\mu$ to be zero. This is exactly what $\mathrm{SLE}_\kappa$ processes are, i.e. they are the random Loewner evolutions driven by $\sqrt{\kappa}B_t$, where $B_t$ is a standard Brownian motion.

Note that we cheated a little bit since we explained Loewner theory in the case of simple curves in $\mathbb{H}$, but the scaling limit of the percolation process turns out to be a curve with many self-intersections. In fact Loewner theory works just as well for families of growing hulls under some assumption of "local gowth" and this includes our case of percolation.

It is by no means easy (for general $\kappa$) to prove that the construction that we have just described indeed constructs a random curve $\gamma$ i.e. that one can construct a random curve $\gamma$ starting from a Brownian motion $\beta = \sqrt{\kappa}B_t$. This is derived in the paper [RS05] by Rohde and Schramm. In this paper, they prove that in the upper half-plane, the SLE curve with parameter $\kappa$ almost surely indeed exists and is continuous. They also show that this curve is simple only for $\kappa \le 4$, but that it has double points and hits the boundary of the half-plane as soon as $\kappa > 4$. Note also that SLE in a general simply connected domain is defined as the image of SLE in the upper half-plane under a conformal map.

To sum up, combining Smirnov's proof of Conformal invariance of triangular critical percolation with Schramm's description of interfaces, one ends up with the fact that percolation interfaces have a scaling limit when the mesh of the lattice goes to zero, and this scaling limit is an $SLE_\kappa$ process for some $\kappa > 0$. Once we know that the scaling limit is an SLE process, it is not hard to see that the parameter $\kappa$ has to be 6. There are several ways to see that $\kappa$ needs to be 6, one of them being that $SLE_6$ is the only SLE process whose growth is local (as it is in the discrete picture). To take an extreme example, if one considers $SLE_0$, this corresponds to a geodesic in the Poincare metric, if one perturbs the domain in which one defines the $SLE_0$, it affects locally the Poincare metric and thus affects the curve; this does not happen for $SLE_6$.

Once conformal invariance of percolation is proved, some non-trivial arguments are still needed to deduce the convergence of the discrete exploration path to $SLE_6$. The first detailed proof appears in Camia and Newman [CN07]. Another approach has been outlined by Smirnov in [Smi06] (details can be found in [Wer07]).

In general, many planar models from statistical physics are believed to exhibit conformal invariance at criticality. This has been proved in several cases, including the followings

- LERW (Loop Erased Random Walk) and UST (Uniform Spanning Trees) have a scaling limit corresponding to $SLE_2$ and $SLE_8$ (and are thus conformally invariant), see [LSW04a].

- The frontier of planar Brownian motion corresponds to $SLE_{8/3}$, [LSW01b] (which in particular implied Mandelbrot's conjecture).

- Level lines of discrete Gaussian Free Fields converge to $SLE_4$, [SchShe06].

- Smirnov recently proved conformal invariance for the Ising model ($SLE_3$) and its corresponding FK percolation ($SLE_{16/3}$); see [Smi06, Smi07].

## 1.3 Critical exponents

Convergence of percolation interfaces to $SLE_6$ on the triangular lattice enables one to prove the existence of certain critical exponents, as well as to compute their values. We will give two examples: the *one-arm* and *four-arms* exponents. For any $R > 1$, let $A_R^1$ be the event that the site 0 is connected to distance $R$ by some open path. Also let $A_R^4$ be the event that there are four "arms" of alternating color from the site 0 (which can be of either color) to distance $R$ (i.e. there are four connected paths, two open, two closed from

0 to radius $R$ and the closed paths lie between the open paths). Figure 1.2 represents two percolation configurations satisfying respectively the events $A_R^1$ and $A_R^4$.

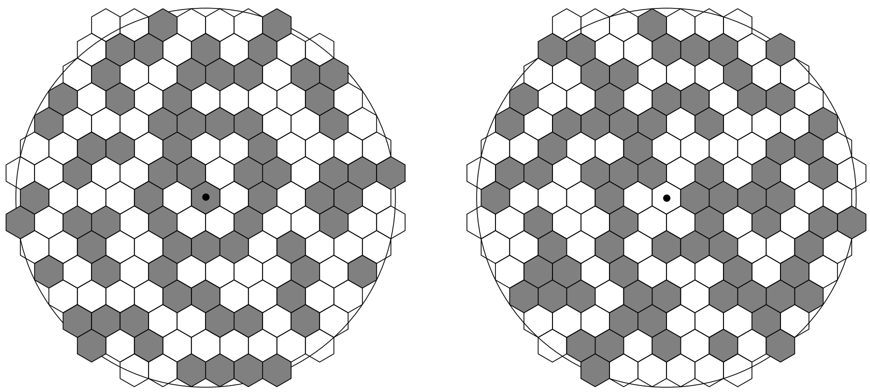


Figure 1.2: The configuration on the left satisfies the one-arm event, while the configuration on the right satisfies the four-arms event

It was proved in [LSW02] that the probability of the one-arm event decays like

$$\mathbb{P}\big[A_R^1\big] := \alpha_1(R) = R^{-\frac{5}{48}+o(1)}\,,$$

where $\frac{5}{48}$ is what we call a *critical exponent*.

For the four-arms event, it was proved by Smirnov and Werner in [SW01] that its probability decays like

$$\mathbb{P}\big[A_R^4\big] := \alpha_4(R) = R^{-\frac{5}{4}+o(1)}\,.$$

The four-arms event will be of key importance throughout this thesis. Indeed suppose the four arms event holds at some site $x \in \mathbb{T}$ up to some large distance $R$. This means that the site $x$ carries an important information about the large scale connectivities in the Ball $B(x,R)$. Changing the status of $x$ will drastically change the "picture" in $B(x,R)$. We call such a point a **pivotal point** up to distance $R$.

Using Kesten's scaling relations [Kes87], this implies (see [Wer07, Nol07]) for the density function $\theta(p)$ on the triangular lattice, the following behavior near $p_c = 1/2$

$$\theta(p) = (p-1/2)^{5/36+o(1)}\,,$$

when $p \to 1/2+$. This is part of the description of *near-critical* percolation.

As mentioned above, critical exponents provide information about the fractal properties of percolation at the scaling limit. For instance if one considers the one-arm exponent, this means that in average there are $R^{91/48+o(1)}$ sites in the square $[-R, R]^2$, which are part of a cluster of diameter at least $R$; since there are only "finitely" many such macroscopic clusters; this means that at the scaling limit, clusters of percolation are some random compact sets whose fractal dimension is a.s. $\frac{91}{48}$ (which can be proved rigorously).

A difficulty on the discrete level arises from the fact that the above probabilities are known at exponents level only. It is for instance not known whether $\alpha_1(R)/R^{-5/48}$ remains bounded or not.

We defined these events for critical triangular percolation, but we could have defined these events on $\mathbb{Z}^2$ as well; for instance we will often use the probability $\alpha_4(R)$ in the context of $\mathbb{Z}^2$. Some information is known about the decay of these rare events; for instance it is know that there are absolute constants $1 < \alpha < \beta < 2$, such that for $R$ large enough

$$R^{-\beta} < \alpha_4(R) < R^{-\alpha}.$$

Still the existence itself of the critical exponents for $\mathbb{Z}^2$ is not known.

## 1.4 Brief overview

The main body of this thesis will consist of four independent chapters:

- The expected area of the planar Brownian loop. In this first chapter, we show that the expected area enclosed inside a planar Brownian loop of time one equals $\frac{\pi}{5}$. In order to determine this expected area, one uses in an essential way the $\mathrm{SLE}_{8/3}$ process which is known to describe the "boundary of Brownian motion". This is one instance where it seems that one has to use SLE processes to determine quantities concerning Brownian motion that seem out of reach of the usual stochastic calculus techniques. This quantity $\pi/5$ has some consequences for fractal properties of the so-called Brownian Loop Soups introduced in [LW04].

- In our second chapter, we prove an analog of Makarov's theorem (about harmonic measure) for the $\mathrm{SLE}_\kappa$ processes. In other words, we study the possible size of the set $\partial D \cap \gamma$ for an SLE in a general domain $D$. We also prove that $\mathrm{SLE}_\kappa$ paths in arbitrary (simply connected) domains are a.s. continuous if $\kappa \in [0,8)$, which was known only for $\kappa \leq 4$ (this is not a trivial statement, because the boundary of a simply connected domain can be wild).

- The Fourier Spectrum of Critical Percolation. In this third chapter one obtains sharp results about noise sensitivity of percolation. Various applications of these results are derived for the model of dynamical percolation. This is the model where the percolation configuration evolves with time and the status of each site is updated after i.i.d. exponential times. One proves in particular that the set of exceptional times of dynamical critical site percolation on the triangular grid in which the origin percolates has dimension $31/36$ a.s. We also prove the existence of such exceptional times in the case of $\mathbb{Z}^2$ percolation.

- Scaling limit of near-critical and dynamical percolation. This last chapter is part of an ongoing project where we plan to prove that near-critical percolation and dynamical percolation, properly renormalized, have a scaling limit. We do not include a full proof of the existence of these scaling limits here but we provide two theorems (of sufficient independent interest to be stated alone) that are key steps in the larger project.

All these chapters are related to two-dimensional conformally invariant objects. The first two chapters are using and studying SLE processes. The last two chapters do not rely directly on SLE techniques, but they use some results that had been derived via SLE. We would like to emphasize that even though both chapters 3 and 4 are related to dynamical percolation, they are in fact completely independent of each other, and address quite different perspectives.

The rest of this introduction is structured as follows: We will first describe the first two chapters. These results can be stated without requiring any further background. But before describing the content of the final two chapters, we have chosen to propose a more detailed introduction to the mathematical objects (such as the Fourier spectrum) that are used there, as we thought that it could be useful to recall them in order to try to give a clearer picture of the results.

## 2 Expected area of the planar Brownian loop

Our first result, in joint work with *José Trujillo Ferreras*, concerns the expected area enclosed in a planar Brownian loop of time one. More precisely, let $B_t, 0 \leq t \leq 1$ be a planar Brownian loop (a Brownian motion in $\mathbb{C}$ conditioned so that $B_0 = B_1$). We consider the compact hull obtained by filling in all the holes of the Brownian loop, i.e. the complement of the unique unbounded connected component of $\mathbb{C} \setminus B[0,1]$. Let $\mathcal{A}$ denotes the area of that hull; in [GT06] we prove the following theorem

**Theorem 2.1.**

$$\mathbb{E}[\mathcal{A}] = \frac{\pi}{5}$$

The higher moments of the random variable $\mathcal{A}$ remain unknown at present. This work was motivated by the so called Brownian Loop Soups introduced in [LW04]; see also [Wer03, Wer05b] for links to $CLEs$ (Conformal Loop Ensembles) which are the natural candidates for the scaling limit of conformally invariant systems (such as Ising, Potts and so on).

More precisely, a Brownian Loop Soup of intensity $c > 0$ in some simply connected domain $\Omega \neq \mathbb{C}$, is a Poisson cloud of rooted Brownian loops (restricted to stay in $\Omega$) of intensity $c\mu^{\text{loop}}$, where the infinite measure $\mu^{\text{loop}}$ is defined as follows

$$\mu^{\text{loop}} := \int_{\mathbb{C}} \int_0^\infty \frac{dt}{2\pi t^2} \mu^\sharp(z,z,t) dt dA(z).$$

Here $\mu^\sharp(z,z,t)$ stands for the probability measure on Brownian loops of time duration $t$ rooted at $z$. For such a Brownian Loop Soup of intensity $c > 0$, let us consider the complement (in $\Omega$) of all the "filled" loops of the Soup. As explained in [Wer05b], this random set in $\Omega$ has the same "structure" as the model of Fractal Percolation introduced by Mandelbrot. With the analogy of Fractal Percolation in mind, if one is willing to compute the Hausdorff dimension of the complement of the Brownian Loop Soup (i.e. the set of points which are not surrounded by any loop), the quantity one needs to know is the first moment of the size of our objects (at some fixed level, say $t = 1$). This quantity is precisely what we computed. Using our result, it can be shown (see [Tha06]) that this dimension a.s. equals $2 - \frac{c}{5}$, where $c$ was the intensity of the Brownian Loop Soup (in particular, when the intensity is above 10, a.s. all the points in $\mathbb{C}$ are surrounded by some loop).

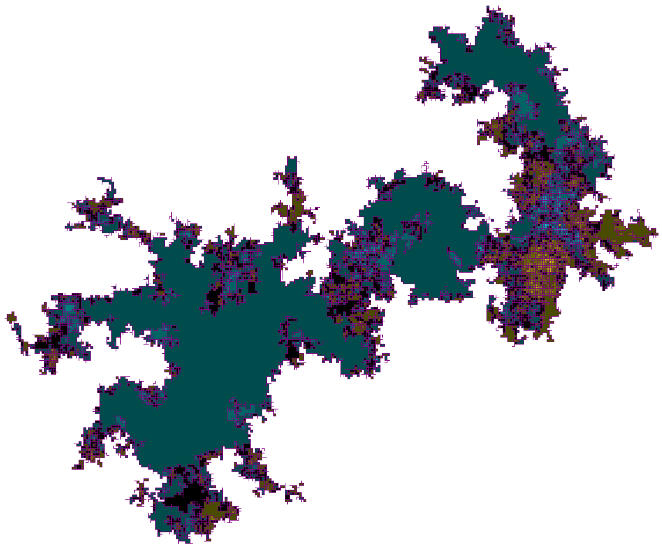


Figure 2.1: Different indices in a random walk of 50000 steps, black areas correspond to index 0.

The proof of Theorem 2.1 relies on $SLE_\kappa$ processes, and more precisely on $SLE_{8/3}$, which is known to describe (at least "locally") the boundary of Brownian hulls (see [LSW01b]). A natural approach to prove Theorem 2.1 without using $SLE$ technology would have been to use Yor's formula ([Yor80]) for the law of the index of a Brownian loop. Let $B_t, 0 \leq t \leq 1$ be a Brownian Loop; define for all $n \in \mathbb{Z} \setminus \{0\}$, $\Omega_n$ to be the (random) open set consisting of all the points in $\mathbb{C}$ which have index $n$ with respect to the loop $B([0,1])$. Let $\mathcal{W}_n$ denote the area of $\Omega_n$, i.e.

$$\mathcal{W}_n = \int_{\mathbb{C}} 1_{n_z = n} dA(z),$$

where $n_z$ is the index of $z$ with respect to the Brownian loop $B([0,1])$. Yor's formula gives the law of the index $n_z$ depending on the position $z$. By integrating this law over the plane $\mathbb{C}$ we found that for all $n \in \mathbb{Z} \setminus \{0\}$,

$$\mathbb{E}\big[\mathcal{W}_n\big] = \int_{\mathbb{C}} \mathbb{P}\big[n_z = n\big] dA(z) = \frac{1}{2\pi n^2}.$$

This result was already obtained in the physics literature ([CDO90]) using

Coulomb Gas methods. Since a point $z$ of index $n_z \neq 0$ is necessarily inside the filled Brownian loop, we have that $\sum_{n\neq 0} \mathcal{W}_n \leq \mathcal{A}$. The points which remain are the points of index 0 which are inside the Brownian Loop. Let $\mathcal{W}_0$ be the area of the points of zero index inside the Loop. Even though Yor's formula gives the probability that some fixed point $z$ is of index $n_z = 0$, it does not "see" whether that point is inside or outside the Brownian Loop (for instance, in this setting a distant point will have high probability to be of index 0). Since the proof of Yor's formula is based on martingales following the angle from the point $z$, there is no chance to add geometric information of the type inside/outside to it. This is why it seems that $SLE$ techniques are needed here. In [CDO90], Comtet, Desbois and Ouvry (who computed the expected area $\mathbb{E}\big[\mathcal{W}_n\big]$ for $n \neq 0$ with Coulomb gas) asked the question to compute the expected area of the points of zero index inside the loop (which is what we called $\mathbb{E}\big[\mathcal{W}_0\big]$). Combining the above results, we obtain

**Theorem 2.2.**

$$\mathbb{E}\big[\mathcal{W}_0\big] = \frac{\pi}{30}.$$

Figure 2.1 represents the different regions $\Omega_n$ colored differently depending on their index $n$. Note that if one wanted to compute the higher moments of $\mathcal{A}$, say the second moment, one of the ingredients needed would be to know the two-point function for $SLE_{8/3}$, i.e. given two points $z_1, z_2 \in \mathbb{H}$, what is the probability that the $SLE_{8/3}$ path is on their left; which is known to be a difficult question.

# 3 An SLE-analog of Makarov Theorem and the continuity of SLE paths in arbitrary domains

This chapter is joint work with *Steffen Rohde* and *Oded Schramm*.

Makarov's theorem on the support of harmonic measure claims that for any simply connected domain $\Omega \subsetneq \mathbb{C}$, there exists a set $E \subset \partial\Omega$ of Hausdorff dimension one such that for any $z \in \Omega$, a.s a Brownian motion starting at $z$ will exit $\Omega$ on $E \subset \partial\Omega$. One considers here the analogous situation for $\mathrm{SLE}_\kappa$ processes. For instance if one considers $\kappa = 6$, this can be described as follows. Let $\Omega \subsetneq \mathbb{C}$ be some simply connected domain and let $z \in \Omega$. Instead of starting a Brownian motion at $z$, we can think of "sending" a continuum critical percolation cluster at $z$, for instance by conditioning on the event of probability 0 that $z$ is connected to the boundary $\partial\Omega$ (it is possible to make sense of this conditioning, see for instance [Kes86]). Since the percolation cluster will hit the boundary at many places, we do not expect to find a set $E$ of dimension one which almost surely “absorbs” all the points on the boundary which are connected to $z$. Does one need the whole boundary to absorb the clusters ? We will prove that there is some absolute constant $1 < d < 2$ so that for any simply connected domain $\Omega$, there exists a set $E \subset \partial\Omega$ of Hausdorff dimension less than $d$ which almost surely absorbs on the boundary all the (macroscopic) percolation clusters in $\Omega$. See figure 3.1 for an illustration.

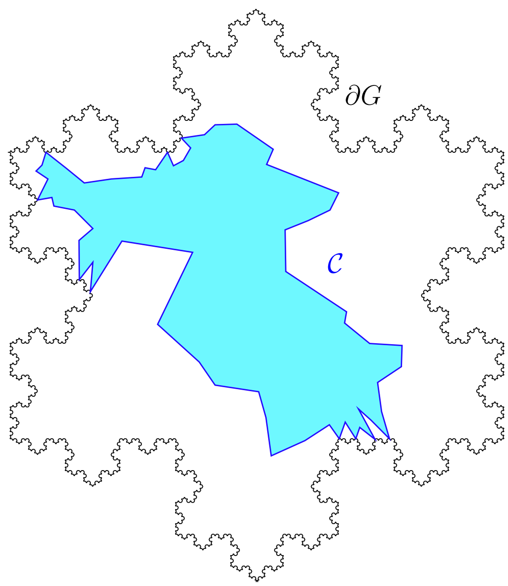



Figure 3.1: A schematic view of a percolation cluster $\mathcal{C}$ (or an $\mathrm{SLE}_6$ hull) inside a fractal domain $\Omega$; the blue curve represents the exterior boundary of the cluster.

In the general case of $\mathrm{SLE}_\kappa$ processes, we run an $\mathrm{SLE}_\kappa$ path in some domain $\Omega$ (say a radial $\mathrm{SLE}_\kappa$ from a point $z \in \Omega$ to some prime end of $\Omega$) and we wonder to what extend the $\mathrm{SLE}_\kappa$ path “enters” the fjords of $\Omega$. We prove the following result

**Theorem 3.1.** *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain, let $a, b$ be two prime ends of $G$, let $z_0 \in \Omega$, and let $\kappa \in (4, 8)$. Then there is a Borel set*

*$E \subset \partial\Omega$ such that the chordal* $\mathrm{SLE}_\kappa$ *trace in $\Omega$ from $a$ to $b$ and the radial* $\mathrm{SLE}_\kappa$ *trace in $\Omega$ from $a$ to $z_0$ almost surely satisfy*

$$\gamma(0,\infty) \cap \partial\Omega \subset E,$$

*and*

$$\dim E \leq d(\kappa) < 2\,,$$

*where $d(\kappa)$ is a constant that depends only on $\kappa$.*

One also shows that the theorem cannot hold with $d(\kappa) = 1$. Furthermore we obtain some quantitative estimates on the dimension $d(\kappa)$; in particular on has that $\lim_{\kappa\to 4} d(\kappa) = 1$.

The techniques used for proving this result allow us to answer a related question about $\mathrm{SLE}_\kappa$ processes, namely the continuity of their paths in arbitrary domains. More precisely, let $\Omega \subsetneqq \mathbb{C}$ be some domain and let $a, b$ be two prime ends of $\Omega$. Let $f : \mathbb{H} \to \Omega$ be some conformal map such that 0 is send to the prime end $a$ and $\infty$ is send to the prime end $b$. The $\mathrm{SLE}_\kappa$ path in $\Omega$ from $a$ to $b$ is defined as the image by $f$ of the $\mathrm{SLE}_\kappa$ path in $\mathbb{H}$. Without restrictions on the domain $\Omega$, one cannot prolong $f$ by continuity to $\overline{\mathbb{H}}$. Since for $\kappa > 4$ the $\mathrm{SLE}_\kappa$ path in $\mathbb{H}$ intersects the boundary, in order to prove that its image in $\Omega$ is still a continuous curve, one needs to show that somehow the SLE path in $\mathbb{H}$ avoids the boundary points where the conformal map $f$ "explodes" (this is still a naive picture since there exist domains $\Omega$ for which the conformal map $f : \mathbb{H} \to \Omega$ cannot be prolonged anywhere at the boundary). We prove the following theorem

**Theorem 3.2.** *Let $\Omega \subsetneqq \mathbb{C}$ be a simply connected domain, let $a, b$ be two prime ends of $G$, let $z_0 \in G$, and let $\kappa \in [0, 8)$. Then the chordal* $\mathrm{SLE}_\kappa$ *trace in $\Omega$ from $a$ to $b$ and the radial* $\mathrm{SLE}_\kappa$ *trace in $\Omega$ from $a$ to $z_0$ are a.s. continuous in $(0,\infty)$.*

Of course this result was already known for $0 \leq \kappa \leq 4$, where the $\mathrm{SLE}_\kappa$ paths are simple curves which do not intersect the boundary.

These results which concern general properties of $\mathrm{SLE}_\kappa$ processes were partly motivated by the following situation. Schramm and Smirnov prove in [SS] that the scaling limit of percolation can be seen as a two-dimensional black noise in the sense of Tsirelson (see [Tsi04]). Being a noise means that if $A$ and $B$ are two smooth open sets, then all the information about the connectivities of the continuum percolation inside $A$ ($\mathcal{F}_A$) plus all the information about the connectivities inside $B$ ($\mathcal{F}_B$) is enough to "reconstruct" all the connectivities inside $\overline{A \cup B}$. This means that the filtration of the percolation process (at the scaling limit) is "factorisable". It is known ([Tsi04]) that black noises are not factorisable as much as whites noises are. In this particular context of percolation, one can illustrate this non-factorisability by asking what is the situation if the open sets $A$ and $B$ are not assumed to be smooth. If one wanted to "glue" the information from $\mathcal{F}_A$ and $\mathcal{F}_B$ "cluster by cluster", one would need to know by how much percolation clusters enter the fjords of a possibly fractal domain $A$ (and $B$), which is linked to the above Theorem 3.1. More precisely there is a theorem about harmonic measure by Bishop, Carleson, Garnett and Jones ([BCGJ89], see also [Roh91]) which shows that there are curves $\gamma$ for which harmonic measure seen from one side and harmonic measure seen from the other side are singular measures. By analogy the same techniques used for the above theorems imply that for any $\kappa \in (4, 8)$ there exists some domain $\Omega = \Omega(\kappa)$ and a set $E \subset \partial\Omega$ such that if $\gamma_1$ and $\gamma_2$ are respectively $\text{SLE}_\kappa$ curves driven inside and outside $\Omega$, then a.s. $\gamma_1(0, \infty) \cap \partial\Omega \subset E$ while $\gamma_2(0, \infty) \cap \partial\Omega \subset E^c$. Applied to $\kappa = 6$, it means that there are some domains $\Omega$ for which (at the continuum limit) inside clusters are invisible to outside clusters.

# 4 The Fourier Spectrum of critical percolation

Before explaining our results in the context of percolation, we give a short survey of noise sensitivity of Boolean functions, and review earlier results on dynamical percolation. The reader acquainted with this notion may wish to skip these subsections.

## 4.1 Noise sensitivity of Boolean functions

Let us start with an example. Imagine that we are trying to study the sensitivity of an election's outcome to small errors in the counting of votes (or said differently, to small level of noise). For simplicity suppose there are only two candidates (+1 and -1) and that each voter independently makes a choice for +1 or -1 uniformly at random. An election scheme corresponds to some Boolean function $f$ from $\{-1, 1\}^n$ to $\{-1, 1\}$, where $n$ is the number of voters. One can assume furthermore that the election scheme is well balanced in the sense that the outcome $+1$ is as likely as -1 (in other words $\mathbb{E}\big[f\big] = 0$). The small level of noise (or errors) is described as follows : suppose that independently for each voter, an error has occurred with probability $\epsilon$, where $\epsilon \in (0,1)$ is some fixed constant. This means that independently for each voter, with probability $\epsilon$ the vote is misrecorded ($+1$ changed into -1 and vice-versa). The sensitivity of the election scheme $f$ corresponds here to the probability that the outcome is affected by the errors. For example simple majority will be less sensitive to noise than several-layers majority (as in the US).

More formally we will work with Boolean functions $f : \{-1,1\}^n \to \{-1,1\}$ (a Boolean function is often written from $\{0,1\}^n$ to $\{0,1\}$ but for symmetry reasons it will be more convenient for us to work from $\{-1,1\}^n$ to $\{-1,1\}$ and more generally from $\{-1,1\}^n$ to $\mathbb{R}$). The properties of Boolean functions are extensively studied in computer science as well as in many other fields (see [KS06] for example).

As motivated above, for some fixed Boolean function $f$ of $n$ bits, we will be primarily interested in the sensitivity of the function $f$ when the data is subjected to some "noise". In computer language one would ask: how robust is the function $f$ with respect to errors (say in the transmission of the data)? More precisely, let $f$ be some Boolean function from $\{-1,1\}^n$ to $\{-1,1\}$. Suppose the hypercube cube $\{-1,1\}^n$ is endowed with the uniform probability measure. The theory can easily be extended to other product measures on $\{-1,1\}^n$, but we will restrict ourself to this (already rich) case. Given a random configuration $x = (x_1,\ldots,x_n) \in \{-1,1\}^n$, let $y = (y_1,\ldots,y_n)$ be a random perturbation of $x$, in the sense that independently for every bit $i \in \{1,\ldots,n\}$, with probability $\epsilon$, $y_i = -x_i$ and with probability $1-\epsilon$, $y_i = x_i$. Here $\epsilon$ is some small fixed constant corresponding to the level of noise. The Boolean function $f$ will be noise sensitive if for all but a small fraction of the configurations $x$, even if one knows the initial data $x$, it will be hard to predict what will be the outcome $f(y)$. This can be encoded by the following quantity :

$$N(f,\epsilon) := \mathrm{var}\left[\mathbb{E}\big[f(y_1,\ldots,y_n) \,\big|\, x_1,\ldots,x_n\big]\right]. \tag{4.1}$$

We will be interested in the asymptotic $n$ goes to $\infty$.

*Definition* 4.1. Let $(n_m)_{m\in\mathbb{N}}$ be some increasing sequence in $\mathbb{N}$. A sequence of Boolean functions $f_m : \{-1,1\}^{n_m} \to \{-1,1\}$ will be called **asymptotically**

**noise sensitive** (or just noise sensitive) if for any $\epsilon > 0$,

$$\lim_{m\to\infty} N(f_m, \epsilon) = 0. \tag{4.2}$$

This can be rephrased by saying that asymptotically the initial data $(x_1, \ldots, x_{n_m})$ gives almost no information about the outcome $f(y_1, \ldots, y_{n_m})$.

The opposite situation corresponds to **noise stability**. A sequence of Boolean functions $f_m : \{-1,1\}^{n_m} \to \{-1,1\}$ is said to be (asymptotically) noise stable if

$$\sup_{m\geq 0} \mathbb{P}\big[f(x_1, \ldots, x_{n_m}) \neq f(y_1, \ldots, y_{n_m})\big] \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Of course noise sensitivity and noise stability are both extreme cases; there are many examples which lie in between. It is the same situation as in Tsirelson's theory of noises where Black and White noises are extreme cases.

In some contexts, other ways to measure sensitivity might be more natural, but in most cases, our measure of sensitivity $N(f, \epsilon)$ controls the other quantities. For instance, it is straightforward using Cauchy-Schwarz to check that for $f : \{-1,1\}^n \to \mathbb{R}$, we control the correlation

$$\big|\mathbb{E}\big[f(x)f(y)\big] - \mathbb{E}\big[f\big]^2\big| \leq \sqrt{N(f,\epsilon)}\sqrt{\mathrm{var}(f)},$$

which in the case of a balanced Boolean function with values in $\{-1,1\}$ can be rephrased into

$$\big|\mathbb{P}\big[f(x) \neq f(y)\big] - \frac{1}{2}\big| \leq \frac{1}{2}\sqrt{N(f,\epsilon)}.$$

The last expression is what we would be interested in about the outcome of some voting process.

It turns out that discrete harmonic analysis will provide very useful tools for the study of noise sensitivity.

## 4.2 Fourier analysis of Boolean functions and application to noise sensitivity

Let us start with an analogy with "classical" Fourier analysis. Say we have some real function $f$ on the circle in $L^2(\mathbb{R}/\mathbb{Z})$. Take some uniform random point $x$ on the circle. Let $y$ be $x$ plus some noise, for instance $y = x + \mathcal{N}(0, \epsilon^2)$ for some small $\epsilon > 0$. One wishes to predict the value of $f(y)$ knowing $x$. For instance if $f(x) = \sin(\pi 2^{100} x)$, one expects that for $\epsilon = 10^{-3}$, the sensitivity will be very high. In general the sensitivity of $f$ can be encoded by

$$N(f, \epsilon) = \mathrm{var}\left[\mathbb{E}\big[f(y)|x\big]\right]. \tag{4.3}$$

It is well known that the Fourier coefficients of $f$ provide information about the "regularity" of $f$. If the spectrum of $f$ is concentrated on small frequencies $f$ will be very regular and not sensitive to noise, while if $f$ has many high frequencies, the output $f(y)$ will be less predictable. One can easily compute $N(f, \epsilon)$ using the Fourier expansion $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2i\pi n x}$, indeed

$$\begin{aligned}
N(f,\epsilon) &= \mathbb{E}\big[[\mathbb{E}\big[f(y) \mid x\big] - \mathbb{E}\big[f\big]]^2\big] \\
&= \int_0^1 \left( \sum_n \hat{f}(n) \mathbb{E}\big[e^{2i\pi n y}|x\big] - \hat{f}(0) \right)^2 dx \\
&= \int_0^1 \left( \sum_{n\neq 0} \hat{f}(n) e^{2i\pi n x} \mathbb{E}\big[e^{2i\pi n \mathcal{N}(0,\epsilon^2)}\big] \right)^2 dx \\
&= \int_0^1 \left( \sum_{n\neq 0} \hat{f}(n) e^{2i\pi n x} e^{-2\pi^2 n^2 \epsilon^2} \right)^2 dx \\
&= \sum_{n\neq 0} |\hat{f}(n)|^2 e^{-4\pi^2 n^2 \epsilon^2} \;\; \text{since } \hat{f}(n) = \overline{\hat{f}(-n)}\,.
\end{aligned}$$

Therefore one can see from the above formula that high frequencies favor noise sensitivity.

One would like to follow the same approach for the study of Boolean functions. There is a well developed theory of Fourier analysis on the hypercube $\{-1,1\}^n$. Let us work in the more general case of the space $L^2(\{-1,1\}^n)$ of real functions from $n$ bits to $\mathbb{R}$, endowed with the inner product :

$$\begin{aligned}
\langle f, g \rangle &= \sum_{x_1,\ldots,x_n} 2^{-n} f(x_1,\ldots,x_n) g(x_1,\ldots,x_n) \\
&= \mathbb{E}\big[fg\big],
\end{aligned}$$

for the uniform probability measure on the hypercube. For any $S \subset \{1, 2\ldots, n\}$, let $\chi_S$ be the function on $\{-1,1\}^n$ defined for any $x = (x_1, \ldots, x_n)$ by

$$\chi_S(x) := \prod_{i\in S} x_i\,. \tag{4.4}$$

It is straightforward to see that this set of $2^n$ functions forms an orthonormal basis of $L^2(\{-1,+1\}^n)$. Thus, any function $f$ can decomposed as

$$f = \sum_{S \subset \{1,\ldots,n\}} \hat{f}(S)\, \chi_S,$$

where $\hat{f}(S)$ are the Fourier coefficients of $f$. They are sometimes called the **Fourier-Walsh** coefficients of $f$. They satisfy

$$\hat{f}(S) = \langle f, \chi_S \rangle = \mathbb{E}\big[f \chi_S\big].$$

Note that $\hat{f}(\emptyset)$ corresponds to the average $\mathbb{E}\big[f\big]$.

Of course one might find many other orthonormal basis for $L^2(\{-1,1\}^n)$, but there are many situations for which this particular set of functions $(\chi_S)_S$ arises naturally. First of all there is a well known theory of Fourier analysis on groups, theory which is particularly simple and elegant on Abelian groups (thus including our special case of $\{-1,1\}^n$, but also $\mathbb{R}/\mathbb{Z}$, $\mathbb{R}$ and so on). For the Abelian groups, what turns out to be relevant for doing harmonic analysis is the set $\hat{G}$ of characters of $G$ (i.e. the group homomorphisms from $G$ to $\mathbb{C}^*$). In our case of $G = \{-1,1\}^n$, the characters are precisely the functions $\chi_S$ indexed by $S \subset \{1,\ldots,n\}$ since obviously $\chi_S(x \cdot y) = \chi_S(x)\chi_S(y)$.

These functions also arise naturally if one performs simple random walk on the hypercube (equipped with the Hamming graph structure), since they are the eigenfunctions of the heat kernel on $\{-1,1\}^n$.

Last but not least, the basis $(\chi_S)$ turns out to be particularly adapted to our study of noise sensitivity. Indeed as we computed for the functions on the circle $\mathbb{R}/\mathbb{Z}$, if $f$ is any function in $L^2(\{-1,1\}^n)$, we have

$$\begin{aligned} N(f,\epsilon) &= \mathbb{E}\big[[\mathbb{E}\big[f(y) \bigm| x\big] - \mathbb{E}\big[f\big]]^2\big] \\ &= \mathbb{E}\big[[\sum_{S \subset \{1,\ldots,n\}} \hat{f}(S)\mathbb{E}\big[\chi_S(y) \bigm| x\big] - \hat{f}(\emptyset)]^2\big]. \end{aligned}$$

But it is easy to check $\mathbb{E}\big[\chi_S(y) \bigm| x\big] = \prod_{i \in S} \mathbb{E}\big[y_i \bigm| x_i\big] = (1-2\epsilon)^{|S|}$, by independence of the bits. Hence using the fact that for $S_1 \neq S_2$, $\chi_{S_1}$ and $\chi_{S_2}$ are orthogonal, one obtains

$$N(f,\epsilon) = \sum_{S \subset \{1,\ldots,n\},\, S \neq \emptyset} \hat{f}(S)^2 (1-2\epsilon)^{2|S|}. \tag{4.5}$$

Therefore, in the setting of Boolean functions, "high frequencies" will correspond to subsets $S$ of $\{1,\ldots,n\}$ with large cardinality. Parseval formula implies

$$\sum_S \hat{f}(S)^2 = \|f\|_2^2.$$

For any Boolean function $f:\{-1,1\}^n \to \mathbb{R}$, one defines its **spectral measure** on the subsets of $[n]$ to be

$$\mathbb{Q}_f\big[\mathscr{S}=S\big] = \mathbb{Q}\big[\mathscr{S}=S\big] := \hat{f}(S)^2,$$

where the "random" set $\mathscr{S}\subset[n]$ (note that $\mathbb{Q}$ is not necessarily a probability measure) will be called the **Fourier spectral sample** of $f$. In the particular case of a Boolean function $f$ with values in $\{-1,1\}$, since $\|f\|_2=1$ one ends up with a **spectral probability measure**,

$$\mathbb{P}_f\big[\mathscr{S}=S\big] = \mathbb{P}\big[\mathscr{S}=S\big] := \hat{f}(S)^2.$$

Note that there is a slight abuse of notation here, $\mathbb{Q}$ and $\mathbb{P}$ are not defined on the same probability space as $x\in\{-1,1\}^n$, so formally one should have used some other notations.

For any Boolean function $f$ (with values in $\mathbb{R}$), one can rewrite its sensitivity $N(f,\epsilon)$ in terms of its spectral measure in the following way

$$\begin{aligned} N(f,\epsilon) &= \sum_{S\subset\{1,\ldots,n\},\, S\neq\emptyset} \hat{f}(S)^2(1-2\epsilon)^{2|S|} \\ &= \sum_{k=1}^{n} \mathbb{Q}\big[|\mathscr{S}|=k\big](1-2\epsilon)^{2k}. \end{aligned}$$

For a Boolean function of $L^2$ norm one, this can be rewritten as

$$N(f,\epsilon) = \sum_{k=1}^{n} \mathbb{P}\big[|\mathscr{S}|=k\big](1-2\epsilon)^{2k} = \mathbb{E}\big[(1-2\epsilon)^{2|\mathscr{S}|}\big],$$

where $\mathbb{E}$ denotes here the expectation with respect to the Fourier spectral sample $\mathscr{S}$. Therefore it shows that a sequence of Boolean functions $(f_m)$ (with values in $\{-1,1\}$) will be asymptotically noise sensitive if and only if, the spectral measures $(\mathbb{P}_{f_m})$ are supported on larger and larger sets leaving no mass on "finite frequencies" (except possibly $\emptyset$). More precisely

**Proposition 4.2.** *A sequence of Boolean functions $f_m$ from $\{-1,1\}^{n_m} \to \{-1,1\}$ is asymptotically noise sensitive if and only if for any finite $N > 0$,*

$$\lim_{m\to\infty} \mathbb{P}_{f_m}\big[0 < |\mathscr{S}| < N\big] = 0.$$

Therefore the distribution of the size of the Spectrum sample encloses all the information we need to study the noise sensitivity of $f$, one might then be tempted to restrict to the information about its size (or cardinal), but it will turn out to be very helpful in Chapter V to think of $\mathscr{S}$ "geometrically".

## 4.3 Some simple examples of Boolean functions

- Let us start with an example linked to the above situation of voting schemes. For any odd integer $n \geq 1$, we define the majority function $\mathrm{MAJ}_n$ on the hypercube $\{-1,1\}^n$ (still with uniform probability measure) in the following way: for any $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$ define
$$\mathrm{MAJ}_n(x) = \mathrm{sign}(\sum_i x_i).$$
For any level of noise $\epsilon > 0$, if one think of $x_1 + \ldots + x_n$ as a simple random walk on $\mathbb{Z}$ of $n$-steps, $y = (y_1, \ldots, y_n)$ will be an $\epsilon$-noised version of the $x$-random walk, and therefore for large $n$, $\frac{1}{\sqrt{n}}(x_1, \ldots, x_n)$ and $\frac{1}{\sqrt{n}}(y_1, \ldots, y_n)$ will be approximately $\sqrt{\epsilon}$ close. So if $x$ is such that $|x_1 + \ldots + x_n| > 100\sqrt{\epsilon}$ (which happens with high probability if $\epsilon$ is small), one can predict $f(y)$ with good accuracy. This implies that the Majority function is (asymptotically) noise stable.

  One can actually exactly compute in this case the distribution of the sizes of the Spectral sample. Let us look at the first level ($|\mathscr{S}| = 1$) of the Fourier distribution. For any bit $i \in \{1, \ldots, n\}$, $\mathbb{P}\big[\mathscr{S} = \{i\}\big] = \mathbb{E}\big[\mathrm{sign}(x_1 + \ldots + x_n)x_i\big]^2$. It is obvious that the only contribution to the expectation comes from the configurations $x$ where $x_1 + \ldots + x_n = \pm 1$, which asymptotically is of probability $\frac{2}{\sqrt{2\pi n}}$. This gives $\mathbb{P}\big[\mathscr{S} = \{i\}\big] = \frac{2}{\pi n} + o(\frac{1}{n})$, hence $\mathbb{P}\big[|\mathscr{S}| = 1\big] = \frac{2}{\pi} + o(1)$. One sees that asymptotically, a positive fraction is concentrated on level-1 Fourier coefficients; all odd levels $k \geq 1$ asymptotically receive positive mass, but the mass does not spread (as $n$ goes to infinity) to $\infty$ (Majority function under reasonable hypothesis is in some sense the most stable Boolean function).

- The Parity function $\mathrm{PAR}_n$: let $n \geq 1$ and consider the function which returns 1 if among the $n$ bits, there are an even numbers of $-1$, and $-1$ else. Therefore the Parity functions can be written for any $x = (x_1, \ldots, x_n) \in \{-1,1\}^n$ as

$$\mathrm{PAR}_n(x) = \prod_{i=1}^{n} x_i = \chi_{\{1,2,\ldots,n\}}(x).$$

Therefore in this special case, the Spectral probability measure is concentrated on the singleton $\delta_{\{1,\ldots,n\}}$. It is the most noise sensitive ("higher frequency") Boolean function one can find on the hypercube (it is intuitively clear that choosing such or such candidate according to "parity" would not be a satisfying voting scheme..).

- Now we turn to the Boolean functions we will be mainly interested in in Chapter V, i.e. crossing events (radial or Left-right crossing in a rectangle) in critical 2-d percolation. For example if one considers $\mathbb{Z}^2$ percolation at $p_c = 1/2$, one can look at the rectangle $n \times (n+1)$. We consider the Boolean functions $f_n$ on the edges of this square, which returns 1 if there is a Left-Right crossing, -1 else. By duality the probability of Left-Right crossing for such a rectangle is 1/2, which makes our Boolean function balanced. We would like to understand how noise sensitive is percolation, or more exactly how its connectivities, clusters and so on are disturbed under noise. If one wanted to compute the Fourier coefficients of $f_{10}$, since there are about 200 bits concerned, one would need to compute about $2^{200}$ Fourier coefficients. There is at present no known way to compute Fourier coefficients of percolation or even sample a spectral set (according to the spectral probability measure) using say Monte-Carlo methods (compare for instance with the situation of $SAW$ where it is hard to count the number of Self Avoiding Walks, but at least it is possible using pivot algorithm to sample them). We computed below the Fourier-Walsh expansion of $f_1$ (there would already be about $2^{13}$ terms for $f_2$).

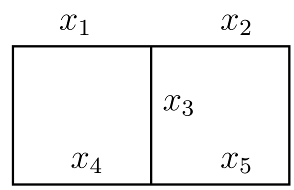



Figure 4.1: Variables for the function $f_1$.

$$\begin{aligned} f_1(x_1, \ldots, x_5) &= \frac{1}{2^5}(12\chi_1 + 12\chi_2 + 4\chi_3 + 12\chi_4 + 12\chi_5) \\ &+ \frac{1}{2^5}(-8\chi_{1,2} + 8\chi_{1,4} + 8\chi_{2,5} - 8\chi_{4,5}) \\ &+ \frac{1}{2^5}\left\{ \begin{array}{l} -4\chi_{1,2,3} - 4\chi_{1,2,4} - 4\chi_{1,3,4} + 4\chi_{2,3,4} - 4\chi_{1,2,5} \\ +4\chi_{1,3,5} - 4\chi_{2,3,5} - 4\chi_{1,4,5} - 4\chi_{2,4,5} - 4\chi_{3,4,5} \end{array} \right. \\ &+ \frac{4}{2^5}\chi_{1,2,3,4,5} \end{aligned}$$

Which makes $\mathbb{P}\big[|\mathscr{S}| = 1\big] = \frac{592}{2^{10}} \approx 0.58$, $\mathbb{P}\big[|\mathscr{S}| = 2\big] = \frac{252}{2^{10}} = 1/4$, $\mathbb{P}\big[|\mathscr{S}| = 3\big] = \frac{160}{2^{10}} \approx 0.156$ and $\mathbb{P}\big[|\mathscr{S}| = 5\big] = \frac{16}{2^{10}} \approx 0.016$.

## 4.4 Results previously obtained about noise sensitivity of percolation

The study of this problem originated in the seminal paper [BKS99]. They proved that the crossing event of an $n \times (n+1)$ square in $\mathbb{Z}^2$ indeed is noise sensitive. More precisely they proved the following Theorem

**Theorem 4.3.** *If $f_n$ corresponds to the indicator function (in $\{-1, 1\}$) of the left-right crossing of an $n \times (n+1)$ rectangle, then for any fixed $N > 0$,*

$$\lim_{n\to\infty} \mathbb{P}_{f_n}\big[0 < |\mathscr{S}_{f_n}| < N\big] = 0. \tag{4.6}$$

The main ingredient of their proof is the hypercontractivity of the noise operator (analog of the Ornstein-Uhlenbeck semigroup which associates to the Boolean function $f(x)$ its conditional expectation $\mathbb{E}\big[f(y) \mid x\big]$). In that paper the authors raised the question to know by how much percolation is sensitive to noise. More precisely instead of fixing the noise level at some fixed $\epsilon > 0$, the amount of noise could decrease to zero with the size of the system. One therefore considers some sequence $(\epsilon_n)$ going to zero and we look at $N(f_n, \epsilon_n)$. Benjamini, Kalai and Schramm asked the following question

*Question* 4.4. Does $N(f_n, n^{-\beta})$ go to zero for some exponent $\beta > 0$ ?

This is equivalent to the question whether $\mathbb{P}_{f_n}\big[0 < |\mathscr{S}| < n^\beta\big]$ converges to zero or not (one is interested in the speed at which the mass of the Spectral measure spreads to infinity). In their paper [BKS99], hypercontractivity techniques already implied that percolation is at least $\epsilon_n = \frac{c}{\log n}$ sensitive, for some constant $c > 0$.

The question was answered positively by Schramm and Steif in [SS05]. They proved the following Theorem

**Theorem 4.5.** *There exists some exponent* $\gamma > 0$ *so that*

$$\lim_{n\to\infty} \mathbb{P}_{f_n}\big[0 < |\mathscr{S}_{f_n}| < n^\gamma\big] = 0.$$

This is equivalent to the fact that the sensitivity $N(f_n, n^{-\gamma})$ goes to zero when the size of the rectangle goes to infinity. In the case of critical percolation on the triangular grid, based on the knowledge of the critical exponents, they obtain quantitative estimates for the sensitivity of crossing events. If $g_n$ denotes the indicator function (in $\{-1, 1\}$) of a left-right crossing in a domain approximating the square of sidelength $n$ (one could also choose a shape more adapted to the triangular grid, for instance a rhombus of sidelength $n$), they prove

**Theorem 4.6.** *For all* $\gamma < 1/8$*,*

$$\lim_{n\to\infty} \mathbb{P}_{g_n}\big[0 < |\mathscr{S}_{g_n}| < n^\gamma\big] = 0.$$

In fact they obtain more refined results about the Spectral measures, since they manage to control the lower tail of the spectrum, which will turn out to be the key thing in the study of dynamical percolation.

Their proof uses very different techniques from [BKS99]. They observed the interesting phenomenon that if a Boolean function can be computed with some randomized algorithm so that every bit is unlikely to be read, then the Boolean function will be noise sensitive (to some extent depending on how "efficient" is the algorithm). If one thinks of the majority function above (which we have seen to be stable), it is intuitively clear that if one wants to compute the outcome of the election by looking at the votes one by one in any random way, we will have to wait at least $n/2$ before knowing the outcome; hence there is no algorithm which computes the Majority function and satisfies that every bit is unlikely to be read. Their techniques are valid for any Boolean function and are thus not restricted to the study of percolation.

Let us conclude this subsection by stressing that if the noise decreases very quickly as the size of the systems goes to infinity, then the noise will have almost no effect on the connectivities of percolation. More precisely let us consider $f_n$, the $\pm 1$ indicator function of the left-right crossing of the $n \times (n+1)$ rectangle in $\mathbb{Z}^2$. Let $\omega$ be an i.i.d. configuration on the set of edges $E_n$ of the rectangle $n \times (n+1)$ rectangle; for $\epsilon_n > 0$ let $\omega^{\epsilon_n}$ be an $\epsilon_n$ noised configuration of $\omega$. This means that there are $N \sim \mathcal{B}(|E_n|, \epsilon_n)$ random edges which are flipped. A way to produce $\omega^{\epsilon_n}$ from $\omega$ is as follows: independently for each edge $e \in E_n$, let $u_e$ be a uniform random variable on the unit interval. If $u_e < \epsilon_n$ flip the edge $e$ in the configuration $\omega$, else keep the same status for $e$. This additional randomness provides an ordering on the set of edges $S = \{e_1, \ldots, e_N\}$ which flip from $\omega$ to $\omega^{\epsilon_n}$. With $\omega_0 = \omega$, we define inductively for $0 \le i < N$, the configuration $\omega_{i+1}$ to be the configuration $\omega_i$ with the edge $e_{i+1}$ flipped. In particular we have $\omega_N = \omega^{\epsilon_n}$. Note that for all $0 \le i \le N$, $\omega_i$ has the law of i.i.d percolation on $E_n$ and the $i^{\text{th}}$ edge $e_i$ is distributed uniformly over $E_n$. Knowing the total number of switches $N$, we obtain that

$$
\begin{aligned}
\mathbb{P}\big[f_n(\omega) \neq f_n(\omega^{\epsilon_n}) \bigm| N\big] &\le \sum_{0 \le i < N} \mathbb{P}\big[f_n(\omega_i) \neq f_n(\omega_{i+1})\big] \\
&= \sum_{0 \le i < N} \mathbb{P}\big[e_i \text{ is pivotal for } \omega_i\big].
\end{aligned}
$$

But since for all $0 \le i < N$, $\omega_i$ follows i.i.d percolation and since $e_i$ is distributed uniformly on the rectangle, all these probabilities are equal and are easily seen to be of order $\alpha_4(n)$; note that there are boundary issues here, but it is a standard calculation to check that near-boundary and near-corner points have a negligible contribution. Therefore, one ends up with $\mathbb{P}\big[f_n(\omega) \neq f_n(\omega^{\epsilon_n})\big] \le O(1)\mathbb{E}\big[N\big]\alpha_4(n) = O(1)\epsilon_n n^2 \alpha_4(n)$. In particular one concludes that if the noise level $(\epsilon_n)$ asymptotically satisfies $\epsilon_n \ll \frac{1}{n^2\alpha_4(n)}$, then crossing events are noise stable. The natural conjecture was that there is a sharp threshold of sensitivity, in the sense that once "we start touching many pivotals", all the information is lost at the limit. In other words, if $\epsilon_n \gg \frac{1}{n^2\alpha_4(n)}$, then crossing events are noise sensitive. This will be part of our results described at the end of this section. On the triangular grid, we have seen that $\alpha_4(n) = n^{-5/4+o(1)}$, therefore the threshold of sensitivity should occur around $\epsilon_n = n^{-3/4+o(1)}$. Compare with the above theorem from [SS05] which showed that on the triangular lattice, crossing events are at least $n^{-1/8+o(1)}$ noise sensitive.

## 4.5 Other instrumental use of the Fourier spectrum

Before explaining in the next subsection how the study of noise sensitivity of percolation enables to understand dynamical percolation, let us briefly mention a few contexts where similar techniques were used.

- In [BKS03], it is proved that the lengths of the geodesics in First passage percolation have fluctuations (in variance) bounded by $O(n/\log(n))$, and therefore are different from gaussian fluctuations. The conjecture (still unsolved) being that the standard deviations for this model are in $n^{1/3}$. Prior to that paper it was known from Kesten ([Kes93]) that the fluctuations were less than $O(n)$, which did not rule out Gaussian behavior. Note that "noise sensitivity" techniques are used here for a different purpose, i.e. understanding the fluctuations of an asymptotic deterministic shape.

- In [FK96], it was proved that any monotone Boolean function of a Random Graph $G(n,p)$, $0 \le p \le 1$ necessarily has a sharp threshold around some critical value $p_c = p_c(n)$ (which could depend upon $n$). This means that for any monotone event $\mathcal{A}$, if one considers the function $f_n : p \mapsto \mathbb{P}_{n,p}\big[\mathcal{A}\big]$, then $f_n$ asymptotically has a "cut-off" shape. In other words any monotone event appears "suddenly" while increasing $p$. The proof of this deep result uses among other things the fact that the Total influence of any monotone event (or its energy) is necessarily large (which again follows from hypercontractivity). This statement (somehow uniformly over $p \in (0,1)$) combined with Russo's Lemma implies their result.

## 4.6 Dynamical percolation

Dynamical percolation consists of a natural dynamic on the space of percolation configurations, and more precisely it is a Markov process on these configurations. It is defined in an elementary way as follows: for any graph $G = (V, E)$, start with some initial percolation configuration $\omega_0$ (sampled for instance according to $\mathbb{P}_p$ where each edge is open with probability $p \in [0, 1]$), and let the status of each edge $e \in E$ evolve according to a Poisson clock of rate 1 (this means that independently for each edge, at rate 1 the status (open or closed) is resampled to be open with probability $p$ and closed with probability $1 - p$). Therefore, Dynamical percolation $(\omega_t)$ is a dynamic where at each fixed time the configuration one sees has the law of ordinary percolation, but the random bits determining the status of the edges follow random independent changes at uniform rate. The model was introduced by Häggström, Peres and Steif in [HPS97]. The main questions one faces are of the following type: does a property which holds almost surely for static percolation also holds at all times of the dynamic ? If the answer turns out to be negative, then along the dynamic there are exceptional times at which the property fails. Since the property was assumed to hold almost surely for static percolation, the set of these exceptional times necessarily has Lebesgue measure zero.

In [HPS97], the authors consider the general case of an infinite connected locally finite graph $G$. Let $p_c = p_c(G)$ denotes its critical value. Let $\mathcal{C}$ be the event that there exists an infinite cluster. They prove that except maybe at the critical point $p_c$, there are no exceptional times for the event $\mathcal{C}$. More precisely they prove that if $p > p_c$, then almost surely (under the Probability measure of the Markov process) the event $\mathcal{C}$ holds for all configurations $(\omega_t)_{t\geq 0}$. As well if $p < p_c$, then almost surely, the event $\neg\mathcal{C}$ holds at all times. Hence the study of dynamical percolation focused on the behavior of the dynamic at criticality. Still in [HPS97], the authors raised the question for bond percolation on $\mathbb{Z}^d$, $d \geq 2$ at the critical point $p_c(\mathbb{Z}^d)$. Using results obtained by Hara and Slade about percolation in high dimension ($d \geq 19$), and in particular the fact that the density of the infinite cluster $\theta_{\mathbb{Z}^d}(p)$ has a finite derivative at $p_c$ (i.e. $\theta_{\mathbb{Z}^d}(p) = \mathbb{P}_p\big[0 \leftrightarrow \infty\big] = O(p - p_c(\mathbb{Z}^d))$), they proved that at $p_c$, there are no exceptional times where an infinite cluster appears. In dimension $d = 2$, the situation is different since when one increases $p$ and passes the critical value $1/2$, the infinite cluster appears in some sense more suddenly ($\frac{d}{dp}\big|_{p_c} \theta_{\mathbb{Z}^2}(p) = \infty$). The question of the existence of exceptional times for infinite clusters on $\mathbb{Z}^2$ at $p_c$ remained open, but in [SS05], Schramm and Steif made a breakthrough contribution proving that there indeed exist such exceptional times on site percolation on the triangular grid (at $p_c = 1/2$). They proved the following Theorem.

**Theorem 4.7.** *Almost surely, the set of exceptional times* $t \in [0,1]$ *such that dynamical critical site percolation on the triangular lattice has an infinite open cluster is nonempty.*

*Furthermore, the Hausdorff dimension of the set of these exceptional times is an almost sure constant in* $[1/6, 31/36]$.

They conjectured that the dimension of these exceptional times is a.s. $31/36$.

Dynamical percolation is intimately related to noise sensitivity of percolation. Indeed, for dynamical percolation on the triangular grid, the configuration $\omega_{t+s}$ at time $t+s$ is an $\epsilon$-noise of $\omega_t$ with $\epsilon = \frac{1}{2}(1-\exp(-s))$; here the factor $1/2$ comes from the fact that we defined the dynamic by resampling each site at rate 1, instead of flipping each site at rate 1 (the first definition being more convenient for general graphs where $p_c \neq 1/2$). As often, it is easier to provide an upper bound for the Hausdorff dimension of the set of exceptional times. On the other hand, if for some event (of static probability 0) we want to prove that there exist exceptional times, then it usually goes through the determination of a positive lower bound for the dimension of the set of these exceptional times, which is the key part.

Let us explain in the above case of the triangular grid where the upper bound $31/36$ comes from. Let $\mathcal{E}$ denote the (random) set of exceptional times $t \in [0,1]$ for which there is an infinite cluster in $\omega_t$, we want to show that a.s. $\dim_H(\mathcal{E}) \leq \frac{31}{36}$. For any site $x$ in the triangular lattice, let $\mathcal{I}_x$ be the event that there is an open path from $x$ to $\infty$, and let $\mathcal{E}_x$ be the set of exceptional times $t \in [0,1]$ for which $x \overset{\omega_t}{\longleftrightarrow} \infty$. By definition we have that $\mathcal{E} = \cup_{x \in TG} \mathcal{E}_x$; since there are countably many lattice points, it is enough to show that the set $\mathcal{E}_0$ of exceptional times where the origin 0 is connected to infinity is almost surely of dimension less than $\frac{31}{36}$. For any large $n \geq 1$, let us partition the unit interval $[0,1)$ into $n$ intervals $I_k = [\frac{k}{n}, \frac{k+1}{n})$, $0 \leq k < n$. For any $0 \leq k < n$, we want to bound the probability that $\mathcal{E}_0 \cap I_k \neq \emptyset$. For this, notice that $\omega_{k/n}$ has the law of critical percolation (p=1/2); now define $\tilde{\omega_k}$ to be the set of open sites in $\omega_{k/n}$ plus all the sites that have switched (at least once) from closed to open during the interval $I_k$. Hence, by definition, for any $t \in I_k$ the percolation configuration $\omega_t$ is dominated by $\tilde{\omega_k}$. But it is easy to see that $\tilde{\omega_k}$ follows exactly the law of i.i.d. percolation with parameter $p = \frac{1}{2} + \frac{1}{4}(1 - e^{-1/n}) \leq \frac{1}{2} + \frac{1}{4n}$. Therefore the probability that there is some time $t \in I_k$ for which 0 is connected to $\infty$ is bounded by the probability that 0 is connected to $\infty$ for $\tilde{\omega_k}$. This is measured by the density function $\theta(\frac{1}{2} + \frac{1}{4n})$. Now, based on the knowledge of the critical exponents in the case of the triangular grid it is known that $\theta(p) = (p - 1/2)^{5/36 + o(1)}$, when $p \to p_c = 1/2$ (see for instance [Wer07]). In particular for any $\alpha > 0$ and $n$ large enough we have that for all $0 \leq k < n$, $\mathbb{P}\big[I_k \cap \mathcal{E}_0\big] \leq (\frac{1}{n})^{5/36 - \alpha}$. This implies that for $n$ large enough the expected number of $1/n$- intervals needed to cover the random set $\mathcal{E}_0$ is bounded by $n^{31/36 - \alpha}$ which (taking $n \to \infty$ and $\alpha \to 0$) proves that a.s. $\dim(\mathcal{E}) = \dim(\mathcal{E}_0) \leq \frac{31}{36}$. See [SS05] for more details.

On the other hand, just proving the existence of such exceptional times turns out to be a much harder task. Indeed, one needs to understand the correlations between configurations $\omega_t$ and $\omega_{t+\epsilon}$. In other words first moment estimates (as above) are enough to imply upper bounds, but for lower bounds one needs at least to control second moments type of estimates (which involves correlations).

Heuristically, if the percolation configuration $\omega_t$ moves very "fast" along the time dynamic $t$, then it will have more chances to create infinite paths at some exceptional times. In other words, if percolation turns out to be very noise sensitive, then the connection properties will decorrelate fast which will help infinite clusters to appear.

More mathematically, for any large radius $R > 1$, we introduce $Q_R$ to be the set of times where 0 is connected to distance $R$:

$$Q_R := \{t \in [0,1] : 0 \overset{\omega_t}{\longleftrightarrow} R\}.$$

Proving the existence of exceptional times boils down to proving that with positive probability $\cap_{R>0} Q_R \neq \emptyset$. Even though the sets $Q_R$ are not closed, with some additional technicality (see [SS05]), it is enough to prove that there is some $c > 0$ such that $\inf_{R>1} \mathbb{P}\big[Q_R \neq \emptyset\big] > c$. This can be done by introducing the amount of time $X_R$ where 0 is connected to distance $R$, more precisely we define

$$X_R := \int_0^1 1_{0 \overset{\omega_t}{\longleftrightarrow} R} \, dt.$$

Since by Cauchy-Schwarz,

$$\mathbb{P}\big[Q_R \neq \emptyset\big] = \mathbb{P}\big[X_R > 0\big] \geq \frac{\mathbb{E}\big[X_R\big]^2}{\mathbb{E}\big[X_R^2\big]},$$

(this is what "second moment method" is referring to) it remains to prove that there is some constant $C > 0$ such that for all $R > 1$, $\mathbb{E}\big[X_R^2\big] < C\mathbb{E}\big[X_R\big]^2$. Note that the second moment can be written

$$\begin{aligned}
\mathbb{E}\big[X_R^2\big] &= \iint_{\substack{0\leq s\leq 1\\ 0\leq t\leq 1}} \mathbb{P}\big[0 \overset{\omega_s}{\longleftrightarrow} R,\, 0 \overset{\omega_s}{\longleftrightarrow} R\big]\, ds dt \\
&\leq 2\int_0^1 \mathbb{P}\big[0 \overset{\omega_0}{\longleftrightarrow} R, 0 \overset{\omega_t}{\longleftrightarrow} R\big]\, dt\,.
\end{aligned}$$

Now let $f_R = f_R(\omega)$ be the indicator function of the event $\{0 \overset{\omega}{\longleftrightarrow} R\}$. $f_R$ can be seen as a Boolean function from the bits in the disk of radius $R$, $B(0,R)$ with values in $\{0,1\}$. One can compute the desired correlation in the following way

$$\begin{aligned}
\mathbb{P}\big[0 \overset{\omega_0}{\longleftrightarrow} R,\, 0 \overset{\omega_t}{\longleftrightarrow} R\big] &= \mathbb{E}\big[f_R(\omega_0) f_R(\omega_t)\big] \\
&= \mathbb{E}\Big[\Big(\sum_{S\subset B(0,R)} \hat{f}_R(S)\chi_S(\omega_0)\Big)\Big(\sum_{S\subset B(0,R)} \hat{f}_R(S)\chi_S(\omega_t)\Big)\Big] \\
&= \mathbb{E}\big[f_R\big]^2 + \sum_{\emptyset\not\subset S\subset B(0,R)} \hat{f}_R(S)^2 \exp(-t|S|) \\
&= \mathbb{E}\big[f_R\big]^2 + \sum_{k\geq 1} \mathbb{Q}\big[|\mathscr{S}| = k\big] e^{-kt}, \qquad (4.7)
\end{aligned}$$

where $\mathbb{Q}$ is the Spectral measure of $f_R$ (it is not a probability measure since $\|f_R\|_2 < 1$). By integrating over the unit interval this gives

$$\mathbb{E}\big[X_R^2\big] \leq 2\,\mathbb{E}\big[X_R\big]^2 + 2\sum_{k\geq 1} \frac{\mathbb{Q}\big[|\mathscr{S}| = k\big]}{k}.$$

Therefore, in order to obtain the desired second moment, one needs to control the Lower Tail of the size of the Spectrum. This was achieved in [SS05] allowing them to prove the existence of exceptional times on the triangular grid. As mentioned above, their control of the lower tail enabled them to obtain the lower bound of 1/6 for the Hausdorff dimension of the set of these exceptional times. In order to reach the upper bound of 31/36, sharp estimates are needed for the spectrum and in particular its lower tail, which will be the results we will describe in the next subsection.

## 4.7 Our contribution

These results are joint work with *Gábor Pete* and *Oded Schramm*.

The following statements hold both for the triangular and square lattices (and they do not rely on SLE techniques). For any $n \geq 1$, $f_n$ will denote the left-right crossing of the $n \times (n+1)$ rectangle if we are on $\mathbb{Z}^2$, and the left-right crossing of a domain approximating the square of sidelength $n$ if we work on the triangular grid. Similarly, $\alpha_4(n)$ will be the probability to observe 4 alternate arms from the origin to distance $n$ on the lattice we are considering.

We have seen above that if $\epsilon_n n^2 \alpha_4(n)$ goes to zero, then the crossing events are noise stable. This means that if $y^n$ is an $\epsilon_n$-noise of $x^n$, then $\mathbb{P}\big[f_n(y^n) \neq f_n(x^n)\big]$ goes to zero. We prove that the transition from stability to sensitivity is sharp:

**Theorem 4.8.** *If the noise level satisfies $\epsilon_n n^2 \alpha_4(n) \to \infty$, then*

$$\lim_{n \to \infty} N(f_n, \epsilon_n) = 0.$$

*In terms of correlations this means that if $y^n$ is an $\epsilon_n$-noise of $x^n$, then we have*

$$\mathbb{E}\big[f_n(y^n) f_n(x^n)\big] - \mathbb{E}\big[f_n\big]^2 \underset{n \to \infty}{\longrightarrow} 0.$$

This theorem is proved by showing that all of the "spectral mass" is concentrated around $n^2 \alpha_4(n)$; i.e. that for any function $\delta(n)$ going to zero (arbitrary slowly) we have that

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| < \delta(n)\, n^2 \alpha_4(n)\big] \underset{n \to \infty}{\longrightarrow} 0.$$

One in fact obtains more refined results about the spectral measure, in particular about its lower tail with the following theorem

**Theorem 4.9.** *The Spectral sample $\mathscr{S}_{f_n}$ of $f_n$ satisfies*

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| < r^2 \alpha_4(r)\big] \asymp \left(\frac{n}{r}\right)^2 \alpha_4(r, n)^2,$$

*for any $r \in [1, n]$ and $\asymp$ denotes equivalence up to multiplicative constants.*

We also prove an analog Theorem about the lower tail of the Spectral measure concerning the radial event, and it is really this radial control that one then applies to dynamical percolation. Note that in the above theorem our control of the lower tail of the spectrum is optimal (up to absolute constants).

In the case of the triangular lattice, using the knowledge of the critical exponents, one has the following variant for the lower tail written in terms of concentration around the mean.

**Proposition 4.10.** *For every* $\lambda \in (0,1]$*, one has*

$$\limsup_{n\to\infty} \mathbb{P}\big[0 < |\mathscr{S}_{f_n}| \le \lambda\, \mathbb{E}\big[|\mathscr{S}_{f_n}|\big]\big] \asymp \lambda^{2/3},$$

*where the constants involved in* $\asymp$ *are absolute constants.*

Note that if one would have followed the setup in [SS05] in order to prove these sharp bounds, we would have needed to find some algorithm for computing $f_n$ with an "optimal" **revealment** $\delta = \delta(n)$ (the revealment being the supremum over all bits of the probability that a bit is "asked" by the algorithm). In the context of our crossing event the general theorem they prove relating algorithms and sensitivity states that for any $k \ge 1$,

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| \le k\big] \le \delta(n) k^2. \tag{4.8}$$

It is clear that in order to evaluate the left right crossing $f_n$, any algorithm has to use at least $n$ sites (in fact with high probability it would need to use at least $n^\beta$ bits with some $\beta > 1$ since "shorter paths" in percolation have a fractal structure). In particular since there are $O(1)n^2$ bits involved, the revealment is necessarily at least $c/n$, for some constant $c > 0$. So the best one can hope, using 4.8 and some optimized algorithm, is to prove that for any $\phi(n) = o(\sqrt{n})$, $\mathbb{P}\big[0 < |\mathscr{S}_{f_n}| < \phi(n)\big]$ goes to zero (i.e. that the spectrum mass spreads at least at speed $\sqrt{n}$). But since we wanted to prove that the mass spreads at speed $n^2\alpha_4(n) = n^{3/4+o(1)}$, we had to use a completely different approach.

Our strategy focuses more on the spatial geometry of the spectral set $\mathscr{S}_{f_n}$. This allows us some additional freedom, for instance one can go beyond the classical setup of noise sensitivity by noising only some subset of the bits (think of a voting scheme where the counting of votes is safer in some state than in some other...). One can prove for example in the case of $\mathbb{Z}^2$, that if one noises only the vertical edges (at some level $\epsilon > 0$), then the crossing event is asymptotically noise sensitive. This situation can be pushed to its extreme (with $\epsilon = 1$), where one resamples (or switches) a fixed (deterministic) set of bits. This answers a conjecture from [BKS99]. Previous techniques could not handle this type of sensitivity with constraints.

As we have seen, one can think of $\mathscr{S}_{f_n}$ as a random subset of the $n \times [n+1]$ rectangle. Gil Kalai suggested to study the scaling limit of the rescaled spectral sample $\frac{1}{n}\mathscr{S}_{f_n}$. Combining Tsirelson's theory of noises and the proof by Schramm and Smirnov that percolation scaling limit can be seen as a noise ([SS]), it follows that $\frac{1}{n}\mathscr{S}_{f_n}$ indeed has a scaling limit. We prove the following theorem

**Theorem 4.11.** *In the setting of the triangular grid, the limit in law of $n^{-1}\mathscr{S}_{f_n}$ exists. It is a.s. a Cantor set of Hausdorff dimension 3/4.*

Note that $\mathscr{S}_{f_n}$ has many properties in common with the random set $\mathcal{P}_n$ of pivotals points for the left right crossing (for instance they have asymptotically the same dimension). However we would like to stress that they are very different from each other (this can be tested via large deviations); we in fact believe that they become asymptotically singular.

We now turn to the description of the results we could achieve concerning dynamical percolation using our sharp control of the spectrum. First of all, our results about the concentration of the Fourier spectrum in the case of $\mathbb{Z}^2$ allow us to prove the following theorem

**Theorem 4.12.** *A.s. there are exceptional times at which dynamical critical bond percolation on $\mathbb{Z}^2$ has infinite clusters, and the Hausdorff dimension of the set of such times is a.s. positive.*

One should point out here that [SS05] came quite close to proving this theorem; and "retrospectively" their techniques would have been sufficient to derive the existence of exceptional times on $\mathbb{Z}^2$.

In the case of the triangular grid, we prove the following theorem.

**Theorem 4.13.** *In the setting of dynamical critical site percolation on the triangular grid, we have the following almost sure values for the Hausdorff dimensions.*

1. *The set of times at which there is an infinite cluster a.s. has Hausdorff dimension* $31/36$.

2. *The set of times at which there is an infinite cluster in the upper half plane a.s. has Hausdorff dimension* $5/9$.

3. *The set of times at which an infinite occupied cluster and an infinite vacant cluster coexist a.s. has Hausdorff dimension at least* $1/9$ *(the conjectured dimension being* $2/3$*).*

The upper bounds were known from [SS05] but even the existence was proved only for the first item (with the lower bound of $1/6$). Note that in the third item, our lower bound does not match with the upper bound. This is due to the lack of monotonicity of the event under consideration, and monotonicity is used at a key step in our way to control the Spectrum.

# 5 Scaling limit of near-critical and dynamical percolation

Let us first stress that in spite of what its title might suggest, this chapter is not a continuation of the previous one and can be seen as a completely independent project. We will nevertheless state a result at the end which links one with the other.

This chapter is part of an ongoing project with *Gábor Pete* and *Oded Schramm*, where the goal is to prove that near-critical (or off-critical) percolation and dynamical percolation properly renormalized have a scaling limit. Even though we do not provide in this thesis a full proof of the existence of these scaling limits, we state and prove two results of independent interest that are essential bricks of this project. In this introduction, we will say some words about the general projects, and describe these two results.

## 5.1 Setup and background

For simplicity, we will restrict our study to the case of site percolation on the triangular grid. Let us first introduce the model of near-critical percolation. One often explains the phase transition in percolation by "increasing the level $p$". This corresponds to defining a natural coupling of percolation configurations $\omega_p$ for all "levels" $p \in [0,1]$ at the same time. A way to do so is to sample independently for all sites $x$ in the triangular grid $\mathbb{T}$ a uniform random variable $u_x$ on the unit interval. For any $p \in [0,1]$, let $\omega_p$ be the configuration corresponding to the set of points $x \in \mathbb{T}$ for which $u_x \le p$. Now, almost surely (under the law of the coupling), when one increases $p$, a (unique) infinite cluster appears from the moment that $p$ exceeds $p_c = 1/2$. We would like, using the recent understanding of critical percolation on $\mathbb{T}$, to understand "how" the infinite cluster suddenly emerges once $p > 1/2$. In other words, we would like to describe the "birth" of the infinite cluster. If one wants to use conformal invariance (i.e. $\mathrm{SLE}_6$), a natural idea is to consider the scaling limit of the whole coupling $(\omega_p)_{0\le p\le 1}$ when the mesh of the triangular lattice goes to zero. So one considers for example the sequence of couplings $(\omega_p^n)$ on the rescaled grids $\frac{1}{n}\mathbb{T}$. The problem one faces with this approach is that for any fixed $p < 1/2$, the connection probabilities of $\omega_p$ decay exponentially fast (i.e. there are some constants $C_1, C_2 > 0$ which depend on $p$, so that the probability that 0 is connected to distance $n$ within $\omega_p$ is less than $C_1 \exp(-C_2 n)$). This implies in particular that if we observe a rescaled subcritical configuration $\omega_p^n$ (on $\frac{1}{n}\mathbb{T}$) in the "window" $[0,1]^2$, then the largest clusters in $\omega_p^n$ will be of diameter $O(1)\frac{\log n}{n}$. Taking $n$ going to infinity we get a trivial limit. The same phenomenon happens in the opposite situation of the supercritical regime $p > 1/2$.

Therefore, if one wants to keep a meaningful transition from subcritical to supercritical regime, while we rescale the lattice, we also need to "slow down" our way to increase the level $p$ (in order to make the transition less "brutal"). So we will rescale our coupling in the following way: for all $n \ge 1$, consider the coupling $(\hat\omega_\lambda^n)_{\lambda\in\mathbb{R}}$ where the configuration $\hat\omega_\lambda^n$ stands for $\omega_p^n$ with $p = 1/2 + \lambda\delta(n)$. Here $\delta(n)$ is the speed at which one slows down the "increase in $p$"; for now on this is just some function going to zero. Note that we defined a coupling for all real values $\lambda \in \mathbb{R}$ (if $\lambda$ is such that $p = 1/2 + \lambda\delta(n) < 0$, then we define $\hat\omega_\lambda^n$ to be the empty configuration, as well for large values of $\lambda$, $\hat\omega_\lambda^n$ will be the full configuration). Using similar ideas as in our previous chapter it is possible to see that if $\delta(n)$ decays too fast, then we end up with a trivial coupling at the scaling limit where all configurations $\hat\omega_\lambda$ coincide with the critical configuration $\hat\omega_0$. This happens if and only if $\lim_{n\to\infty} \delta(n) n^2 \alpha_4(n) = 0$ (which severely reminds us the threshold of sensitivity...). On the other hand, if the "increase in $p$" is not slowed down enough (i.e. if $\delta(n)$ decays too slowly to 0) we also obtain a trivial coupling at the limit, the same that the one we obtained with the original coupling $(\omega_p)$ (i.e. without slowing down). This happens if and only if $\lim_{n\to\infty} \delta(n)\, n^2\alpha_4(n) = \infty$. See [NW08] for more details. This question of "scaling window" had in fact been studied extensively before in various contexts and many ideas in the percolation model go back to Kesten [Kes87].

To sum up the above discussion, if one wants to keep a nontrivial coupling at the scaling limit, we are forced to choose a speed $\delta(n)$ which satisfies $\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$.

In [NW08], the authors consider percolation interfaces (say for instance the standard one in $\mathbb{H}$ starting at 0) on the rescaled triangular lattice $\frac{1}{n}\mathbb{T}$ at the parameter $p_n = 1/2 + \delta(n)$, where $\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$. Let $\gamma^n$ denote the standard percolation interface on $\frac{1}{n}\mathbb{T}$ at $p = p_n$ starting at 0, and remaining in $\mathbb{H}$ until it exits the disk of radius 1. They prove that $(P_n)_{n\geq 1}$, the family of laws which govern the rescaled interfaces $\gamma^n$ is tight (for the topology induced by a well chosen metric on the space of interfaces). In particular there are subsequential scaling limits $\gamma^{n_k} \overset{\text{law}}{\to} \gamma$ converging to laws on continuous interfaces $\gamma$ on $\mathbb{H}\cap\mathbb{D}$. Their main theorem asserts that any such subsequential scaling limit is singular with respect to the $\text{SLE}_6$ measure on interfaces. This means that the "pictures" (at least the interfaces) one sees in the off-critical regime are different from the pictures one sees in the critical regime.

This implies that something "interesting" has to occur at the limit $n \to \infty$ in the only remaining case ($\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$) for our rescaled coupling configurations $(\hat\omega^n_\lambda)_{\lambda\in\mathbb{R}}$.

We will define in chapter VI a natural topology $\mathcal{T}$ on the space $\mathscr{H}$ of all percolation configurations. In order to work on the same space $\mathscr{H}$, both at the discrete and continuum levels, we will associate to a percolation configuration $\omega$ the set of all quads (or "tubes") which have a left-right crossing for $\omega$. So roughly speaking an element of $\mathscr{H}$ consists of a set of quads, see chapter VI for more precise definitions. It is proved in [SS] that the topological space $(\mathscr{H},\mathcal{T})$ is compact. The process one is interested in, i.e. the coupling of configurations $(\hat\omega^n_\lambda)_{\lambda\in\mathbb{R}}$, can be seen as a probability measure on the càdlàg processes $\mathbb{R} \to \mathscr{H}$ (càdlàg comes from our choice to consider some site $x$ open if and only if $u_x \leq p$ instead of $u_x < p$). We equip this space of paths with the topology of locally uniform convergence, let $\widehat{\mathcal{T}}$ denote this topology.

One of the main Theorems of our ongoing project states that in the interesting regime $\delta(n) \asymp \frac{1}{n^2\alpha_4(n)}$, there is essentially a **unique** (up to scaling) subsequential scaling limit for the coupling $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$. More precisely if $\delta(n) := \frac{1}{n^2\alpha_4(n)}$, we plan to establish the following:

**Theorem 5.1.** *The rescaled couplings of percolation configurations $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$ seen as random cádlág processes on $\mathscr{H}$ have a scaling limit when n goes to infinity. They converge in law under the topology of locally uniform convergence ($\widehat{\mathcal{T}}$) to a coupling of continuum percolation $(\hat{\omega}_\lambda)_{\lambda\in\mathbb{R}}$.*

*In particular, for any fixed level $\lambda \neq 0$, near-critical percolation configurations $\omega^n$ on $\frac{1}{n}\mathbb{T}$ at $p_n = 1/2 + \frac{\lambda}{n^2\alpha_4(n)}$ have a scaling limit.*

For dynamical percolation, if one wishes to rescale the lattice in order to obtain some scaling limit of dynamical percolation, for the same reason we need to slow down the time evolution. Indeed, if we would rescale the lattice while keeping the same rate for the poisson clocks on the sites, because of the sensitivity of percolation, one would obtain a scaling limit where at it each time $t \in \mathbb{R}$, one would "see" a completely independent copy of continuous percolation. Following the same discussion as for near-critical percolation, if we want to keep a nontrivial scaling limit, while we rescale the space by $n$, we also need to slow down the time by $\delta(n) \asymp 1/(n^2\alpha_4(n))$. More precisely for each $n \geq 1$, let $(\omega^n_t)_{t\geq 0}$ be a dynamical percolation on $\frac{1}{n}\mathbb{T}$, where each site $x \in \frac{1}{n}\mathbb{T}$ is updated according to a Poisson clock of rate $q_n := 1/(n^2\alpha_4(n)) = n^{-3/4+o(1)}$. As for near-critical percolation, we plan to prove:

**Theorem 5.2.** *The rescaled dynamical percolation processes $(\omega^n_t)_{t\geq 0}$ seen as random cádlág processes on $\mathscr{H}$ have a scaling limit when n goes to infinity. They converge in law under the topology of locally uniform convergence ($\widehat{\mathcal{T}}$) to a continuum dynamical percolation $(\omega_t)_{t\geq 0}$.*

Let us briefly explain how we plan to prove the existence of these scaling limits. We roughly follow a program proposed by Camia, Fontes and Newman in [CFN06]. Their idea was to construct the scaling limit of the entire near-critical coupling $(\hat{\omega}_\lambda)_{\lambda\in\mathbb{R}}$ as well as the scaling limit of dynamical percolation $(\omega_t)_{t\in\mathbb{R}}$, out of the critical slice $\hat{\omega}_0 = \omega_0$. Let us explain this program in the case of the scaling limit of near-critical percolation. In order to sample $\hat{\omega}_\lambda$ (for some level $\lambda > 0$, say) using $\hat{\omega}_0$, many "sites" will switch in some random way from closed to open. But since we are at the scaling limit, there are no "sites" any more. Nevertheless some sites are somehow still visible: the set $\mathcal{P}$ of all pivotal points. In [CFN06] the authors explain that it should be enough to follow the status of these pivotal points in order to follow "along $\lambda$" the configuration $\hat{\omega}_\lambda$. Note here that we follow the status of points which were "initially" (for the configuration $\hat{\omega}_0$) pivotal; it could be that the configuration $\hat{\omega}_\lambda$ "moves" in such a way that its set of pivotal points is not preserved; hence that part of the program needs some proof already. But even assuming that it is enough (and also makes sense) to follow the status of these initial pivotal points, one faces a difficulty: if $\mathcal{P}$ is the set of all the initial pivotal points (i.e. for $\hat{\omega}_0$), then for any $\lambda > 0$, in any given compact, say in the window $[0,1]^2$, infinitely many pivotal points in $\mathcal{P}$ will switch from closed to open between the configurations $\hat{\omega}_0$ and $\hat{\omega}_\lambda$! So it makes it hard to reconstruct the configuration $\hat{\omega}_\lambda$ out of the $\hat{\omega}_0$ configuration plus this "infinite" amount of information.

## 5.2 Results proved in Chapter VI

This is the reason why we introduce a "cut-off": instead of considering all the pivotal points at once, we only consider the pivotal points whose status matters at least up to distance $\epsilon$, for some $\epsilon > 0$. A point $x$ will be called $\epsilon$-important if the four-arms event is satisfied in $B(x, \epsilon)$. For any $\epsilon > 0$, let $\mathcal{P}_\epsilon$ denote the set of all pivotal points which are initially (for $\hat{\omega}_0$) at least $\epsilon$-important.

We will prove in chapter VI, that if we want to predict with good accuracy the "outcome" $\hat{\omega}_\lambda$, then it is indeed enough to follow the status of the $\epsilon$-important points $\mathcal{P}_\epsilon$, the cut-off $\epsilon$ being chosen small enough depending on the degree of "accuracy" we want. In order to prove this result, we will need to rule out configurations where "cascades of importance" happen; i.e. dynamics (in $\lambda$), where some points initially of very low importance get promoted to a much higher importance along the dynamic $\lambda$ and also switch their status (we do not follow the status of these initially low important points, so if they switch from closed to open, we do not "see" it). This "non-cascade" property is our first main theorem in Chapter VI.

Once we know that it is enough to follow the $\epsilon$-important points, we still need to find a way to sample which points in $\mathcal{P}_\epsilon$ will switch (in any compact set, only finitely many will switch, this was the purpose of our cut-off). As argued in [CFN06], this random set of points should be a certain "Poissonian" cloud over the set $\mathcal{P}_\epsilon$, under some measure, which on the discrete level would simply be the counting measure on the set $\mathcal{P}_\epsilon^n$ of $\epsilon$-important points (renormalized by $n^2\alpha_4(n)$). Therefore for any $\epsilon > 0$, if we have some continuum critical percolation $\omega$, we need to define a Borel measure $\mu^\epsilon = \mu^\epsilon(\omega)$ which is a natural analog of the counting measure on the discrete level. More precisely for any $n \geq 1$, let $\mu_n^\epsilon$ be the counting measure on the set of $\epsilon$-important points $\mathcal{P}_\epsilon^n$ renormalized by $n^2\alpha_4(n)$. Hence $\mu_n^\epsilon$ is defined as follows

$$\mu_n^\epsilon = \mu_n^\epsilon(\omega^n) = \frac{1}{n^2\alpha_4(n)} \sum_{x\in\frac{1}{n}\mathbb{T} \text{ is } \epsilon\text{-important}} \delta_x \, .$$

The following result is the second main theorem in chapter VI.

**Theorem 5.3.** *When the mesh $1/n$ vanishes, the random variable $(\omega^n, \mu_n^\epsilon)$ converges in law to some $(\omega, \mu^\epsilon)$, where $\omega$ is the scaling limit of critical percolation, and the Borel measure $\mu^\epsilon = \mu^\epsilon(\omega)$ is a measurable function of the continuum percolation $\omega$.*

Our proof can also be applied to other random objects concerning percolation, for instance one can prove that the counting measure on the exploration process (see figure 1.1) properly renormalized (by $n^2\alpha_2(n)$) converges in law to a "natural" parametrization of the $\mathrm{SLE}_6$ curve. The question of constructing a natural parametrization of $\mathrm{SLE}_\kappa$ curves is natural and was recently addressed by Lawler and Sheffield in [LS]. In our case we obtain natural parametrizations only for $\mathrm{SLE}_{8/3}$ and $\mathrm{SLE}_6$ curves, but with the nice feature that they arise from the discrete model as limits of the discrete counting measures.

## 5.3 Outlook

One should point out that this program not only leads to the proof of the scaling limits of near-critical and dynamical percolation, but that it also describes the limiting couplings. For example, this proof implies that the scaling limit of dynamical percolation $(\omega_t)_{t\geq 0}$ is a Markov process on the space of percolation configurations $\mathscr{H}$; this is obvious at the discrete level, but far from obvious once in the scaling limit.

Also, this allows us to use and apply the near-critical (or dynamical) percolation model in a very flexible way, for instance by making the rate of switching vary over the space. In this direction we prove in chapter VI that the limiting counting measure defined above has nice conformal covariance properties. Plugged into our setup, this will imply in our ongoing project a

Conformal Covariance structure for near-critical percolation. More precisely, we plan to prove that if $\hat{\omega}_\lambda$ is a near-critical percolation at level $\lambda$ in some domain $\Omega$, then if $f : \Omega \to \widetilde{\Omega}$ is some conformal map, $f(\hat{\omega}_\lambda)$ is a "generalized" near-critical configuration where the level $\tilde{\lambda}$ depends on the position in the following way: for any $z \in \Omega$, $\tilde{\lambda}(f(z)) = |f'(z)|^{-3/4}\lambda$.

In this work in progress, we plan to apply the above results to the following models which, as is well-known, are related to near-critical percolation:

- We will prove that Minimal Spanning Tree (defined on the triangular grid) has a scaling limit, is rotationally invariant (it is not believed to be conformally invariant), and we will describe some of its asymptotic properties.
- The front in gradient percolation has a scaling limit.
- The process of invasion percolation has a scaling limit

We would like to conclude by the following remark. As we argued, chapters V and VI are largely independent from each other. However, combining their results, implies that for dynamical percolation when $t$ goes to infinity, $\omega_t$ is getting independent of $\omega_0$. Roughly speaking, as time goes on, we forget the initial configuration.

By considering the percolation dynamic (at the scaling limit) as a process in the space $\mathscr{H}$, we can actually prove using the techniques of chapter V the following theorem.

**Theorem 5.4.** *The process $t \mapsto \omega_t$ is* **ergodic** *in the space of configurations $\mathscr{H}$.*

# Chapter III

# The expected area of the filled planar Brownian loop is $\frac{\pi}{5}$

Joint work with *José Trujillo Ferreras.*

Let $B_t, 0 \leq t \leq 1$ be a planar Brownian loop (a Brownian motion conditioned so that $B_0 = B_1$). We consider the compact hull obtained by filling in all the holes, i.e. the complement of the unique unbounded component of $\mathbb{C} \setminus B[0, 1]$. We show that the expected area of this hull is $\pi/5$. The proof uses, perhaps not surprisingly, the Schramm Loewner Evolution ($SLE$). Also, using Yor's result [Yor80] about the law of the index of a Brownian loop, we show that the expected areas of the regions of index (winding number) $n \in \mathbb{Z} \setminus \{0\}$ are $\frac{1}{2\pi n^2}$. As a consequence, we find that the expected area of the region of index zero inside the loop is $\pi/30$; this value could not be obtained directly using Yor's index description.

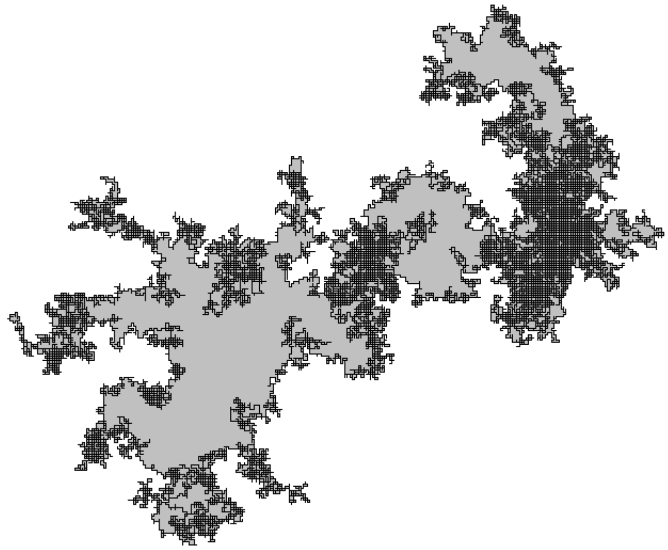


Figure 0.1: Random walk loop of 50000 steps and corresponding hull.

# 1 Introduction

In the abundant literature about planar Brownian motion, there are certainly results dealing with the question of area. Paul Lévy's stochastic area formula describing the algebraic area "swept" by a Brownian motion will likely come to the mind of many readers. Our result, however, is very different from this classical theorem, firstly because Levy's area is a signed area, but mainly because of the following : in order to apprehend Lévy's area it is enough to follow the Brownian curve locally without paying attention to the rest of the curve. In our case, one needs to consider the curve globally.

Aside from the fact that the question we address is a very natural one for Brownian motion, we have been motivated by related results in the Physics literature. In [Car94], using methods of conformal field theory, Cardy has shown that the ratio of the expected area enclosed by a self-avoiding polygon of perimeter $2n$ to the expected squared radius of gyration for a polygon of perimeter $2n$ converges as $n$ goes to infinity to $4\pi/5$. We note that self-avoiding polygons are supposed to have the same asymptotic shape as filled Brownian loops (see, for example, [Ric04] and references therein). However, studying this relationship is hard basically for the following reason. The boundary of the Brownian loop is of $SLE_{8/3}$-type, but, unfortunately, there does not exist a good way of "talking about the length" of $SLE$ curves at this moment.

Our result gives interesting information regarding the Brownian loop soups introduced in [LW04]. This conformally invariant object plays an important role in the understanding and description of $SLE$ curves (see, e.g. [LW04, Wer05a, LSW03]). It can be viewed as a Poissonian cloud (of intensity $c$) of *filled* Brownian loops in subdomains of the plane. Among other things, it is announced in [Wer03] that the dimension of the set of points in the complement of the loop soup (i.e. the points that are in the inside of no loop) can be shown to be equal to $2 - c/5$, using consequences of the restriction property. A detailed proof of this statement has never been published, and in fact, our result implies the corresponding first moment estimate (i.e. the mean number of balls of radius $\varepsilon$ needed to cover the set). The other arguments needed to derive the result announced in [Wer03] will be detailed in [Tha06].

Let us make precise what we mean by area enclosed by a Brownian loop. Let $B$ denote a Brownian bridge in $\mathbb{C}$ of time duration 1. I.e. the law of $B_t$, $0 \leq t \leq 1$ is the same as the law of $W_t - tW_1$, $0 \leq t \leq 1$, where $W$ is just a standard Brownian motion in $\mathbb{C}$. $\mathbb{C} \setminus B[0,1]$, i.e. the complement of the path, has a unique infinite connected component $H$. The hull $T$ generated by the Brownian loop is by definition $\mathbb{C} \setminus H$. Let $\mathcal{A}$ be the random variable whose value is the area of $T$. In this paper, we will prove

**Theorem 1.1.**

$$\mathbb{E}(\mathcal{A}) = \frac{\pi}{5}.$$

We would like to explain now how this result is related to the problem of windings of a Brownian loop. In [Yor80], Yor gave an explicit formula for the law of the index of a Brownian loop around a fixed point $z$. A point with a non-zero index has to be inside the loop. Using this fact, it is almost possible to describe the probability that a point is inside the loop, modulo the problem of the zero index; indeed, there are some regions inside the Brownian loop which are of index zero. It seems hard to control the influence of these zero-index points inside the curve. In the last section, using our main result, theorem 1.1, combined with the law of the index given by Yor, we find that the expected area of the set of points inside the loop that have index zero is $\frac{\pi}{30}$. We also compute the expected areas of the regions of index $n \in \mathbb{Z} \setminus \{0\}$.

In [CDO90], using physics methods, Contet, Desbois and Ouvry obtained the values of the expected areas for the non-zero regions. In their paper, they noted the different nature of the $n = 0$ sector (the points in the plane of zero index) and emphasized that "it would be interesting to distinguish in the $n = 0$ sector, curves which do not enclose the origin from curves which do enclose the origin but an equal number of times clockwise and anticlockwise". Their values in the case $n \neq 0$ agree with our results; they argue that the 0-case cannot be treated within the scope of their analysis.

From a probabilistic viewpoint, it also appears that usual techniques for Brownian motion are not strong enough to obtain the expected area of the Brownian loop or the expected area of the 0-index region inside the Brownian loop. However, the computation of the expected area of the $n$-index region for $n \neq 0$ was within reach using the result of Yor. To our knowledge this computation had not been carried out in a mathematical way before.

Let us briefly explain why usual techniques for Brownian motion seem unable to tackle the problem of the expected area of the Brownian loop. Basically, the enclosed area depends only on the boundary of the hull generated by the Brownian loop. The frontier of the Brownian loop concerns only a small subset of the time duration $[0, 1]$. In some sense, on certain time-intervals, the enclosed area does not depend much on the behavior of the Brownian motion. So, this problem needs a good description of the frontier of a Brownian loop. Recently, Lawler, Schramm and Werner proved a conjecture of Mandelbrot that the Hausdorff dimension of the Brownian frontier is 4/3. For this purpose they used the value of intersection exponents computed with the help of $SLE$ curves, see for instance [LSW01b] and references therein. The description of the Brownian frontier via $SLE$ can be done in a slightly different way using the conformal-restriction point of view, see [LSW03]. We will use this approach, and so will present to the reader the facts needed about conformal restriction measures in the next section.

The paper gives another striking example of a simple result concerning planar Brownian motion that seemed out of reach using the usual stochastic calculus approach, but that can be derived using conformal invariance and $SLE$. For a thorough account on $SLE$ processes, see [Law05, Wer04].

**Acknowledgments :** We wish to thank Greg Lawler and Wendelin Werner for suggesting the problem and for fruitful discussions, and Wendelin Werner for pointing out the link with the paper of Yor [Yor80].

# 2 Preliminaries

Conformal restriction measures in $\mathbb{H}$ are measures supported on the set of closed subsets $K$ of $\mathbb{H}$ such that $\overline{K} \cap \mathbb{R} = \{0\}$, $K$ is unbounded and $\mathbb{H} \setminus K$ has two infinite connected components, that satisfy the conformal restriction property : for all simply connected domains $H \subset \mathbb{H}$ such that $\mathbb{H} \setminus H$ is bounded and bounded away from the origin, the law of $K$ conditioned on $K \subset H$ is the law of $\Phi(K)$, where $\Phi$ is any conformal transformation from $H$ to $\mathbb{H}$ preserving 0 and $\infty$ (this law doesn't depend of the choice of $\Phi$). It is proved in [LSW03] that there is only one real parameter family of such restriction measures, $\mathbb{P}_\alpha$ where $\alpha \geq 5/8$. These measures are uniquely described by the following property : for all closed $A$ in $\mathbb{H}$ bounded and bounded away from 0,

$$\mathbb{P}_\alpha[K \cap A = \emptyset] = \Phi_A^{'}(0)^\alpha\,, \tag{2.1}$$

where $\Phi_A$ is a conformal transformation from $\mathbb{H} \backslash A$ into $\mathbb{H}$ such that $\Phi_A(z)/z \to 1$, when $z \to \infty$. To aid with the notation for the rest of the paper whenever we write $\Phi_A$ we will be assuming that we have chosen the translate with the additional property $\Phi_A(0) = 0$. $\mathbb{P}_{5/8}$ is the law of chordal $SLE_{8/3}$, and $\mathbb{P}_1$ can be constructed by filling the closed loops of a Brownian excursion in $\mathbb{H}$ (Brownian motion started at 0 conditioned to stay in $\mathbb{H}$). An important property of these conformal restriction measures is that using two independent restriction measures $\mathbb{P}_{\alpha_1}$ and $\mathbb{P}_{\alpha_2}$, we can construct $\mathbb{P}_{\alpha_1+\alpha_2}$ by filling the "inside" of the union of $K_1$ and $K_2$. This "additivity" property and the construction of $\mathbb{P}_{5/8}$ and $\mathbb{P}_1$ give the good description of the Brownian motion in terms of $SLE$ curves, namely, 8 $SLE_{8/3}$ give the same hull as 5 Brownian excursions.

Since, we want to describe the boundary of loops of time duration 1, we will first create loops with the use of the infinite hulls described above. Restriction measures are conformally invariant (Brownian excursion, $SLE_{8/3}$,..), so we had better use conformal maps. There is obviously no conformal map which sends both $\infty$ and 0 to 0, so the natural idea is to consider a Möbius transformation preserving $\mathbb{H}$ which maps 0 to 0, and $\infty$ to $\varepsilon$. We can choose

$$\mathrm{m}_\varepsilon(z) = \frac{\varepsilon z}{z+1}$$

$$\mathrm{m}_\varepsilon^{-1}(z) = \frac{z}{\varepsilon - z}\,.$$

The limit when $\varepsilon$ goes to zero of the measures $\mathrm{m}_\varepsilon(\mathbb{P}_1)$ is the dirac measure at $\{0\}$. The good renormalization to keep something interesting is in $\varepsilon^2$. Hence, we define the Brownian bubble measure in $\mathbb{H}$ as :

$$\mu^{\text{bub}} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{m}_\varepsilon(\mathbb{P}_1) \,.$$

This measure was introduced in [LSW03], and it is an important tool for studying the link between $SLE$ curves and the Brownian loop soup (see [LW04]). It was already noted in [LSW03, LSW04b], as an easy consequence of the "additivity" property described above, that

$$\frac{5}{8}\mu^{\text{bub}} = \frac{5}{8} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{m}_\varepsilon(\mathbb{P}_1) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathrm{m}_\varepsilon(\mathbb{P}_{5/8}) \,.$$

The last measure can be seen as an infinite measure on "$SLE_{8/3}$ loops", let us call this measure $\mu^{\text{sle}}$. Recall, that we are interested in a Brownian loop of time duration 1. We have the following time decomposition for $\mu^{\text{bub}}$, (see [LW04],[Law05])

$$\mu^{\text{bub}} = \int_0^\infty \frac{dt}{2\,t^2} \mathbb{P}_t^{\text{br}} \times \mathbb{P}_t^{\text{exc}} \,, \tag{2.2}$$

where $\mathbb{P}_t^{\text{br}}$ is the law of a *one-dimensional* Brownian bridge of time duration $t$, and $\mathbb{P}_t^{\text{exc}}$ is the law of an Itô Brownian excursion re-normalized to have time $t$. $\mathbb{P}_t^{\text{br}} \times \mathbb{P}_t^{\text{exc}}$ is the law of an $\mathbb{H}$-Brownian bridge of time duration $t$, by considering the one dimensional bridge as the $x$ coordinate of the curve, and the excursion as the $y$ coordinate. Unfortunately, it is hard to compute fixed-time quantities with $SLE$ techniques. Thus, we will compute a "geometric quantity" using $SLE_{8/3}$, and then extract $\mathbb{E}(\mathcal{A})$ from this geometric value by using the relation $\mu^{\text{bub}} = 8/5\mu^{\text{sle}}$ and the decomposition 2.2.

Let us explain in a few words why we need to deal with Brownian bridges in $\mathbb{H}$ and cannot work directly with bridges in $\mathbb{C}$. The underlying idea is the fact that one needs to choose a starting point on the boundary of the Brownian loop for the $SLE$ loop representation. A natural choice is the (almost surely) unique lower point, this is why we are interested in $\mathbb{H}$ quantities. So let $\mathcal{A}^{\mathbb{H}}$ be the random variable giving the area of an $\mathbb{H}-$Brownian bridge of time duration one. Working with $\mathcal{A}^{\mathbb{H}}$ will turn out not to be a problem since, as the reader might already suspect, the random variables $\mathcal{A}$ and $\mathcal{A}^{\mathbb{H}}$ have the same law.

For the geometric quantity, we could choose to compute $\int A(\gamma) d\mu^{\text{sle}}$, where $A(\gamma)$ is the area enclosed in $\mathbb{H}$ by the "curve" $\gamma$, but this integral is infinite. Let $\gamma^*$ be the radius of the curve $\gamma$, that is, $\gamma^* = \sup_{0\le t\le t_\gamma} |\gamma(t)|$. We may consider the "expected" area under the law $\mu^{\text{sle}}$ "conditioned" on $\gamma^* = 1$. Here, $\mu^{\text{sle}}$ is not a probability measure so the term "expected value" is not correct, and the conditioning is on a set of $\mu^{\text{sle}}-$measure equal to 0. But we have the following rigorous definition :

$$\mu^{\text{sle}}(A|\gamma^* = 1) = \lim_{\delta\downarrow 0} \frac{\int A(\gamma) 1_{\{\gamma^*\in[1,1+\delta)\}} d\mu^{\text{sle}}}{\mu^{\text{sle}}\{\gamma^* \in [1, 1+\delta)\}} \,. \tag{2.3}$$

Using $\mu^{\text{sle}} = 5/8\mu^{\text{bub}}$, we can write in the same way :

$$\mu^{\text{sle}}(A|\gamma^* = 1) = \lim_{\delta\downarrow 0} \frac{\int A(\gamma) 1_{\{\gamma^*\in[1,1+\delta)\}} d\mu^{\text{bub}}}{\mu^{\text{bub}}\{\gamma^* \in [1, 1+\delta)\}} \,. \tag{2.4}$$

Thus, $\mu^{\text{sle}}(A|\gamma^* = 1)$ represents at the same time the "expected" area of an $SLE_{8/3}$ loop conditioned to touch the half circle of radius one and the expected area of a Brownian bubble with the same conditioning. With the use of the restriction property for $SLE_{8/3}$, we will be able to compute in the last section $\mu^{\text{sle}}(A|\gamma^* = 1)$. Before, in the coming section, we will find the relationship between $\mathbb{E}(\mathcal{A})$ and $\mu^{\text{sle}}(A|\gamma^* = 1)$.

# 3 Extraction of $\mathbb{E}(\mathcal{A})$ from $\mu^{\text{sle}}(A|\gamma^* = 1)$

In this section we will prove the following

**Lemma 3.1.**

$$\mathbb{E}(\mathcal{A}) = 2\mu^{sle}(A|\gamma^* = 1).$$

**Proof.** First of all, by using the definition of $\mu^{\text{bub}}$ in terms of $\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^2} \mathrm{m}_\varepsilon(\mathbb{P}_1)$ and the restriction property of $\mathbb{P}_1$, it is easy to show that $\mu^{\text{bub}}\{\gamma^* \ge r\} = \frac{1}{r^2}$, hence

$$\mu^{\text{bub}}\{\gamma^* \in [1, 1+\delta)\} = 1 - 1/(1+\delta)^2 = 2\delta + O(\delta^2),$$

and thus, from (2.4), we have

$$\begin{aligned}\mu^{\text{sle}}(A|\gamma^* = 1) &= \lim_{\delta\downarrow 0} \frac{\int A(\gamma) 1_{\{\gamma^*\in[1,1+\delta)\}} d\mu^{\text{bub}}}{\mu^{\text{bub}}\{\gamma^* \in [1, 1+\delta)\}} \\ &= \lim_{\delta\downarrow 0} \frac{\int_0^\infty \frac{dt}{2t^2} \mathbb{E}_t(A(\gamma) 1_{\{\sup_{0\le u\le t} |\gamma(u)|\in[1,1+\delta)\}})}{2\delta + O(\delta^2)}.\end{aligned}$$

Here $\mathbb{E}_t$ is the expectation according to the law of an $\mathbb{H}$-Brownian bridge in time $t$. By Brownian scaling we have

$$\mathbb{E}_t(A(\gamma)\,1_{\{\sup_{0\le u\le t}|\gamma(u)|\in[1,1+\delta)\}}) = t * \mathbb{E}_1(A(\gamma)\,1_{\{\sup_{0\le u\le 1}|\gamma(u)|\in[\frac{1}{\sqrt{t}},\frac{1}{\sqrt{t}}+\frac{\delta}{\sqrt{t}})\}}).$$

Therefore :

$$\begin{aligned}
\mu^{\mathrm{sle}}(A|\gamma^*=1) &= \lim_{\delta\downarrow 0}\int_0^\infty \frac{dt}{4\,t(\delta+O(\delta^2))}\mathbb{E}_1(A\,1_{\{\gamma^*\in[\frac{1}{\sqrt{t}},\frac{1}{\sqrt{t}}+\frac{\delta}{\sqrt{t}})\}})\\
&= \lim_{\delta\downarrow 0}\int_0^\infty \frac{dt}{4\,t^{3/2}}\frac{\mathbb{P}_1\{\gamma^*\in[\frac{1}{\sqrt{t}},\frac{1}{\sqrt{t}}+\frac{\delta}{\sqrt{t}})\}}{\frac{\delta}{\sqrt{t}}(1+O(\delta))}\mathbb{E}_1(A\,|\gamma^*\in[\frac{1}{\sqrt{t}},\frac{1}{\sqrt{t}}+\frac{\delta}{\sqrt{t}}))\\
&= \lim_{\delta\downarrow 0}\frac{1}{2}\int_0^\infty du(1+O(\delta))\frac{\mathbb{P}_1\{\gamma^*\in[u,u+\delta u)\}}{\delta u}\mathbb{E}_1(A\,|\gamma^*\in[u,u+\delta u))\,,
\end{aligned}$$

using the change of variables $u=\frac{1}{\sqrt{t}}$. Let $\eta_1$ be the density on $\mathbb{R}_+$ of the random variable $\gamma^*$ under the $\mathbb{H}$-Brownian bridge of time duration one. As for the one dimensional bridge (law of the maximum of the bridge), this density decays exponentially fast at infinity. Thus, we can interchange the limit and the integral to obtain :

$$\mu^{\mathrm{sle}}(A|\gamma^*=1) = \frac{1}{2}\int_0^\infty \eta_1(u)\mathbb{E}_1(A|\gamma^*=u)du = \frac{1}{2}\mathbb{E}(\mathcal{A}^{\mathbb{H}})\,.$$

Hence, the proof of the lemma will be concluded as soon as we establish

$$\mathbb{E}(\mathcal{A}^{\mathbb{H}}) = \mathbb{E}(\mathcal{A}).$$

There is a (almost sure) one to one correspondence between $\mathbb{C}$-Brownian bridges and $\mathbb{H}$-Brownian bridges. The idea is to start the Brownian loop from its lowest point. More precisely, if $B_t, 0\le t\le 1$ is a Brownian bridge in $\mathbb{C}$, with probability one, there is a unique $\bar{t}\in[0,1]$ such that $\mathrm{Im}(B_{\bar{t}})\le \mathrm{Im}(B_t)$, for all $t\in[0,1]$. We associate to the Brownian Bridge $B_t$ the process $(Z_t)_{0\le t\le 1}$ in $\overline{\mathbb{H}}$, defined by this simple space-time translation :

$$Z_t = \begin{cases} B_{\bar{t}+t}-B_{\bar{t}} & ,0\le t\le 1-\bar{t}\,,\\ B_{\bar{t}+t-1}-B_{\bar{t}} & ,1-\bar{t}\le t\le 1\,. \end{cases} \tag{3.1}$$

Now, we have to identify the law of $Z_t$ with $\mathbb{P}_1^{\mathrm{exc}}\times\mathbb{P}_1^{\mathrm{br}}$. The real and imaginary parts of $B_t$ are two independent one-dimensional Brownian bridges. The law of the random variable $\bar{t}$ is independent of $\mathrm{Re}(B_t)$, so in the space-time change 3.1, $\mathrm{Re}(Z_t)$ is still a one-dimensional bridge independent of the imaginary part of $Z_t$. $\mathrm{Im}(Z_t)$ has the law of a one-dimensional Brownian bridge viewed from its (almost sure) unique lowest point. By the Vervaat Theorem (see [Ver79]), this gives the law of an Itô excursion renormalized to have time one. Thus $Z_t$ has the law of an $\mathbb{H}$-Brownian bridge of time one. Our space-time transformation obviously preserves the area, hence $\mathbb{E}(\mathcal{A}^{\mathbb{H}}) = \mathbb{E}(\mathcal{A})$.

# 4 Computation of $\mu^{\mathbf{sle}}(A|\gamma^* = 1)$, and proof of theorem 1.1

In this section we prove lemma 4.1, the proof provides a good example of the use of standard techniques for $SLE_{8/3}$. We have chosen to leave out some algebraic details in order to allow the reader to focus on the main ideas.

**Lemma 4.1.**

$$\mu^{sle}(A|\gamma^* = 1) = \frac{\pi}{10}.$$

Note that theorem 1.1 follows immediately from this lemma and lemma 3.1.

**Proof.** Recall (2.3) :

$$\mu^{\mathrm{sle}}(A|\gamma^* = 1) = \lim_{\delta \downarrow 0} \frac{\int A(\gamma) 1_{\{\gamma^* \in [1,1+\delta)\}} d\mu^{\mathrm{sle}}}{\mu^{\mathrm{sle}}\{\gamma^* \in [1, 1+\delta)\}}\,. \tag{4.1}$$

By using the definition $\mu^{\mathrm{sle}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \mathrm{m}_\varepsilon(\mathbb{P}_{5/8})$, we can rewrite 4.1 as :

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathbb{E}_\varepsilon(A(\gamma)|\gamma^* \in [1, 1+\delta)), \tag{4.2}$$

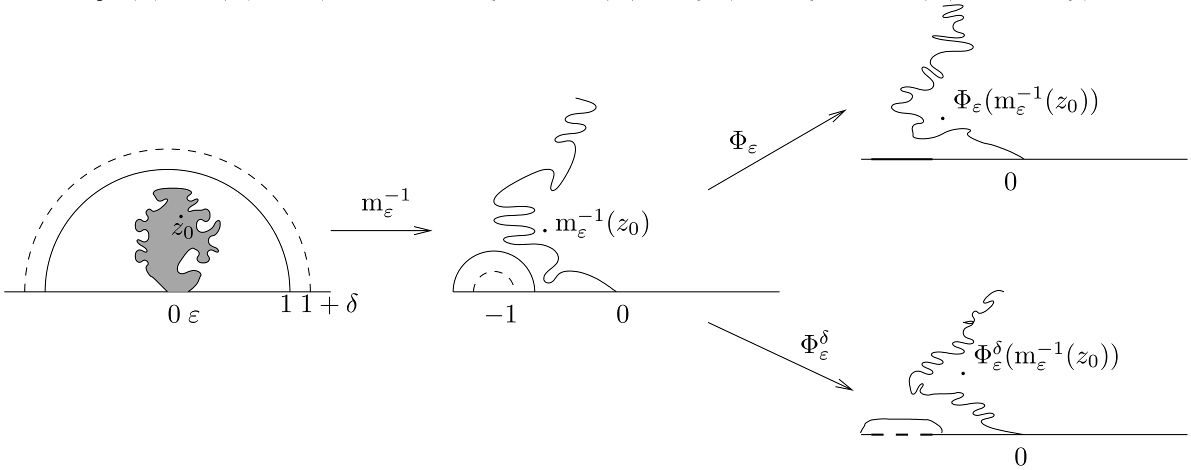
where $\mathbb{E}_\varepsilon$ is a more appealing notation for the expected value under the law of $\mathrm{m}_\varepsilon(\mathbb{P}_{5/8})$ (this law, in simpler words, is the law of a chordal $SLE_{8/3}$ in $\mathbb{H}$ from 0 to $\varepsilon$). Recall that $A(\gamma)$ is the area of the bounded set in $\mathbb{H}$ enclosed by the curve $\gamma$. $A(\gamma)$ can be written as $\int_{\mathbb{H}} 1_{\{z \text{ inside}\}} d\mathrm{A}(z)$, where $\{z \text{ inside}\}$ means that $z$ is in the component bounded by $\gamma$. Thus (4.2) can be written as :

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathbb{E}_\varepsilon \left( \int_{(1+\delta)\mathbb{D}^+} 1_{\{z \text{ inside}\}} d\mathrm{A}(z) | \gamma^* \in [1, 1+\delta) \right), \tag{4.3}$$

where $\mathbb{D}^+$ is $\mathbb{D} \cap \mathbb{H}$. Since everything is nicely bounded, we can interchange the limits and the integral. This gives us :

$$\mu^{\mathrm{sle}}(A|\gamma^* = 1) = \int_{\mathbb{D}^+} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathbb{P}_\varepsilon\{z \text{ inside } |\gamma^* \in [1, 1+\delta)\}\, d\mathrm{A}(z). \tag{4.4}$$

Therefore, what remains to be done is to compute, for a fixed z, the "probability" that this $z$ is inside an "$SLE_{8/3}$ loop" conditioned to have radius exactly 1. So let us fix $z_0$ in $\mathbb{D}^+$. Let $D_\varepsilon$ ( resp $D_\varepsilon^\delta$) denote the image under $\mathrm{m}_\varepsilon^{-1}(z) = z/(\varepsilon - z)$ of the set $\{z \in \mathbb{H} : |z| \geq 1\}$ (resp $\{z \in \mathbb{H} : |z| \geq 1 + \delta\}$).




We warn the reader that $\gamma$ will denote two different kinds of curves in $\mathbb{H}$ : a curve from 0 to $\infty$, or a curve from 0 to $\varepsilon$. Let $F_\varepsilon$ be the event $\{\gamma[0,\infty) \cap D_\varepsilon = \emptyset\}$, and, similarly, let $F_\varepsilon^\delta$ be the analogous event for $D_\varepsilon^\delta$. Then,

$$\mathbb{P}_\varepsilon\{z_0 \text{ inside } |\gamma^* \in [1, 1+\delta)\} = \mathbb{P}_{5/8}\{\mathrm{m}_\varepsilon^{-1}(z_0) \text{ is to the right of } \gamma \;|(F_\varepsilon)^c \cap F_\varepsilon^\delta\}\,.$$

Recall that $\mathbb{P}_{5/8}$ is the law of a chordal $SLE_{8/3}$ from 0 to $\infty$ in $\mathbb{H}$, henceforth, we will simply call it $\mathbb{P}$. In order to make the formulas more concise we will denote the event $\{z \text{ is to the right of } \gamma\}$ by $R(z)$. Then,

$$\mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(z_0))|(F_\varepsilon)^c \cap F_\varepsilon^\delta\} = \frac{\mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(z_0))|F_\varepsilon^\delta\}\mathbb{P}\{F_\varepsilon^\delta\} - \mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(z_0))|F_\varepsilon\}\mathbb{P}\{F_\varepsilon\}}{\mathbb{P}\{F_\varepsilon^\delta\} - \mathbb{P}\{F_\varepsilon\}}\,. \tag{4.5}$$

The reason for this last step is that now all the probabilities involved can be computed using the *restriction* property for $SLE_{8/3}$, and a simple formula, see lemma 4.2, for the probability that a point is to the right of an $SLE_{8/3}$ path from 0 to $\infty$ in $\mathbb{H}$. This requires (cf. section 2) to know the unique conformal map $\Phi_\varepsilon = \Phi_{D_\varepsilon}$ from $\mathbb{H} \setminus D_\varepsilon$ into $\mathbb{H}$, with $\Phi_\varepsilon(0) = 0$, $\Phi_\varepsilon(\infty) = \infty$ and $\Phi'_\varepsilon(\infty) = 1$ (with a similar statement for $D^\delta_\varepsilon$). Thus by restriction, the law of the chordal $SLE_{8/3}$ in $\mathbb{H}$ conditioned not to touch $D_\varepsilon$ is the inverse image of the chordal $SLE$ in $\mathbb{H}$ by $\Phi_\varepsilon$. This implies for the quantities we need to compute :

$$\begin{aligned} \mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(z_0))|F_\varepsilon\} &= \mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(\Phi_\varepsilon(z_0)))\} \\ \mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(z_0))|F^\delta_\varepsilon\} &= \mathbb{P}\{R(\mathrm{m}_\varepsilon^{-1}(\Phi^\delta_\varepsilon(z_0)))\} \,. \end{aligned}$$

Note that $\mathrm{m}_\varepsilon^{-1}$ is a Möbius transformation, which maps $\infty$ to $-1$. Therefore, $D_\varepsilon$ and $D^\delta_\varepsilon$ are half disks whose centers are very close to -1. The fact that they are not exactly centered at -1 is due to the lack of symmetry in the problem : an $SLE$ from 0 to $\varepsilon$ in a half disk $\mathbb{D}^+$ centered in 0. Nevertheless, for the computation of $\Phi_\varepsilon(z)$ and $\Phi^\delta_\varepsilon(z)$, we can think of $D_\varepsilon$ and $D^\delta_\varepsilon$ as two half disks centered at -1 with radii respectively $\varepsilon$ and $(1-\delta)\varepsilon$. If we carried out the computations with the actual disks (straightforward but tedious), we would see that our approximation is of order $O(\varepsilon^2 + \varepsilon^2\delta^2/|z+1| + \varepsilon^4/|z+1|^2)$, when $z$ goes to -1. In this way, we have

$$\begin{aligned} \Phi_\varepsilon(z) &= z - \varepsilon^2 + \frac{\varepsilon^2}{z+1} + O(\varepsilon^2 + \frac{\varepsilon^4}{|z+1|^2}) \\ \Phi^\delta_\varepsilon(z) &= z - \varepsilon^2(1-\delta)^2 + \frac{\varepsilon^2(1-\delta)^2}{z+1} + O(\varepsilon^2 + \frac{\varepsilon^2\delta^2}{|z+1|} + \frac{\varepsilon^4}{|z+1|^2}) \,. \end{aligned}$$

We now have to evaluate these functions at the point $\mathrm{m}_\varepsilon^{-1}(z_0) = z_0/(\varepsilon - z_0) = -1 - \frac{\varepsilon}{z_0} + O(\varepsilon^2)$ (recall $z_0$ is fixed). The approximations $O(\varepsilon^4/|z+1|^2)$ and $O(\varepsilon^2\delta^2/|z+1|)$ at the point $\mathrm{m}_\varepsilon^{-1}(z_0)$ are of order $O(\varepsilon^2)$ and $O(\varepsilon\delta^2)$, respectively; this gives us :

$$\begin{aligned} \Phi^\delta_\varepsilon(\mathrm{m}_\varepsilon^{-1}(z_0)) &= -1 - \frac{\varepsilon}{z_0} + \frac{\varepsilon^2(1-\delta)^2}{-\varepsilon/z_0 + O(\varepsilon^2)} + O(\varepsilon\delta^2 + \varepsilon^2) \\ &= -1 - \varepsilon(z_0 + \frac{1}{z_0}) + 2\varepsilon\delta z_0 + O(\varepsilon\delta^2 + \varepsilon^2) \,. \end{aligned}$$

Using the Taylor series for the logarithm, and then taking the imaginary part, we see that

$$\arg\left(\Phi^\delta_\varepsilon(\mathrm{m}_\varepsilon^{-1}(z_0))\right) = \pi + \varepsilon \mathrm{Im}(z_0 + \frac{1}{z_0}) - 2\varepsilon\delta \mathrm{Im}(z_0) + O(\varepsilon\delta^2 + \varepsilon^2).$$

Now, using lemma 4.2, and the Taylor series for cosine we see that

$$\mathbb{P}\{R(\Phi_\varepsilon^\delta(\mathrm{m}_\varepsilon^{-1}(z_0)))\} = \frac{\varepsilon^2}{4}\left[\left(\mathrm{Im}(z_0+\frac{1}{z_0})\right)^2 - 4\delta\mathrm{Im}(z_0+\frac{1}{z_0})\mathrm{Im}(z_0)\right] + O(\varepsilon^2\delta^2+\varepsilon^3). \tag{4.6}$$

In particular, if we set $\delta = 0$ we obtain,

$$\mathbb{P}\{R(\Phi_\varepsilon(\mathrm{m}_\varepsilon^{-1}(z_0)))\} = \frac{\varepsilon^2}{4}\left(\mathrm{Im}(z_0+\frac{1}{z_0})\right)^2 + O(\varepsilon^3). \tag{4.7}$$

Also, by (2.1), we have (our approximation doesn't change significantly the derivative at 0 which is far away from small disks centered at -1) :

$$\begin{aligned}\mathbb{P}\{F_\varepsilon^\delta\} = \mathbb{P}_{5/8}\{\gamma[0,\infty)\cap D_\varepsilon^\delta = \emptyset\} &= (\Phi_\varepsilon^\delta)'(0)^{5/8} \\ &= (1-\varepsilon^2(1-2\delta+O(\delta^2)))^{5/8} + O(\varepsilon^3) \\ &= 1 - \frac{5}{8}\varepsilon^2 + \frac{5}{4}\varepsilon^2\delta + O(\varepsilon^2\delta^2+\varepsilon^3)\end{aligned}$$

Similarly, $\mathbb{P}\{F_\varepsilon\} = 1 - 5/8\varepsilon^2 + O(\varepsilon^3)$, which gives

$$\mathbb{P}\{F_\varepsilon^\delta\} - \mathbb{P}\{F_\varepsilon\} = \frac{5}{4}\varepsilon^2\delta + O(\varepsilon^2\delta^2+\varepsilon^3)\,. \tag{4.8}$$

Hence, by combining this last expression, 4.5, 4.6, 4.7 and using the fact that both $\mathbb{P}\{F_\varepsilon\}$ and $\mathbb{P}\{F_\varepsilon^\delta\}$ are $1+O(\varepsilon^2)$, we obtain :

$$\begin{aligned}&\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\mathbb{P}_\varepsilon\{z_0 \text{ inside } |\{\gamma^*\in[1,1+\delta)\}\} \\ &\qquad = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{\frac{\varepsilon^2}{4}(-4\delta\mathrm{Im}(z_0+\frac{1}{z_0})\mathrm{Im}(z_0)) + \varepsilon^2 O(\delta^2) + O(\varepsilon^3)}{\frac{5}{4}\varepsilon^2\delta + O(\varepsilon^2\delta^2+\varepsilon^3)} \\ &\qquad = -\frac{4}{5}\mathrm{Im}(z_0+\frac{1}{z_0})\mathrm{Im}(z_0).\end{aligned}$$

Therefore, by (4.4), and using polar coordinates to evaluate the integral, we get :

$$\begin{aligned}\mu^{\mathrm{sle}}(A|\gamma^*=1) &= \int_{\mathbb{D}^+} -\frac{4}{5}\mathrm{Im}(z+\frac{1}{z})\mathrm{Im}(z)d\mathrm{A}(z) \\ &= \frac{\pi}{10}\,.\end{aligned}$$

This concludes the proof of the lemma.

Below we state a result that we have used extensively in our proof; for the reader's sake we will sketch a proof. This lemma gives an equivalent expression to the one given by Schramm, in [Sch01].

**Lemma 4.2.** *Let $\gamma$ be chordal $SLE_\kappa$ in $\mathbb{H}$ with $\kappa \leq 4$, and let $z = re^{i\theta}$ be a point in $\mathbb{H}$. If we let $f(z) = \mathbb{P}\{z$ is to the right of $\gamma[0,\infty)\}$, then $f$, which by scaling depends only on $\theta$, is given by*

$$f(\theta) = \frac{1}{\int_0^\pi (\sin u)^{\frac{2(4-\kappa)}{\kappa}} du} \int_\theta^\pi (\sin u)^{\frac{2(4-\kappa)}{\kappa}} du.$$

*In particular, for $\kappa = \frac{8}{3}$ :*

$$\mathbb{P}\{z \textit{ is to the right of } \gamma[0,\infty)\} = 1/2 + 1/2\cos(\theta).$$

**Proof.** *(sketch)*

As already mentioned, by scale invariance of $SLE$, the probability that a point $z = re^{i\theta}$ is to the right of the curve only depends on the angle $\theta$. Thus, this probability is a certain function $f$ of the angle $\theta$. $SLE$ curves satisfy also a conformal-type Markov property. Thus, if $X_t$ is the unique conformal map from $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$ satisfying $X_t(\infty) = \infty$ , $X_t'(\infty) = 1$ and $X_t(\gamma(t)) = 0$, we get :

$$\mathbb{P}(z \text{ is on the right } |\mathcal{F}_t) = \mathbb{P}(X_t(z) \text{ is on the right}) = f(\theta_t)\,,$$

where $\theta_t$ is the continuous argument of $X_t(z)$. This shows that $f(\theta_t)_{t\geq 0}$ is a martingale in $(0,1)$. Using that $X_t(z) = g_t(z) - \sqrt{\kappa} dB_t$, where $g_t$ is defined by :

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa}B_t}\,, \quad g_0(z) = z \qquad z \text{ in } \mathbb{H},$$

we have

$$\begin{aligned} dX_t &= dg_t(z) - \sqrt{\kappa}dB_t = \frac{2}{X_t}dt - \sqrt{\kappa}dB_t\,, \\ d\log X_t &= \frac{2}{X_t^2}dt - \frac{\sqrt{\kappa}}{X_t}dB_t - \frac{\kappa}{2X_t^2}dt = \frac{(4-\kappa)}{2X_t^2}dt - \frac{\sqrt{\kappa}}{X_t}dB_t\,, \end{aligned}$$

and by taking the imaginary part :

$$d\theta_t = \frac{\kappa - 4}{2|X_t|^2}\sin(2\theta_t)dt + \frac{\sqrt{\kappa}}{|X_t|}\sin(\theta_t)dB_t\,.$$

Now suppose $f$ is a $\mathcal{C}^2$ function, and apply Itô's formula to $f(\theta_t)$. We want this process to be a martingale, so the $dt$ term in the expression for $df(\theta_t)$ has to be 0. This gives a simple second order deterministic differential equation. Moreover we have the boundary conditions $f(0) = 1$ and $f(\pi) = 0$. There is a unique solution, given in the lemma, which indeed is $\mathcal{C}^2$.

**Remark :** We would like to point out that the $1/5$ in the final result, comes from the $8/5$ in the *restriction* formula 2.1.

# 5 Decomposition of the expected area of the Brownian loop into the expected areas of the regions with fixed winding number

Let $z \in \mathbb{C} \setminus \{0\}$ be fixed, and $(B_t)_{0 \leq t \leq 1}$ a Brownian loop in $\mathbb{C}$ starting at 0. Almost surely $z \notin \{B_s : 0 \leq s \leq 1\}$, and therefore we can define its index $n_z$. More precisely, $\forall s \in [0,1], B_s - z = R_s^z \exp(i\theta_s^z)$, where $R_s^z = |B_s - z|$ and $\theta_s^z$ is any continuous representative of the argument. The index $n_z$ is by definition $\frac{\theta_1^z - \theta_0^z}{2\pi}$ ; this is the number of times that the Brownian particle winds around $z$. For each $n \in \mathbb{Z}$, $n \neq 0$, let $\mathcal{W}_n$ denote the area of the open set of points of index $n_z = n$. This random variable can be written as :

$$\mathcal{W}_n = \int_{\mathbb{C}} 1_{\{n_z = n\}} d\mathrm{A}(z)\,.$$

Let $\mathcal{W}_0$ be the area of the open set of points inside the loop that have index zero :

$$\mathcal{W}_0 = \int_{\mathbb{C}} 1_{\{n_z = 0\} \cap \{z \text{ is inside}\}} d\mathrm{A}(z)\,.$$

Since the Brownian curve is of Lebesgue measure zero, we have the following decomposition of the area $\mathcal{A}$ inside the Brownian loop (basically, the Brownian path does not take much place inside its hull)

$$\mathcal{A} = \sum_{n \in \mathbb{Z}} \mathcal{W}_n$$

Hence :

$$\mathbb{E}(\mathcal{A}) = \frac{\pi}{5} = \sum_{n \in \mathbb{Z}} \mathbb{E}(\mathcal{W}_n).$$

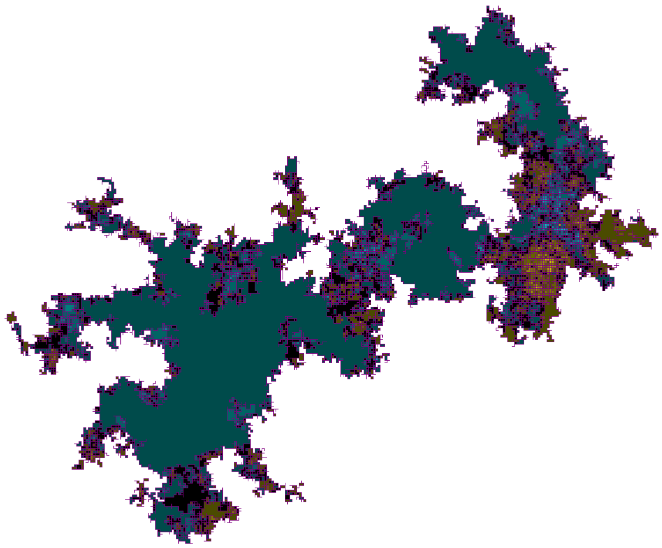


Figure 5.1: Different indices in a random walk of 50000 steps, black areas correspond to index 0.

Using results of Yor [Yor80], it will be straightforward to compute $\mathbb{E}(\mathcal{W}_n)$ for $n \neq 0$. And, hence, by subtracting from $\pi/5$, one can obtain the value of $\mathbb{E}(\mathcal{W}_0)$.

**Theorem 5.1.**

$$\mathbb{E}(\mathcal{W}_n) = \begin{cases} \frac{\pi}{30} & n = 0\,, \\ \frac{1}{2\pi n^2} & n \neq 0,\, n \in \mathbb{Z}\,. \end{cases} \tag{5.1}$$

**Remark :** This result is consistent with the asymptotic result obtained by Werner in [Wer94], about the area $A_n^t$ of the set of points around which the planar Brownian motion (not the loop) winds around $n$ times on $[0,t]$. It is indeed proved that $A_n^t$ is equivalent (in the $L^2$-sense) to $\frac{t}{2\pi n^2}$ as $n$ goes to infinity. Very roughly the area of the $n$-sector for large $n$ comes from local contributions along the path, hence the global picture of the hull is not relevant; that is why, both Brownian motion and Brownian bridge should have the same asymptotics. Werner's proof requires to compute the asymptotics of the first and second moments. This present paper gives exact computations for the first moments in the case of the loop, but it does not provide any information about the second moments.

**Proof.** We start by computing $\mathbb{E}(\mathcal{W}_n)$ for $n \neq 0$. For this purpose we use theorem 5.2, which was proved by Yor [Yor80]. Thus, for each $n \neq 0$, using polar coordinates :

$$\begin{aligned}
\mathbb{E}(\mathcal{W}_n) &= \int_{\mathbb{C}} \mathbb{P}(n_z = n) d\mathrm{A}(z) \\
&= 2\pi \int_0^\infty r dr e^{-r^2} \left[ \int_0^\infty dt e^{-r^2 \cosh(t)} \left( \frac{2n-1}{t^2 + (2n-1)^2\pi^2} - \frac{2n+1}{t^2 + (2n+1)^2\pi^2} \right) \right] \\
&= 2\pi \int_0^\infty dt \left( \frac{2n-1}{t^2 + (2n-1)^2\pi^2} - \frac{2n+1}{t^2 + (2n+1)^2\pi^2} \right) \int_0^\infty r e^{-r^2(1+\cosh(t))} dr \\
&= \pi \int_0^\infty \frac{dt}{1+\cosh(t)} \left( \frac{2n-1}{t^2 + (2n-1)^2\pi^2} - \frac{2n+1}{t^2 + (2n+1)^2\pi^2} \right) \\
&= \frac{1}{2\pi n^2} .
\end{aligned}$$

We sketch one possible way to see how to obtain the last line in the above chain of equalities.

It is slightly more convenient to generalize a bit, so thinking of $2n$ as $x$ and using the symmetry of the integrand, we consider the function

$$F(x) = \int_{-\infty}^\infty \frac{dt}{1+\cosh(t)} \left( \frac{x-1}{t^2 + (x-1)^2\pi^2} - \frac{x+1}{t^2 + (x+1)^2\pi^2} \right) .$$

In this new notation what we want to prove is that $F(x) = \frac{4}{\pi^2 x^2}$ (for $x \geq |2|$). Since, $F$ is symmetric about 0, it is enough to study the case of $x$ positive; furthermore, since $F$ is real analytic on $\{x : x > 1\}$, we can allow ourselves to assume that $x$ is not an integer. Now, for $x > 1$ and $x$ not an integer, a simple residue computation with appropriate contours yields

$$F(x) = -\frac{8}{\pi^2} \sum_{k=1}^\infty \left( \frac{(2k-1)(x-1)}{((x-1)^2 - (2k-1)^2)^2} - \frac{(2k-1)(x+1)}{((x+1)^2 - (2k-1)^2)^2} \right) .$$

In order to evaluate this sum, it is enough to notice that using partial fractions one can obtain

$$\sum_{k=1}^\infty \frac{(2k-1)w}{(w^2 - (2k-1)^2)^2} = -\frac{1}{16} \sum_{k=0}^\infty \left( \frac{1}{(k + w/2 + 1/2)^2} - \frac{1}{(k - w/2 + 1/2)^2} \right) ,$$

and substituting $x-1$ and $x+1$ for $w$, and noticing the telescoping cancellations one readily obtains

$$F(x) = \frac{4}{\pi^2 x^2},$$

hence, $\mathbb{E}(\mathcal{W}_n) = \frac{1}{2\pi n^2}$.

Finally, using the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, and the fact that the area of the Brownian loop is $\pi/5$ we conclude $\mathbb{E}(\mathcal{W}_0) = \frac{\pi}{30}$. This finishes the proof of the theorem.

**Theorem 5.2.** *Fix $z = re^{i\theta}$, with $r \neq 0$. Under the law of a Brownian loop of time duration one, starting at 0, we have the following probabilities :*

$$\mathbb{P}(n_z = n) = e^{-r^2}[\Psi_r((2n-1)\pi) - \Psi_r((2n+1)\pi)] \text{ if } n \in \mathbb{Z} \setminus 0\,, \quad (5.2)$$
$$\mathbb{P}(n_z = 0) = 1 + e^{-r^2}[\Psi_r(-\pi) - \Psi_r(\pi)]\,, \quad (5.3)$$

*where $\forall x \neq 0$,*

$$\Psi_r(x) = \frac{x}{\pi}\int_0^\infty e^{-r^2\cosh(t)}\frac{dt}{t^2+x^2}\,.$$

# Chapter IV

# Continuity of the SLE trace in simply connected domains

Joint work with *Steffen Rohde* and *Oded Schramm*.

We prove that the $\mathrm{SLE}_\kappa$ trace in any simply connected domain $G$ is continuous (except possibly near its endpoints) if $\kappa < 8$. We also prove an SLE analog of Makarov's Theorem about the support of harmonic measure.

# 1 Introduction

The stochastic Loewner evolution (SLE) describes a collection of random curves that are related to scaling limits of two-dimensional statistical physics systems. In [RS05, Theorem 5.1] it was shown that the chordal SLE trace in the upper half plane $\mathbb{H}$ is a well defined continuous path. For other simply connected domains $G \subsetneqq \mathbb{C}$, the SLE in $G$ is defined via a conformal homeomorphism $f : \mathbb{H} \to G$. The situation with radial SLE is similar, except that the "standard" domain is the unit disk $\mathbb{D}$. Our first theorem extends this continuity result, as follows.

**Theorem 1.1.** *Let $G \subsetneqq \mathbb{C}$ be a simply connected domain, let $a, b$ be two prime ends of $G$, let $z_0 \in G$, and let $\kappa \in [0, 8)$. Then the chordal* $\mathrm{SLE}_\kappa$ *trace in $G$ from $a$ to $b$ and the radial* $\mathrm{SLE}_\kappa$ *trace in $G$ from $a$ to $z_0$ are a.s. continuous in* $(0, \infty)$.

Besides the intrinsic interest in this result, it is also useful in the general theory of SLE and the related scaling limits. For example, the construction of the conformal loop ensembles in [She06, Section 4.1] would have been simpler if this theorem was available.

If the boundary $\partial G$ is a smooth curve (more generally, if it is locally connected), then the conformal map $f$ to $G$ extends continuously to the closure of $\mathbb{H}$ (respectively $\mathbb{D}$), and the continuity of the trace $f \circ \gamma$ follows at once from the continuity of the trace $\gamma$. But if $\partial G$ contains boundary points at which it looks like the topologist's sine curve, then $f$ is not continuous at the corresponding points, and the continuity of $f \circ \gamma$ is no longer obvious when $\kappa > 4$. In fact, this non-continuity could happen at *every* boundary point: there are simply connected domains $G$ for which the limit set $f(z) := \{w : \exists (z_k \to z), \lim f(z_k) = w\}$ equals $\partial G$, for **all** $z \in \partial\mathbb{D}$ [Kue74]! On the other hand, for every conformal map $f : \mathbb{D} \to \mathbb{C}$, the radial limit $lim_{r\to 1} f(re^{it})$ exists for a.e. $t \in [0, 2\pi]$. A celebrated theorem of Makarov asserts that there is a set $A \subset \partial\mathbb{D}$ of full measure such that the set of radial limits $f(A)$ has Hausdorff dimension 1 (even sigma-finite length). Equivalently, for every simply connected domain $G \subsetneqq \mathbb{C}$ there is a set $B \subset \partial G$ of Hausdorff dimension 1 such that a Brownian motion started inside $G$ will a.s. exit $G$ through $B$. However, under a mild assumption on the geometry of $G$ (precisely, if $G$ is a John domain), reflected Brownian motion in $G$ intersects the boundary in a set of full dimension [BCR04]. In particular, there is no nontrivial upper bound on the dimension of the trace of reflected Brownian motion on $\partial G$. The situation is different for SLE:

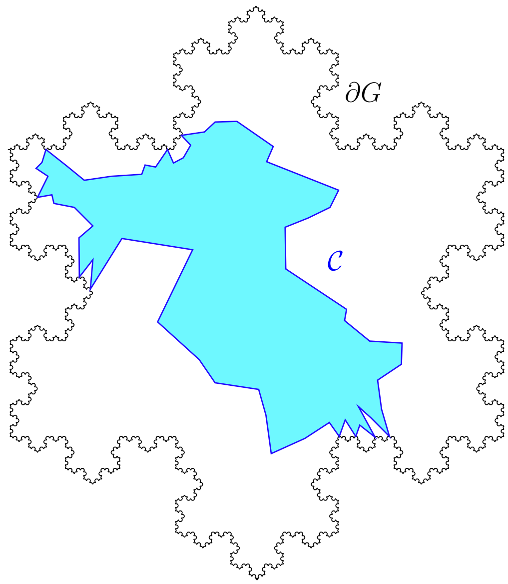



Figure 1.1: A schematic view of a percolation cluster $\mathcal{C}$ (or an $\mathrm{SLE}_6$ hull) inside a fractal domain $G$; the blue curve represents the exterior boundary of the cluster.

**Theorem 1.2.** *Let $G \subsetneq \mathbb{C}$ be a simply connected domain, let $a, b$ be two prime ends of $G$, let $z_0 \in G$, and let $\kappa \in (4, 8)$. Then there is a Borel set (actually a $F_\sigma$ set) $B \subset \partial G$ such that the chordal $\mathrm{SLE}_\kappa$ trace in $G$ from $a$ to $b$ and the radial $\mathrm{SLE}_\kappa$ trace in $G$ from $a$ to $z_0$ almost surely satisfy*

$$\gamma(0, \infty) \cap \partial G \subset B,$$

*and*

$$\dim B \leq d(\kappa) < 2\,,$$

*where $d(\kappa)$ is a constant that depends only on $\kappa$.*

In the case $\kappa = 6$, this theorem can be thought of as a Makarov theorem for percolation. Indeed, instead of starting a Brownian motion at $z_0$ inside $G$, think of "sending" a critical percolation cluster from $z_0$ in the following way: already at the scaling limit, condition $z_0$ to be connected via some open cluster to the boundary $\partial G$ (though this is an event of probability 0, it is possible to make sense of this conditioning; see [Kes86]). Theorem 1.2 then implies that there is a subset $B \subset \partial G$ of dimension $d \leq d(6) < 2$ which almost surely "absorbs" all the points on $\partial G$ which are connected to $z_0$ within $G$. See figure 1.1 for an illustration of this. Of course one cannot hope to find such a set $B$ of dimension one, as in Makarov's theorem, since the percolation cluster is much "thicker" than the Brownian path stopped on the boundary. Following this intuition, we will also show that the smallest bound $d(\kappa)$ necessarily satisfies $d(\kappa) > 1$ for all $\kappa \in (4, 8)$.

The case $\kappa = 4$ should nearly correspond to the setting of Makarov's theorem (for instance if one considers chordal $\mathrm{SLE}_4$ in the disc with a random initial point). Therefore we expect that for $\kappa$ close to 4, $d(\kappa)$ should be close to one. We indeed prove the following estimates on $d(\kappa)$:

**Proposition 1.3.** *There are absolute constants $C_1, C_2 > 0$ such that for any $\kappa \in (4, 8)$,*

$$d(\kappa) \leq \big(2 - C_1(8-\kappa)\big) \wedge \big(1 + C_2\sqrt{\kappa - 4}\big)\,. \tag{1.1}$$

Finally, in Section 4 we will relate the integral means spectrum of a conformal map to the dimension of the SLE trace on the boundary of a domain, and show existence of nice Jordan curves such that two independent $\mathrm{SLE}_\kappa$, run in the two complimentary domains of the curve, are almost surely disjoint.

**Acknowledgments**. We wish to thank Jeff Steif for pointing out an error in the statement of Theorem 1.2.

# 2 Uniform Continuity

For sets $A \subset \mathbb{C}$, denote

$$H_p(A) = \inf\Big\{\sum r_i^p : A \subset \bigcup_i B(x_i, r_i)\Big\},$$

which is called the $p$-dimensional Hausdorff content of $A$, where the infimum is over all covers of $A$ by discs with positive radius. The following is an adaptation of [KR97, Proposition 3.3].

**Lemma 2.1.** *Let $G \subsetneq \mathbb{C}$ be a simply connected domain and $f : \mathbb{D} \to G$ a conformal homeomorphism. For every $0 < p < 1$ and $\epsilon > 0$ there is $D \subset \mathbb{D}$ and $C > 0$ such that*

$$H_p(\mathbb{D} \setminus D) < \epsilon$$

*and*

$$|f(z) - f(z')| \leq C\,|z - z'|^{p/2} \tag{2.1}$$

*for all $z, z' \in D$.*

For our present application, the case where $p$ is small is the most relevant. The proof shows that we can choose the centers of the discs to lie on the unit circle.

**Proof.** We first assume that $G$ is bounded. Consider the collection $\mathcal{Q}$ of dyadic "squares"

$$Q = Q_{n,k} = \Big\{re^{it} : 1 - 2^{-n} \leq r < 1, \frac{k}{2^n} \leq \frac{t}{2\pi} \leq \frac{k+1}{2^n}\Big\},$$

where $n \geq 1$ and $k = 0, 1, 2, ..., 2^n - 1$. Denote by $\ell(Q) := 2^{-n}$, the size of $Q$, and $T(Q) := \{z \in Q : 1 - 2^{-n} \leq r \leq 1 - 2^{-(n+1)}\}$, the "inner half" of $Q$. Fix $N > 1$, to be determined later, and let $\mathcal{L}$ be the collection of those $Q \in \mathcal{Q}$ for which

$$\ell(Q) \leq 2^{-N}$$

and

$$\int_{T(Q)} |f'|^2 > \ell(Q)^p.$$

Set

$$D = \mathbb{D} \setminus \bigcup_{Q \in \mathcal{L}} Q\,.$$

We claim that

$$\forall_{z \in D} \qquad |f'(z)| \leq C\,\frac{1}{(1 - |z|)^{1-p/2}} \tag{2.2}$$

with $C$ depending on $N$ and $p$ only. Indeed, let $z \in D$. If $|z| < 1 - 2^{-N}$, we get (2.2) simply by choosing $C$ large enough. Else, suppose that $|z| \geq 1 - 2^{-N}$. Let $Q$ be such that $z \in T(Q)$ and notice that $Q \notin \mathcal{L}$. Hence $\int_{T(Q)} |f'|^2 \leq \ell(Q)^p$. By the Koebe distortion theorem [Pom92], $|f'|$ is essentially constant in $T(Q)$ and (2.2) follows.

We now claim that $f$ satisfies (2.1) for $z, z' \in D$, with possibly a different constant $C$. Consider any $z, z' \in D$. First, suppose that $z = s\,z'$, where $s > 1$. Note that the interval $[z, z']$ is contained in $D$. Then we may integrate the estimate (2.2) over $[z, z']$ to obtain (2.1). Next, suppose that $z = r\,e^{i\theta_1}$ and $z' = r\,e^{i\theta_2}$, where $|\theta_1 - \theta_2|/(2\,\pi) \le 1 - r$. In that case, the path $r\,e^{i\theta}$, $\theta \in [\theta_1, \theta_2]$, is contained in the union of some $Q \in \mathcal{Q}$ satisfying $z \in T(Q)$ and a possibly different $Q' \in \mathcal{Q}$ satisfying $z' \in T(Q')$. Therefore, this path is in $D$, and we get (2.1) in the same way. In general, suppose that $z = r_1\,e^{i\theta_1}$ and $z' = r_2\,e^{i\theta_2}$ with $|\theta_1 - \theta_2| \le \pi$. Then take $\rho := \min\{r_1, r_2, 1 - |\theta_1 - \theta_2|/(2\,\pi)\}$. We then use the above cases and $|f(z) - f(z')| \le |f(z) - f(\rho\,e^{i\theta_1})| + |f(\rho\,e^{i\theta_1}) - f(\rho\,e^{i\theta_2})| + |f(\rho\,e^{i\theta_2}) - f(z')|$, to obtain (2.1).

To estimate the $p$-content of $\mathbb{D} \setminus D$, just notice that the interiors of the sets $T(Q)$, $Q \in \mathcal{Q}$, are disjoint and

$$\sum_{Q \in \mathcal{L}} \ell(Q)^p \le \sum_{Q \in \mathcal{L}} \int_{T(Q)} |f'|^2 \le \operatorname{area}\{f(z) : 1 - 2^{-N} < |z| < 1\}\,,$$

which can be made arbitrarily small by choosing $N$ large. This completes the proof in the case where $G$ is bounded.

The case of unbounded $G$ requires a few minor adaptations. Set

$$\phi(z) := \left|\frac{f'(z)}{\max\bigl\{|f(z)| \log |f(z)|, 1\bigr\}}\right|^2,$$

and redefine $\mathcal{L}$ to be the set of $Q \in \mathcal{Q}$ such that $\ell(Q) \le 2^{-N}$ and $\int_{T(Q)} \phi > \ell(Q)^p$. Note that a simple change of variables gives

$$\int_{\mathbb{D}} \phi = \int_G \max\{|z| \log |z|, 1\}^{-2} < \infty$$

and so we get $\sum_{Q \in \mathcal{L}} \ell(Q)^p < \epsilon$ by taking $N$ sufficiently large, as above. The Koebe distortion theorem implies that $\phi$ is essentially constant in $T(Q)$. Therefore, we get

$$\forall_{z \in D} \qquad \left|\frac{f'(z)}{\max\bigl\{|f(z)| \log |f(z)|, 1\bigr\}}\right| \le C\,\bigl(1 - |z|\bigr)^{-1+p/2}. \tag{2.3}$$

Set

$$g(z) := \int_0^{|z|} \frac{ds}{\max\{s \log s, 1\}}\,.$$

Then $\frac{d}{dr}\, g\big(f(r\, e^{i\theta})\big)$ is bounded by the left hand side of (2.3) with $z = r\, e^{i\theta}$. Therefore, since $D$ is star-shaped about 0, we have for $z \in D$,

$$g \circ f(z) \leq g \circ f(0) + C \int_0^1 \big(1 - r\big)^{-1+p/2}\, dr < \infty\,.$$

Thus $g \circ f$ is bounded on $D$. Since $\lim_{z\to\infty} g(z) = \infty$, it follows that $f$ is bounded on $D$. Thus (2.3) implies (2.2) with possibly a different constant. This gives (2.1), as before.

# 3 Remaining proofs

**Proof of Theorems 1.1 and 1.2.** Let $\gamma$ be the chordal $\mathrm{SLE}_\kappa$ trace in $\mathbb{D}$ from 1 to $-1$, and denote by $f$ a conformal map from $\mathbb{D}$ to $G$ sending 1 and $-1$ to $a$ and $b$. If $\kappa \leq 4$, then $\gamma$ is continuous on $(0, \infty)$ and $\gamma(0, \infty) \subset \mathbb{D}$ almost surely [RS05, Theorems 5.1 and 6.1]. The continuity of $f \circ \gamma$ follows at once. Now let $4 < \kappa < 8$, let $\delta, \epsilon > 0$ and $\delta < |t| < \pi - \delta$. We have

$$\mathbb{P}\big[\gamma(0,\infty) \cap B(e^{it}, r) \neq \emptyset\big] \leq C\, r^{\frac{8}{\kappa}-1} \tag{3.1}$$

for some constant $C = C(\kappa, \delta)$ and for all $r < \delta/2$; see, e.g., [SZ07, Proposition 2.3] or [AK08, Theorem 3.2]. Let $p = \frac{8}{\kappa} - 1$ and let $D$ be as in Lemma 2.1. It follows from that lemma that there are discs $B(x_i, r_i)$ with $\mathbb{D} \setminus D \subset \bigcup_i B(x_i, r_i)$ and $\sum r_i^p < \epsilon$, and we may and will assume $x_i \in \partial\mathbb{D}$. From (3.1) we obtain

$$\mathbb{P}\Big[\big(\gamma(0,\infty) \setminus (B(1, 2\,\delta) \cup B(-1, 2\,\delta))\big) \cap \bigcup_i B(x_i, r_i)) \neq \emptyset\Big] \leq C\epsilon.$$

Since $f$ is continuous on $\overline{D}$, it follows that with probability at least $1 - C\epsilon$ $f \circ \gamma$ is continuous at every $t$ such that $\gamma(t) \notin B(1, 2\,\delta) \cup B(-1, 2\,\delta)$. Now Theorem 1.1 in the chordal case follows by first letting $\epsilon \to 0$, then letting $\delta \to 0$, and using the transience of $\gamma$ [RS05, Theorem 7.1]. The radial case is similar.

To prove Theorem 1.2, consider the domain

$$D_\epsilon = \mathbb{D} \setminus \bigcup_i T(x_i, r_i),$$

where $x_i$ and $r_i$ are as above and $T(x,r)$ is the triangular region bounded by the circular arc $B(x, 2r) \cap \partial\mathbb{D}$ and the two line segments joining the endpoints of the arc with the point $(1-2r)\,x$. (Alternatively, let $D_\epsilon$ be the domain $D_\epsilon = \mathbb{D} \setminus \bigcup_{Q\in\mathcal{L}} Q$ from Lemma 2.1.) Then $D_\epsilon$ is a John domain (meaning that $D_\epsilon$ is bounded and there is a number $M > 1$ such that every Jordan arc $\gamma \subset D_\epsilon$ with endpoints on $\partial D_\epsilon$ decomposes $D_\epsilon$ into two subdomains at least one of which has diameter $\leq M \operatorname{diam}\gamma$) with uniformly bounded John constant $M$. Therefore, any conformal map $\varphi : \mathbb{D} \to D_\epsilon$ is Hölder continuous with some universal exponent $\alpha = \alpha(M) > 0$; see [Pom92], Chapter 5.2, or [GM05], Chapter 7. Thus (2.1) implies

$$|f \circ \varphi(z) - f \circ \varphi(z')| \leq C\,|z - z'|^{\alpha p/2}$$

for all $z, z' \in \mathbb{D}$, with a constant $C$ depending on $\epsilon$, but with exponent $\alpha p/2$ depending on $\kappa$ only.

If $\alpha\,p/2$ happened to be greater than $1/2$, then the result would follow from the trivial estimate for the change of Hausdorff dimension by the reciprocal of the Hölder exponent. Since it is not the case (recall $p = 8/\kappa - 1$), we need to use some more advanced results. Here is a Theorem by Jones and Makarov (Theorem C.2 in [JM95], see Corollary 3.2 in [KR97] for a different proof of this Theorem).

**Theorem 3.1** (Jones-Makarov)**.** *Let $\eta \in (0,1)$ and let $\Omega$ be some Hölder domain with exponent $\eta$ (i.e., $\Omega$ is a Jordan domain so that any conformal mapping $f : \mathbb{D} \to \Omega$ is $\eta$-Hölder in the disk). Then*

$$\dim \partial\Omega \leq 2 - c\,\eta\,,$$

*where $c > 0$ is an absolute constant.*

Applied to our setting, this theorem implies that

$$\dim \partial f(D_\epsilon) = \dim \partial\big(f \circ \varphi(\mathbb{D})\big) \leq 2 - c\alpha p/2.$$

Above, we have seen that for every $0 < t < T < \infty$ we almost surely have $\gamma[t,T] \subset D_\epsilon$ for some $\epsilon > 0$, and therefore

$$f\big(\gamma(0,\infty) \cap \partial\mathbb{D}\big) \subset \bigcup_n \partial f(D_{1/n}).$$

Setting $B = \partial G \cap \bigcup_n \partial f(D_{1/n})$, Theorem 1.2 follows at once with $d(\kappa) = 2 - c\alpha p/2$. Notice here that $B$ is indeed a $F_\sigma$ set since by (2.1), $f$ is uniformly continuous on $D_{1/n}$. □

Notice that $\alpha$ can be chosen independently of $\kappa$, therefore there is a constant $C_1 > 0$ such that

$$d(\kappa) \leq 2 - C_1\,(8 - \kappa)\,, \tag{3.2}$$

which proves part of Proposition 1.3.

In order to show that we cannot choose $d(\kappa) \leq 1$ in general, fix $\kappa \in (4, 8)$ and choose $\alpha \in (\frac{8}{\kappa} - 1, 1)$. Then there is a simply connected domain $G$ whose boundary has dimension greater than 1 and such that any conformal map $f$ from $\mathbb{D}$ onto $G$ is Hölder continuous with exponent $\alpha$. In fact, any sufficiently "flat" snowflake curve, or the bounded Fatou component of the quadratic polynomial $z^2 + \lambda z$ with $0 < |\lambda| < (1 - \sqrt{\alpha})/(1 + \sqrt{\alpha})$ will do, see [AIM], Chapter 13.3. If $B \subset \partial G$ is a Borel set which almost surely contains the intersection $\gamma(0, \infty) \cap \partial G$ of the chordal SLE$_\kappa$ trace $\gamma$ from $a$ to $b$ with the boundary of the domain, then almost surely the chordal SLE$_\kappa$ in $\mathbb{D}$ does not intersect $L = \partial\mathbb{D} \setminus f^{-1}(B)$. We claim that $L$ has to be of Hausdorff dimension at most $a := \frac{8}{\kappa} - 1$. We briefly sketch the proof, which is based on estimates and arguments from [SZ07]. Consider the chordal SLE$_\kappa$ path from 0 to $\infty$ in the upper half plane. Let $L' \subset [1, 2]$ be Borel-measurable and have Hausdorff dimension larger than $a$. By Theorem 8.8 in [**?**], there exists a Frostman measure $\mu$ supported on a compact subset $A$ of $L'$ whose $a$-energy is finite; namely, $\mu$ is supported on $A \subset L'$, $\mu(A) > 0$, and

$$\iint_{A\times A} \frac{d\mu(x)d\mu(y)}{|x - y|^a} < \infty\,.$$

(See [**?**, Section 8] for background on Frostman measures.) Let $C_\epsilon$ be defined as in [SZ07, Section 2]. We now apply a second moment argument to the random variable $\mu(C_\epsilon)$. By Propositions 2.3 and 2.4 in [SZ07] we have for $1 \leq x < y \leq 2$ and $\epsilon < 1$ that $\mathbb{P}\big[x \in C_\epsilon\big]$ is comparable to $\epsilon^a$ and $\mathbb{P}\big[x, y \in C_\epsilon\big] \leq C\,\epsilon^{2a}\,(y - x)^{-a}$. Hence,

$$\mathbb{E}\big[\mu(C_\epsilon)\big] = \int_{[1,2]} \mathbb{P}\big[x \in C_\epsilon\big]\,d\mu(x)$$

is of order $\epsilon^a$ and

$$\mathbb{E}\big[\mu(C_\epsilon)^2\big] = \iint_{[1,2]^2} \mathbb{P}\big[x, y \in C_\epsilon\big]\, d\mu(x)\, d\mu(y)$$
$$\leq C\, \epsilon^{2a} \iint_{[1,2]^2} |x-y|^{-a}\, d\mu(x)\, d\mu(y)\,.$$

By the choice of $\mu$, the latter is bounded by a constant times $\epsilon^{2a}$. Thus, $\mathbb{E}\big[\mu(C_\epsilon)^2\big] \leq C\,\mathbb{E}\big[\mu(C_\epsilon)\big]^2$. The standard second moment argument (i.e., Cauchy Schwarz) therefore implies that $\mathbb{P}[\mu(C_\epsilon) > 0]$ is bounded away from 0 independently of $\epsilon$. Thus, with positive probability $\mu(C_\epsilon) > 0$ for every $\epsilon > 0$. But since the support of $\mu$ is compact and contained in $L'$, this implies that the SLE path hits $L'$ with positive probability, which clearly implies our claim that the Hausdorff dimension of $L = \partial\mathbb{D} \setminus f^{-1}(B)$ is at most $a$.

As $\dim L \leq a$ and $f$ is $\alpha$-Hölder, it follows that $\dim f(L) \leq a/\alpha < 1$. Since $B = \partial G \setminus f(L)$, it follows that $\dim B = \dim \partial G > 1$, as required.

**Proof of Proposition 1.3.** We already noticed one part of the inequality in (3.2). It remains to bound $d(\kappa)$ when $\kappa$ is close to 4. We will follow the same plan of proof, but instead of using the quantities $\int_Q |f'|^2$ we will refine Lemma 2.1 by using $\int_Q |f'|^t$ for some well chosen $t = t(\kappa)$ close to zero. What allowed us to conclude the proof of Lemma 2.1 was the fact that for bounded domains, $\int_{\mathbb{D}} |f'|^2 < \infty$. Here we will use instead the known bounds on the Integral means spectrum of univalent functions (in particular, we do not need to assume $G$ bounded in this proof).

Let us briefly recall some facts about the *Integral Means Spectrum* (see [Pom92]). Let $f$ be an univalent function in the unit disc $\mathbb{D}$. For any $t \in \mathbb{R}$, let

$$\beta_f(t) := \inf\Big\{\beta \in \mathbb{R} : \lim_{r\to 1}(1-r)^\beta \int_{|z|=r} |f'(z)|^t\, |dz| = 0\Big\}.$$

The *universal integral means spectrum* $B(t)$ of univalent functions is defined as

$$B(t) = \sup_f \beta_f(t),$$

where the supremum is over all univalent functions (often one restricts the supremum to bounded univalent functions, resulting in a slightly different spectrum). Much is known about this spectrum, see [Pom92] and references therein. We will use the following upper bound on the spectrum ([Pom92], Theorem 8.5):

$$B(t) < 4t^2 \text{ for all } t \in \mathbb{R} \setminus \{0\} .$$

We prove the following Lemma (which is a refinement of Lemma 2.1 for $p$ close to 1), from which Proposition 1.3 will easily follow.

**Lemma 3.2.** *Let $G \subsetneq \mathbb{C}$ be a simply connected domain and $f : \mathbb{D} \to G$ a conformal homeomorphism. For every $0 < p < 1$, and $\epsilon > 0$ there is $D \subset \mathbb{D}$ and $C > 0$ such that*

$$H_p(\mathbb{D} \setminus D) < \epsilon,$$

*and*

$$|f(z) - f(z')| \le C|z - z'|^{1-6\sqrt{1-p}}, \tag{3.3}$$

*for all $z, z' \in D$.*

**Proof.** Notice that the lemma is relevant only when $p$ is close enough to 1. As in the proof of Lemma 2.1, we consider the collection $\mathcal{Q}$ of dyadic squares. For each $Q \in \mathcal{Q}$, denote by $L(Q)$ the inner "segment" $\{z \in Q : |z| = 1 - 2^{-n}\}$. Fix the parameters $N > 1, t > 0$ and $0 < \delta < p$, to be determined later, and let $\mathcal{L}$ be the collection of squares $Q \in \mathcal{Q}$ for which

$$l(Q) \le 2^{-N}$$

and

$$\int_{L(Q)} |f'|^t > l(Q)^{p-\delta}.$$

Set as before

$$D = \mathbb{D} \setminus \bigcup_{Q \in \mathcal{L}} Q .$$

Using the integral means spectrum, one can easily estimate the Hausdorff $p$-content of $\mathbb{D} \setminus D$. Indeed,

$$\begin{aligned} H_p(\mathbb{D} \setminus D) \le \sum_{Q \in \mathcal{L}} l(Q)^p &\le \sum_{Q \in \mathcal{L}} l(Q)^\delta \int_{L(Q)} |f'|^t \\ &\le \sum_{n \ge N} 2^{-n\delta} \int_{|z|=1-2^{-n}} |f'|^t \\ &\le \sum_{n \ge N} C\, 2^{-n(\delta - 4t^2)}, \end{aligned}$$

since $\beta_f(t) \leq B(t) < 4t^2$, where $C > 0$ may depend on $f$. Therefore one needs to choose $\delta > 4t^2$; let $\delta := 5t^2$. By taking $N$ large enough, we get $H_p(\mathbb{D} \setminus D) \leq \epsilon$.

As in Lemma 2.1, to conclude the proof, it is enough to check that there is some $C = C(f, N, p)$ such that for all $z \in D$,

$$|f'(z)| \leq C \frac{1}{(1-|z|)^{6\sqrt{1-p}}} . \tag{3.4}$$

Let $z \in D$, $|z| \geq 1 - 2^{-N}$ (for $|z| < 1 - 2^{-N}$, we choose $C$ large enough so that (3.4) is satisfied). Let $Q$ be such that $z \in T(Q)$ and notice that $Q \notin \mathcal{L}$. Therefore $\int_{L(Q)} |f'|^t \leq l(Q)^{p-5t^2}$. By Koebe's distortion theorem, $|f'|$ fluctuates by at most a multiplicative constant with is essentially constant within $T(Q)$, and hence

$$|f'(z)| \leq O(1) \left(\frac{1}{1-|z|}\right)^{\frac{1-p+5t^2}{t}} .$$

By choosing our last parameter $t := \sqrt{1-p}$, this leads to (3.4).

Now, following Lemma 2.1, i.e., integrating along appropriate arcs, this proves Lemma 3.2. □

Proposition 1.3 follows from Lemma 3.2 with $p = 8/\kappa - 1$ in the following way: a.s. the $\mathrm{SLE}_\kappa$ trace remains in $D = D_\epsilon$ for some $\epsilon$ small enough. Moreover the map $f$ from $D_\epsilon$ to $f(D_\epsilon)$ is $\eta$-Hölder with $\eta = 1 - 6\sqrt{1-p}$. Hence, by the obvious bound (here we do not need the above Theorem of Jones and Makarov),

$$\dim \partial f(D_\epsilon) \leq \frac{1}{\eta} \dim \partial D_\epsilon = \frac{1}{1 - 6\sqrt{1-p}} \leq 1 + C_2\sqrt{\kappa - 4}$$

($p = 8/\kappa - 1$), which together with (3.2) implies Proposition 1.3. □

# 4 Related results

In [SZ07, AS08], it is proved that for $\kappa \in (4, 8]$, the chordal $\mathrm{SLE}_\kappa$ in $\mathbb{H}$ a.s. satisfies $\dim(\mathrm{SLE}_\kappa \cap \mathbb{R}) = 2 - 8/\kappa$ (the same holds of course for radial SLE in the unit disc $\mathbb{D}$). What happens in the case of a general simply connected

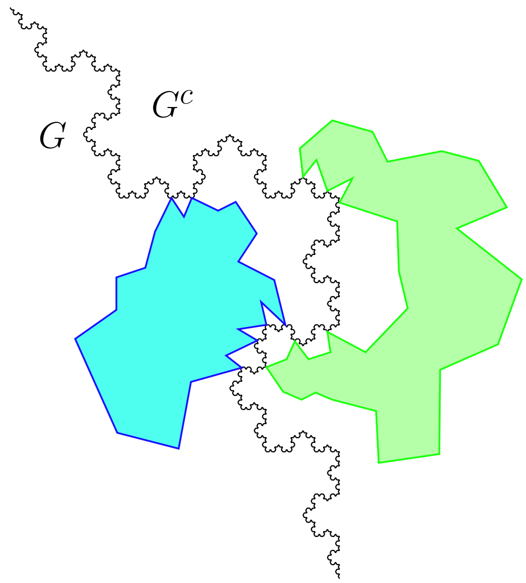



Figure 4.1: For some domains $G$, (at the continuum limit) percolation clusters inside $G$ are "invisible" to percolation clusters inside $G^c$.

domain $G$? That is, what is the Hausdorff dimension of $\mathrm{SLE}_\kappa \cap \partial G$ ? By Theorem 1.2 we know that it is $\leq d(\kappa) < 2$. If $G$ is a John domain, we can generally do much better. Notice that if one takes $\kappa = 8$, then $\mathrm{SLE}_\kappa$ is space filling and thus $\mathrm{SLE}_\kappa \cap \partial G = \partial G$. If $G$ is a John domain, it is known ([Pom92], Theorem 10.17) that the dimension $d$ of the boundary is the unique solution to the equation $\beta_f(d) = d - 1$, where $f$ is a conformal map from $\mathbb{D}$ to $G$ and $\beta_f(d)$ is the integral means spectrum of $f$. We will now sketch a proof that for general $\kappa \in (4, 8]$ and John domains $G$, the dimension of $\mathrm{SLE}_\kappa \cap \partial G$ is bounded from above by the solution $d$ of the equation $\beta_f(d) = d - (2 - 8/\kappa)$. Fix $\kappa \in (4, 8)$, and $t > 0$ such that $\beta_f(t) < t - (2 - 8/\kappa)$. Let $\gamma$ be a chordal $\mathrm{SLE}_\kappa$ from $-1$ to $1$ in $\mathbb{D}$. We are interested in $\dim f(\gamma(0,\infty)) \cap \partial G$. Since we assumed $G$ to be a John domain, there is a constant $C$ such that $C \cdot f(T(Q)) \supset f(Q)$ for each dyadic square $Q \in \mathcal{Q}$. Cover $f(\gamma(0,\infty)) \cap \partial G$ by $\bigcup C \cdot f(T(Q))$, where the union is over those $Q \in \mathcal{Q}$ for which $\gamma \cap 2 \cdot Q \neq \emptyset$ and $l(Q) \leq 2^{-N}$ with $N$ large enough. By (3.1), the expected Hausdorff $t$-content of $f(\gamma(0,\infty)) \cap \partial G$ is thus bounded by

$$\sum_{n \geq N} \sum_{l(Q)=2^{-n}} O(1)(2^{-n}|f'(z_Q)|)^t (2^{-n})^{8/\kappa - 1} = \sum_{n \geq N} O(1) 2^{-n(t-2+8/\kappa)} \int_{|z|=1-2^{-n}} |f'|^t,$$

where $z_Q$ is any point in $T(Q)$. This sum converges since $\beta_f(t) < t-(2-8/\kappa)$; so by letting $N$ going to $\infty$, the expected Hausdorff $t$-content of $f(\gamma(0,\infty))\cap\partial G$ is equal to zero, and the result follows.

If $\kappa > 16/3$, then $\dim(\mathrm{SLE}_\kappa \cap \mathbb{R}) = 2 - 8/\kappa > 1/2$ and hence two independent $\mathrm{SLE}_\kappa$, one in the upper half plane and one in the lower, will intersect a.s. This is not true any more for general Jordan domains: For each $\kappa \in (4,8)$, there exists a John domain (actually a quasidisc) $G = G(\kappa)$ and a set $E \subset \partial G$ such that if $\gamma_1$ and $\gamma_2$ are respectively $\mathrm{SLE}_\kappa$ curves driven inside and outside $G$, then a.s. $\gamma_1(0,\infty)\cap\partial G \subset E$ while $\gamma_2(0,\infty)\cap\partial G \subset E^c$. Indeed, given $\kappa \in (4,8)$ and choosing $0 < \varepsilon < 8/\kappa - 1$, by [Tha06] and [Roh91] there is a quasidisc $G$ and a subset $A \subset \partial\mathbb{D}$ with $\dim A < \varepsilon$ and $\dim \partial\mathbb{D} \setminus f_c^{-1}(f(A)) < \varepsilon$, where $f$ and $f_c$ are conformal maps from $\mathbb{D}$ to $G$ and $G^c$. It follows that $\mathrm{SLE}_\kappa$ in $\mathbb{D}$ will a.s. be disjoint from both $A$ and $\partial\mathbb{D} \setminus f_c^{-1}(f(A))$, and the claim follows with $E = f(\partial\mathbb{D} \setminus A)$. This can be viewed as an SLE analog of the Theorem by Bishop, Carleson, Garnett and Jones about harmonic measure (see [BCGJ89, Roh91]). Figure 4.1 is an illustration of this property in the case of percolation clusters.

# Chapter V

# Fourier Spectrum of Critical Percolation

Joint work with *Gábor Pete* and *Oded Schramm*.

Consider the indicator function $f$ of a two-dimensional percolation crossing event. In this paper, the Fourier transform of $f$ is studied and sharp bounds are obtained for its lower tail in several situations. Various applications of these bounds are derived. In particular, we show that the set of exceptional times of dynamical critical site percolation on the triangular grid in which the origin percolates has dimension 31/36 a.s., and the corresponding dimension in the half-plane is 5/9. It is also proved that critical bond percolation on the square grid has exceptional times a.s. Also, the asymptotics of the number of sites that need to be resampled in order to significantly perturb the global percolation configuration in a large square is determined.

## Contents

# 1 Introduction

## 1.1 Some general background

The Fourier expansion of functions on $\mathbb{R}^d$ is an indispensable tool with numerous applications. Likewise, the harmonic analysis of functions defined on the discrete cube $\{-1,1\}^d$ has found a host of applications; see the survey [KS06]. Yet the Fourier expansion of some functions of interest is rather poorly understood. Here, we study the harmonic analysis of functions arising from planar percolation and answer most if not all of the previously posed problems regarding their Fourier expansion. We also derive some applications to the behavior of percolation under noise and in the study of dynamical percolation. It is hoped that some of the techniques introduced here will be helpful in the harmonic analysis of other functions.

Let $\mathcal{I}$ be some finite set, and let $\Omega = \{-1,1\}^{\mathcal{I}}$ be endowed with the uniform measure. The Fourier basis on $\Omega$ consists of all the functions of the form $\chi_S(\omega) := \prod_{i\in S}\omega_i$, where $S \subset \mathcal{I}$. (These functions are also sometimes called the Walsh functions.) It is easily seen to be an orthonormal basis with respect to the inner product $\mathbb{E}\big[f\,g\big]$. Therefore, for every $f:\Omega\to\mathbb{R}$, we have

$$f = \sum_{S\subset\mathcal{I}} \widehat{f}(S)\,\chi_S\,, \tag{1.1}$$

where $\widehat{f}(S) := \mathbb{E}[f\, \chi_S]$. If $\mathbb{E}[f^2] = 1$, then the random variable $\mathscr{S} = \mathscr{S}_f \subset \mathcal{I}$ with distribution given by

$$\mathbb{P}\big[\mathscr{S} = S\big] = \widehat{f}(S)^2$$

will be called the **Fourier spectral sample** of $f$. Due to Parseval's formula, this is indeed a probability distribution. The idea to look at this as a probability distribution was proposed in [BKS99], though the study of the weights $\widehat{f}(S)^2$ is "ancient", and boils down to the same questions in a different lingo. As noted there, important properties of the function $f$ are encoded in the law of the spectral sample. For example, suppose that $f : \{-1, 1\}^{\mathcal{I}} \to \{-1, 1\}$. Let $x \in \{-1, 1\}^{\mathcal{I}}$ be random and uniform, and let $y$ be obtained from $x$ by resampling[1] each coordinate with probability $\epsilon$ independently, where $\epsilon \in (0, 1)$. Then $y$ is referred to as an $\epsilon$-noise of $x$. Since for $i \in \mathcal{I}$ we have $\mathbb{E}\big[x_i\, y_i\big] = 1-\epsilon$ it follows that $\mathbb{E}\big[\chi_S(x)\, \chi_S(y)\big] = (1-\epsilon)^{|S|}$ for $S \subset \mathcal{I}$ and hence it easily follows by using the Fourier expansion (1.1) that

$$\mathbb{E}\big[f(x)\, f(y)\big] = \mathbb{E}\big[(1-\epsilon)^{|\mathscr{S}_f|}\big]. \tag{1.2}$$

Thus, the stability or sensitivity of $f$ to noise is encoded in the law of $|\mathscr{S}|$.

One mathematical model in which noise comes up is that of the Poisson dynamics on $\Omega$, in which each coordinate is resampled according to a Poisson process of rate 1, independently. This is, of course, just the continuous time random walk on $\Omega$. If $x_t$ denotes this continuous time Markov process started at the stationary (uniform) measure on $\Omega$, then $x_t$ is just $\epsilon$-noise of $x_0$, where $\epsilon = 1 - e^{-t}$. Indeed, the Markov operator defined by

$$T_t f(x) = \mathbb{E}\big[f(x_t) \,\big|\, x_0 = x\big]$$

is diagonalized by the Fourier basis:

$$T_t \chi_S = e^{-|S|} \chi_S\,.$$

It is therefore hardly surprising that the behavior of $|\mathscr{S}_f|$ will play an important role in the study of the generally non-Markov process $f(x_t)$. These types

[1] In the definition of [BKS99], each bit is flipped with probability $\epsilon$, rather than resampled. This accounts for some discrepancies involving factors of 2. The present formulation generally produces simpler formulas.

of questions have been under investigation in the context of Bernoulli percolation [HPS97], [PS98], [HP99], [PSS07], [Kho08], other percolation type processes [VMW97], [BS98], [BS06], and also more generally [BHPS03], [JS06], [Hof06], [KLMH06]. Estimates of the Fourier coefficients played an important role in the proof that the dynamical version of critical site percolation on the triangular grid a.s. has percolation times [SS05]. These estimate can naturally be phrased in terms of properties of the random variable $|\mathscr{S}|$.

Recall that the (random) set of **pivotals** of $f$ is the set of $i \in \mathcal{I}$ such that flipping the value of $\omega_i$ also changes the value of $f(\omega)$. It is easy to see [KKL88] that the first moment of the number of pivotals of $f$ is the same as the first moment of $|\mathscr{S}_f|$. Gil Kalai (personal communication) observed that the same is true for the second moment, but not for the higher moments. (We will recall the easy proof of this fact in Section 2.3.) This often facilitates an easy estimation of $\mathbb{E}\big[|\mathscr{S}_f|\big]$ and $\mathbb{E}\big[|\mathscr{S}_f|^2\big]$.

It is often the case that $\mathbb{E}\big[|\mathscr{S}|^2\big]$ is of the same order of magnitude as $\mathbb{E}\big[|\mathscr{S}|\big]^2$, and this implies that with probability bounded away from zero, the random variable $|\mathscr{S}|$ is of the same order of magnitude as its mean. However, what turns out to be much harder to estimate is the probability that $|\mathscr{S}|$ is positive and much smaller than its mean. (In particular, this is much harder than the analogous result for pivotals.) This is very relevant to applications; as can be seen from (1.2), the probability that $|\mathscr{S}|$ is small is what matters most in understanding correlation of $f(x)$ with $f$ evaluated on a noisy version of $x$. Likewise, in the dynamical setting, the lower tail of $|\mathscr{S}|$ controls the switching rate of $f$. Indeed, the primary purpose of this paper is getting good estimates on $\mathbb{P}\big[0 < |\mathscr{S}_f| < s\big]$ for indicators of crossing events in percolation and deriving the consequences of such bounds.

## 1.2 The main result

We consider two percolation models: critical bond percolation on the square grid $\mathbb{Z}^2$ and critical site percolation on the triangular grid. See [Wer07] for background. These two models are believed to behave essentially the same, but the mathematical understanding of the latter is significantly superior to the former due to Smirnov's theorem [Smi01] and its consequences. Fix some large $R > 0$, and consider the event that (in either of these percolation models) there is an open (i.e., occupied) left-right crossing of the square $[0, R]^2$. Let $f_R$ denote the $\pm 1$ indicator function of this event; that is, $f_R = 1$

if there is a crossing and $f_R = -1$ when there is no crossing. In this case, $\mathcal{I}$ is the relevant set of bonds or sites, depending on whether we are in the bond or site model. The probability space is $\Omega = \{-1, 1\}^{\mathcal{I}}$ with the uniform measure. Here, it is convenient that $p_c = 1/2$ for both models, and so the relevant measure on $\Omega$ is uniform.

The paper [BKS99] posed the problem of studying the law of $\mathscr{S}_{f_R}$. There, it was proved that $\mathbb{P}\big[0 < |\mathscr{S}_{f_R}| < c \log R\big] \to 0$ as $R \to \infty$ for some $c > 0$. This had the implication that $f_R$ is asymptotically noise sensitive; that is, $\lim_{R\to\infty} \mathbb{E}\big[f_R(x)\, f_R(y)\big] - \mathbb{E}\big[f_R(x)\big]\mathbb{E}\big[f_R(y)\big] = 0$, when $y$ is an $\epsilon$-noisy version of $x$ and $\epsilon > 0$ is fixed. It was also asked in [BKS99] whether $\mathbb{P}\big[0 < |\mathscr{S}| < R^{\delta}\big] \to 0$ for some $\delta > 0$. This was later proved in [SS05] with $\delta = 1/8 + o(1)$ in the setting of the triangular lattice and with an unspecified $\delta > 0$ for the square lattice. While certainly a useful step forward, these results were far from being sharp. But the issue is more than just a quantitative question. The most natural hypothesis is that $|\mathscr{S}|$ is always proportional to its mean when it is nonzero, or more precisely that

$$\sup_{R>1} \mathbb{P}\Big[0 < |\mathscr{S}_{f_R}| < t\, \mathbb{E}|\mathscr{S}_{f_R}|\Big] \to 0 \qquad \text{as } t \searrow 0\,. \tag{1.3}$$

To illustrate the fact that (1.3) is not a universal principle, we note that it does not hold, e.g., for the $\pm 1$-indicator function of the event that there is a left-right percolation in the square $[0, R]^2$ and this square has more open sites [or edges] than closed sites [or edges]. We prove (1.3) in the present paper, and give useful bounds that are sharp up to constants on the left hand side of (1.3). This is the content of our first theorem.

**Theorem 1.1.** *As above, let $R > 1$ and let $f_R$ be the $\pm 1$ indicator function of the left-right crossing event of the square $[0, R]^2$ in critical bond percolation on $\mathbb{Z}^2$ or site percolation on the triangular grid. The spectral sample of $f_R$ satisfies*

$$\mathbb{P}\Big[0 < |\mathscr{S}_{f_R}| < \mathbb{E}|\mathscr{S}_{f_r}|\Big] \asymp \Big(\frac{\mathbb{E}|\mathscr{S}_{f_R}|/R}{\mathbb{E}|\mathscr{S}_{f_r}|/r}\Big)^2 \tag{1.4}$$

*holds for every $r \in [1, R]$, and $\asymp$ denotes equivalence up to positive multiplicative constants.*

To make the sharp bound (1.4) more explicit, one needs to discuss estimates for $\mathbb{E}|\mathscr{S}_{f_r}|$. The value of $\mathbb{E}|\mathscr{S}_{f_r}|$ is estimated by the probability of the so called "alternating 4-arm event", which we will treat in detail in Subsection 2.2. For now, let us just mention that it is known that $\mathbb{E}|\mathscr{S}_{f_R}|/\mathbb{E}|\mathscr{S}_{f_r}| \le (R/r)^{1-\delta}$ for some $\delta > 0$, and that for the triangular lattice

$$\mathbb{E}|\mathscr{S}_{f_R}| \,/\, \mathbb{E}|\mathscr{S}_{f_r}| \asymp (R/r)^{3/4+o(1)} \tag{1.5}$$

as $R/r \to \infty$ while $r \ge 1$ follows from Smirnov's theorem [Smi01] and the SLE-based analysis of the percolation exponents in [SW01]. (This will be proved in Section 7.2.) From (1.5) and (1.4), we get for the triangular grid

$$\mathbb{P}\big[0 < |\mathscr{S}_{f_R}| \le \lambda\, \mathbb{E}|\mathscr{S}_{f_R}|\big] \asymp \lambda^{2/3+o(1)}, \tag{1.6}$$

where $\lambda$ may depend on $R$, but is restricted to the range $\big[(\mathbb{E}|\mathscr{S}_{f_R}|)^{-1}, 1\big]$. Here, the $o(1)$ represents a function of $\lambda$ and $R$ that tends to 0 as $\lambda \to 0$, uniformly in $R$. This answers Problem 5.1 from [Sch07]. For either lattice, (1.3) follows from Theorem 1.1. Below, we prove (7.6), which is a variant of (1.6) with slightly different asymptotics.

There is nothing particularly special about the square with regard to Theorem 1.1. The proof applies to every rectangle of a fixed shape (with the implied constants depending on the shape). For percolation crossings in more general shapes, Theorem 7.1 gives bounds on the behavior of $\mathscr{S}$ away from the boundary, and we also prove Proposition 7.4, which is some analog of (1.3). However, we chose not to go into the complications that would arise when trying to prove (1.4) in this general context.

We also remark that $\mathbb{P}\big[\mathscr{S}_f = \emptyset\big] = \mathbb{E}[f]^2 = \widehat{f}(\emptyset)^2$ is generally easy to compute. The level 0 Fourier coefficient $\widehat{f}(\emptyset)$ has more to do with the way the function is normalized than with its fundamental properties. For this reason, $|\mathscr{S}| = 0$ is separated out in bounds such as (1.4).

At this point we mention that Theorem 7.3 gives the bound analogous to Theorem 1.1, but dealing with the spectrum of the indicator function for a percolation crossing from the origin to distance $R$ away.

## 1.3 Applications to noise sensitivity

Figure 1.1 illustrates a sequence of percolation configurations, where each configuration is obtained from the previous one by applying some noise. The effect on the interfaces can be observed. Theorem 1.1 implies the following sharp noise sensitivity estimate regarding such perturbations:

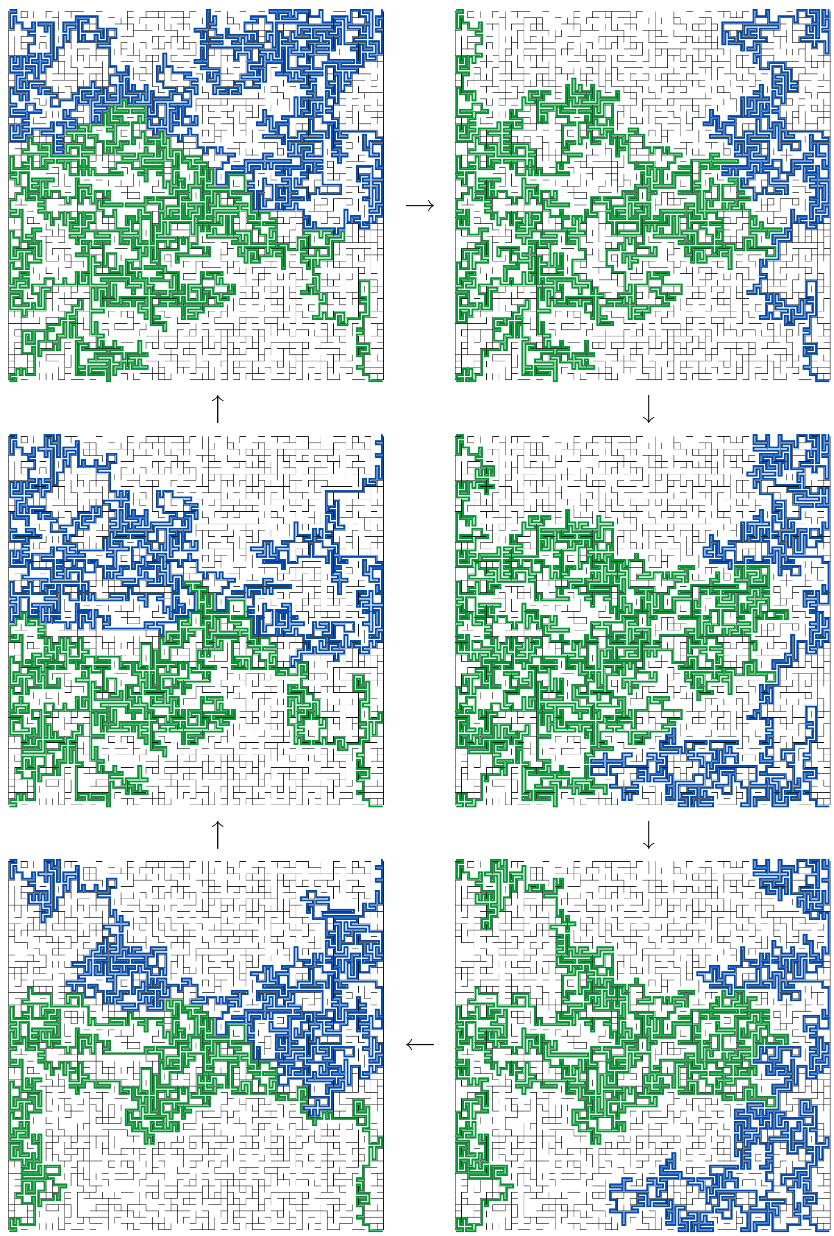


Figure 1.1: Interfaces in percolation on $\mathbb{Z}^2$. Each successive pair of configurations are related by a noise of about 0.04, which results in about one in every 50 bits being different. Each square is about $60 \times 60$ lattice squares.

**Corollary 1.2.** *Suppose that* $y$ *is an* $\epsilon_R$*-noisy version of* $x$*, where* $\epsilon_R \in (0,1)$ *may depend on* $R$*. If* $\lim_{R\to\infty} \mathbb{E}|\mathscr{S}_{f_R}|\,\epsilon_R = \infty$*, then*

$$\lim_{R\to\infty} \mathbb{E}\big[f_R(x)\, f_R(y)\big] - \mathbb{E}\big[f_R(x)\big]\mathbb{E}\big[f_R(y)\big] = 0\,, \tag{1.7}$$

*while if* $\lim_{R\to\infty} \mathbb{E}|\mathscr{S}_{f_R}|\,\epsilon_R = 0$*, then*

$$\lim_{R\to\infty} \mathbb{E}\big[f_R(x)\, f_R(y)\big] - \mathbb{E}\big[f_R(x)^2\big] = 0\,. \tag{1.8}$$

The second of these statements is actually obvious from (1.2) and Jensen's inequality, and is brought here only to complement the first claim. Of course, (1.7) just means that having a crossing in $x$ is asymptotically uncorrelated with having a crossing in $y$, while (1.8) means that with probability going to 1, a crossing in $x$ occurs if and only if there is a crossing in $y$. Although $f_R(x)^2 = 1$, we find the form of (1.8) more suggestive, since this is the statement that generalizes to other situations. Likewise, $\lim_{R\to\infty} \mathbb{E}\big[f_R(x)\big] = 0$, but (1.7) is more suggestive.

In Corollary 8.1, we prove a generalization of Corollary 1.2 for the crossing function $f_{R\mathcal{Q}}$, where $\mathcal{Q}$ is an arbitrary fixed "quad", i.e., a domain homeomorphic to a disk with four marked points on its boundary.

In a forthcoming paper on the scaling limit of dynamical percolation [GPS], we plan to show that for critical percolation on the triangular grid, whenever $\lim_{R\to\infty} \mathbb{E}|\mathscr{S}_{f_R}|\,\epsilon_R$ exists and is in $(0,\infty)$, then $\lim_R \mathbb{E}\big[f_R(x)\, f_R(y)\big]$ also exists and is strictly between the limits of $\mathbb{E}\big[f_R(x)\big]^2$ and $\mathbb{E}\big[f_R(x)^2\big]$.

The following theorem proves Conjecture 5.1 from [BKS99]. With minor adaptations, it follows from the proof of Theorem 1.1.

**Theorem 1.3.** *Consider bond percolation on* $\mathbb{Z}^2$*. Let* $x$ *be a critical percolation configuration, and let* $z$ *be another critical percolation configuration, which equals* $x$ *on the horizontal edges, but is independent from* $x$ *on the vertical edges. Then having a left-right crossing in* $x$ *is asymptotically independent from having a left-right crossing in* $z$*. Moreover, the same holds true if "horizontal" and "vertical" are interchanged.*

## 1.4 Applications to dynamical percolation

First, recall that in dynamical percolation the random bits determining the percolation configuration are refreshed according to independent Poisson clocks of rate 1. Dynamical percolation was proposed in 1992 by Itai

Benjamini, though the first paper on the subject is by Häggström, Peres and Steif [HPS97]. Since then, dynamical percolation and other dynamical random processes have been the focus of several research papers; see the references in Section 1.1. In response to a question in [HPS97] it was proved in [SS05] that critical dynamical site percolation on the triangular grid a.s. has times at which the origin is in an infinite connected percolation component. Such times are called "exceptional", since they necessarily have measure zero. The paper [SS05] also showed that the dimension of the set of exceptional times is a.s. in $\left[\frac{1}{6}, \frac{31}{36}\right]$, and conjectured that it is a.s. 31/36. Likewise, it was conjectured there that the set of times at which an infinite occupied as well as an infinite vacant cluster coexist is a.s. 2/3, but [SS05] only proved that 2/3 is an upper bound, without establishing the existence of such times. Similarly, [SS05] proved that the set of times at which there is an infinite percolation component in the upper half plane has Hausdorff dimension at most 5/9, but did not prove that the set is nonempty. There were numerous other lower and upper bounds of this type in [SS05], some of them having to do with dynamical percolation in wedges and cones, but they will not be discussed presently. We can now prove most of these conjectures regarding the Hausdorff dimensions of exceptional times in the setting of the triangular grid, and for each case corresponding to a monotone event, the Hausdorff dimension a.s. equals the previously known upper bound. In particular, we have

**Theorem 1.4.** *In the setting of dynamical critical site percolation on the triangular grid, we have the following a.s. values for the Hausdorff dimensions.*

1. *The set of times at which there is an infinite cluster a.s. has Hausdorff dimension* 31/36*.*
2. *The set of times at which there is an infinite cluster in the upper half plane a.s. has Hausdorff dimension* 5/9*.*
3. *The set of times at which an infinite occupied cluster and an infinite vacant cluster coexist a.s. has Hausdorff dimension at least* 1/9*.*

The reason that in 1 and 2 the results agree with the conjectured upper bound from [SS05] is that the upper bound is dictated by $\mathbb{E}\big[|\mathscr{S}_f|\big]$ (which is generally not hard to compute), while the tail estimate given in (1.4) and its analogs give sufficient estimates to bound the probability that $|\mathscr{S}_f|$ is much smaller than its expectation. Here, $f$ is the indicator function of some crossing event, which may vary from one application to another. We cannot calculate the exact dimension in item 3 because we use the monotonicity of $f$ in an essential way (though at only one point), and the event that both vacant and occupied percolation crossings occur is not monotone.

See Section 9 for further results and an explanation as to how these numbers are calculated.

The paper [SS05] came quite close to proving that exceptional times exist for dynamical critical bond percolation on $\mathbb{Z}^2$, but was not able to do it. Now, we close this gap.

**Theorem 1.5.** *A.s. there are exceptional times at which dynamical critical bond percolation on $\mathbb{Z}^2$ has infinite clusters, and the Hausdorff dimension of the set of such times is a.s. positive.*

There is one more application to dynamical percolation that we will presently mention. This has to do with the scaling limit of dynamical percolation, as introduced in [Sch07], and whose existence we plan to show in [GPS]. In this scaling limit, time and space are both scaled, and the relationship between their scaling is chosen in such a way that the event of the existence of a percolation crossing of the unit square at time 0 and at one unit of time later have some fixed correlation strictly between 0 and 1. Consequently, as space is shrinking, time is expanding. We leave it as an excercise to the reader to verify that the ratio between the scaling of time and of space can be worked out directly from the law of $|\mathscr{S}_{f_R}|$. An easy consequence of (1.3) is that in the dynamical percolation scaling limit, the correlation between having a left-right crossing of the square at time 0 and at time $t$ goes to zero as $t \to \infty$; see (8.6). In fact, based on [SS] and estimates such as (1.3) and its generalizations to other domains, it can be shown that the dynamical percolation scaling limit is ergodic. These results answer Problem 5.3 from [Sch07].

## 1.5 The scaling limit of the spectral sample

The study of the scaling limit of $\mathscr{S}$ was suggested by Gil Kalai [Sch07, Problem 5.2] (see also [BKS99, Problem 5.4]). The idea is that we can think of $\mathscr{S}_{f_R}$ as a subset of the plane, and consider the existence of the weak limit as $R \to \infty$ of the law of $R^{-1}\,\mathscr{S}_{f_R}$. Boris Tsirelson [Tsi04] addressed this problem more generally within his theory of noises, dealing with various functions $f$

that are not necessarily related to percolation. It follows from Tsirelson's theory and from [SS] that the scaling limit of $\mathscr{S}_{f_R}$ exists. In Section 10, we explain this, and prove

**Theorem 1.6.** *In the setting of the triangular grid, the limit in law of* $R^{-1}\,\mathscr{S}_{f_R}$ *exists. It is a.s. a Cantor set of dimension* $3/4$.

The conformal invariance of the scaling limit of $\mathscr{S}_{f_R}$ in the setting of the triangular grid is also proved in Section 10. These results answer a problem posed by Gil Kalai [Sch07, Problem 5.2].

## 1.6 A rough outline of the proof

The proof of Theorem 1.1 does not follow the same general strategy as the proof of the non-sharp bounds given in [SS05]. The lower bound on the left hand side in (1.4) is rather easy, and so we only discuss here the proof of the upper bound. Fix some $r \in [1, R]$ and subdivide the square $[0, R]^2$ into subsquares of sidelength $r$ (suppose that $r$ divides $R$, say). Let $\mathscr{S}(r)$ denote the set of these subsquares that intersect $\mathscr{S}_{f_R}$. In Section 4 we estimate the probability that $|\mathscr{S}(r)| = k$ when $k$ is small (for example, $k = O(\log(R/r))$). The argument is based on building a rough geometric classification of all the possible configurations of $\mathscr{S}(r)$, applying a bound for each class, and summing over the different classes. The bound obtained this way is

$$\mathbb{P}\big[|\mathscr{S}(r)| = k\big] \le \exp\big(O(1)\,\log^2(k+2)\big)\Big(\frac{\mathbb{E}|\mathscr{S}_{f_R}|/R}{\mathbb{E}|\mathscr{S}_{f_r}|/r}\Big)^2, \tag{1.9}$$

and has the optimal dependence on $R$ and $r$, but a rather bad dependence on $k$.

Here is a naive strategy for getting from (1.9) to (1.4), which does not seem to work. Fix some $r \times r$ square $B$. Suppose that we are able to show that conditioned on the intersection of $\mathscr{S}$ with some set $W$ in the complement of the $r$-neighborhood of $B$, and conditioned on $\mathscr{S}$ intersecting $B$, we have a probability bounded away from 0 that $|\mathscr{S} \cap B| > \mathbb{E}|\mathscr{S}_{f_r}|$. We can then restrict to a sublattice of $r \times r$ squares that are at mutual distance at least $r$ and easily show by induction that the probability that $\mathscr{S}$ intersects at least $k'$ of the squares in the sublattice but has size less than $\mathbb{E}|\mathscr{S}_{f_r}|$ is exponentially small in $k'$. We may then take a bounded set of such sublattices, which covers every one of the $r \times r$ squares in our initial tiling of $[0,R]^2$, and thereby obtain the required bound on $\mathbb{P}\big[0 < |\mathscr{S}| < \mathbb{E}|\mathscr{S}_{f_r}|\big]$ from (1.9) and the exponentially small bound in $k'$. The reason that this strategy fails is that there are presently no good tools to understand the conditional law of $B \cap \mathscr{S}$ given $B \cap W$. Refusing to give up, we observe that, as explained in Section 2.3, the law of $B \cap \mathscr{S}$ conditioned on $\mathscr{S} \cap W = \emptyset$ can be described. Based on this, we amend the above strategy, as follows. We pick a random set $W \subset \mathcal{I}$ independent from $\mathscr{S}$, where each $i \in \mathcal{I}$ is put in $W$ with probability about $1/\mathbb{E}|\mathscr{S}_{f_r}|$ independently. Then, we can hope to get a good upper bound on $\mathbb{P}\big[\mathscr{S} \neq \emptyset = W \cap \mathscr{S}\big]$, which would almost immediately give a constant times the same bound on $\mathbb{P}\big[0 < |\mathscr{S}| < \mathbb{E}|\mathscr{S}_{f_r}|\big]$.

In Section 5 we show that for an $r \times r$ square $B$, if we condition on $\mathscr{S} \cap B \neq \emptyset$ and on $\mathscr{S} \cap B^c \cap W = \emptyset$, then with probability bounded away from 0 we have $\mathscr{S} \cap B' \cap W \neq \emptyset$, where $B'$ is a square of $1/3$ the sidelength that is concentric with $B$; namely,

$$\mathbb{P}\big[\mathscr{S} \cap B' \cap W \neq \emptyset \,\big|\, \mathscr{S} \cap B^c \cap W = \emptyset, \mathscr{S} \cap B \neq \emptyset\big] > a > 0\,, \tag{1.10}$$

for some constant $a$. This is based on a second moment argument, but to carry it through we have to resort to rather involved percolation arguments. A key observation here is to interpret these conditional events for the spectrum sample in terms of percolation events for a coupling of two configurations (which are independent on the set $W$ but coincide elsewhere). An important step is to prove a quasi-multiplicativity property for arm-events in the case of this system of coupled configurations.

Again, there is a simple naive strategy based on (1.10) and (1.9) to get an upper bound for $\mathbb{P}\big[\mathscr{S} \neq \emptyset = W \cap \mathscr{S}\big]$. One may try to check sequentially if $B' \cap W \cap \mathscr{S} \neq \emptyset$ for each of the $r \times r$ squares, and as long as a nonempty intersection has not been found, the probability to detect a nonempty intersection is proportional to the conditional probability that $\mathscr{S} \cap B \neq \emptyset$. However, the trouble with this strategy is that the conditional probability of $\mathscr{S} \cap B \neq \emptyset$ varies with time, and the bound (1.9) does not imply a similar bound for the sum of the conditional probabilities, since each time the conditioning is different.

The substitute for this naive strategy is a large deviations estimate that we state and prove in Section 6, namely, Proposition 6.1. This result is somewhat in the flavor of the Lovász local lemma, since it gives estimates for probabilities of events with a possibly complicated dependence structure. The proposition deals with random variables $x, y \in \{0,1\}^n$. In the application, $x_i$ is the indicator of the event that $\mathscr{S}$ intersects the square of sidelength

$r/3$ concentric with the $i$'th $r \times r$ square and $y_i$ is indicator of the event that $\mathscr{S} \cap W$ intersects the $i$'th $r \times r$ square. The assumption (1.10) then translates to

$$\mathbb{P}\big[y_j = 1 \bigm| y_i = 0 \,\forall i \in I\big] \geq a\, \mathbb{P}\big[x_j = 1 \bigm| y_i = 0 \,\forall i \in I\big]\,, \qquad j \notin I \subset [n]\,,$$

and the proposition tells us that under these assumptions we have

$$\mathbb{P}\big[y = 0 \bigm| X > 0\big] \leq a^{-1}\, \mathbb{E}\big[e^{-aX/e} \bigm| X > 0\big]\,, \tag{1.11}$$

where $X = \sum_i x_i$. In our application $X = |\mathscr{S}(r)|$, and thus (1.11) combines with (1.10) to yield the desired bound. The proof of Theorem 1.1 is completed in this way in Section 7.

**Acknowledgments**. We are grateful to Gil Kalai for inspiring conversations and for his observation (2.11), and to Jeff Steif for insights regarding the dimensions of exceptional times in dynamical percolation. We thank Vincent Beffara for permitting us to include Proposition 12.1 and Pierre Nolin and Wendelin Werner for enlightening discussions.

# 2 Some basics

## 2.1 A few general definitions

In this paper we consider site percolation on the triangular grid as well as bond percolation on $\mathbb{Z}^2$, both at the critical parameter $p = 1/2$.

In the case of site percolation on the triangular grid $T$, a percolation configuration $\omega$ is just the set of sites which are open. However, we often think of $\omega$ as a coloring of the plane by two colors: in the hexagonal grid dual to $T$, a hexagon is colored white if the corresponding site is in $\omega$, while the other hexagons are colored black. If $A$ and $B$ are subsets of the plane, we say that there is a **crossing** in $\omega$ from $A$ to $B$ if there is a continuous path with one endpoint in $A$ and the other endpoint in $B$ that is contained in the closure of the union of the white hexagons. Likewise, a **dual crossing** corresponds to a path contained in the closure of the black tiles.

In the case of bond percolation on $\mathbb{Z}^2$, there is a similar coloring of the plane by two colors which has the "correct" connectivity properties. In this case, we color by white all the points that are within $L_\infty$ distance of $1/4$ from all the vertices of $\mathbb{Z}^2$ and all the points that are within $L_\infty$ distance of

$1/4$ from the edges in $\omega$, and color by black the closure of the complement of the white colored points.

Regardless of the grid, the set of points whose color is determined by $\omega_x$ will be called the **tile** of $x$. In the case of the square grid, we also have tiles with deterministic color, namely, each square of sidelength $1/2$ centered at a vertex of $\mathbb{Z}^2$ and each square of sidelength $1/2$ concentric with a face of $\mathbb{Z}^2$. Thus, in either case we have a tiling of the plane by hexagons or squares where each tile consists of a connected component of a set of points whose colors always agree.

A **quad** $\mathcal{Q}$ is a subset of the plane homemorphic to the closed unit disk together with a distinguished pair of disjoint closed arcs on $\partial\mathcal{Q}$. We say that $\omega$ has a **crossing** of $\mathcal{Q}$ if the two distinguished arcs can be connected by a white path inside $\mathcal{Q}$.

If $\mathcal{A}$ is an event, then the $\pm 1$ **indicator** function of $\mathcal{A}$ is the function $2 \cdot 1_{\mathcal{A}} - 1$, which is 1 on $\mathcal{A}$ and $-1$ on $\neg\mathcal{A}$. The $\pm 1$ indicator function for the event that a quad $\mathcal{Q}$ is crossed will be denoted by $f_{\mathcal{Q}}$.

We use $\mathcal{I}$ to denote the set of bits in $\omega$; that is, in the context of the triangular grid $\mathcal{I}$ is the set of vertices of the grid, and in the context of $\mathbb{Z}^2$ it denotes the set of edges. Although $\mathcal{I}$ is not finite in these cases, the functions we consider will only depend on finitely many bits in $\mathcal{I}$, and so the Fourier-Walsh expansion (1.1) still holds. Moreover, for $L^2$ functions depending on infinitely many bits we still have (1.1), except that the summation is restricted to finite $S \subset \mathcal{I}$.

Since we will be considering $\mathscr{S}$ as a geometric object, we find it convenient to think of $\mathcal{I}$ as a set in the plane. In the context of the triangular grid, this is anyway the case, but for the square grid we will implicitly associate each edge of $\mathbb{Z}^2$ with its center; so $\mathcal{I}$ can be considered as the set of centers of the relevant edges. This way, any subset of the plane also represents a subset of the bits. Note however that e.g. the crossing function $f_{\mathcal{Q}}$ usually depends on more bits than the ones contained in $\mathcal{Q}$.

For $z \in \mathbb{R}^2$ and $r \geq 0$, the set $z + [-r, r)^2$ will be called the square of radius $r$ centered at $z$. Furthermore, we let $B(z, r)$ denote the union of the tiles whose center is contained in $z + [-r, r)^2$, and will refer to $B(z, r)$ as a **box** of radius $r$. One reason for using these boxes (instead of round balls, say) is that the plane can be tiled with them perfectly.

## 2.2 Multi-arm events for percolation

In many different studies of percolation, the multi-arm events play a central role. We now define these events (a word of caution — there are a few different natural variants to these definitions), and discuss the asymptotics of their probabilities.

Let $A \subset \mathbb{R}^2$ be some topological annulus in the plane, and let $j \in \mathbb{N}_+$. If $j$ is even, then the **$j$-arm event** in $A$ is the event that there are $j$ disjoint monochromatic paths joining the two boundary components of $A$, and these paths in circular order are alternating between white and black. If $j$ is odd, the definition is similar, except that the order of the colors is required to be (in circular order) alternating between white and black with one additional white crossing.

In most papers, the restriction that the colors are alternating is relaxed to the requirement that not all crossings are of the same color. Indeed, it is known that if $A$ is an annulus, $A = B(0,R) \setminus B(0,r)$, then in the setting of critical site percolation on the triangular grid the circular order of the colors effects the probability of the event by at most a constant factor (which may depend on $j$), provided that in the case $j > 1$ there is at least one required crossing from each color [ADA99]. However, since it appears that the corresponding result for the square grid has not been worked out, we have opted to impose the alternating colors restriction.

We let $\alpha_j(A)$ denote the probability of the $j$-arm event in $A$. For the case $A = B(0,R) \setminus B(0,r)$, write $\alpha_j(r,R)$ for $\alpha_j(A)$. Note that $\alpha_j(r,R) = 0$ if $r << j < R$. We use $\alpha_j(R)$ as a shorthand for $\alpha_j(2\,j, R)$. We will also adopt the convention that $\alpha_j(r,R) = 1$ if $r \geq R$.

We now review some of the results concerning these arm events. The Russo-Seymour-Welsh (RSW) estimates imply that

$$s^{-a_j}/C_j \leq \alpha_j(r, s\,r) \leq C_j s^{-1/a_j} \tag{2.1}$$

for all $r, s > 1$, where $C_j, a_j > 1$ depend only on $j$. Another important property of these arm events is **quasi-multiplicativity**, namely,

$$\alpha_j(R)/C_j \leq \alpha_j(r)\,\alpha_j(r,R) \leq C_j\,\alpha_j(R) \tag{2.2}$$

for $1 < r < R$, where, again, $C_j > 0$ depends only on $j$. This was proved in [Kes87]; see [SS05, Proposition A.1] and [Nol07, Section 4] for concise proofs.

The above properties in particular give for $r < r' < R' < R$ that

$$C_j^{-1}\Big(\frac{R\,r'}{R'\,r}\Big)^{a_j^{-1}}\alpha_j(r,R) \le \alpha_j(r',R') \le C_j\,\Big(\frac{R\,r'}{R'\,r}\Big)^{a_j}\alpha_j(r,R)\,, \tag{2.3}$$

with possibly different constants $C_j$.

Of the multi-arm events, the most relevant to this paper is the 4-arm event, due to its relation to pivotality for the crossing event in a quad $\mathcal{Q}$. In particular, for closed $B \subset \mathbb{R}^2$, we will use $\alpha_\Box(B,\mathcal{Q})$ to denote the probability of having four arms in $\mathcal{Q} \setminus B$, the white arms connecting $\partial B$ to the two distinguished arcs on $\partial\mathcal{Q}$ and the black arms to the complementary arcs. If $B \cap \partial\mathcal{Q} \neq \emptyset$, then the arms connecting $B$ to the arcs of $\partial\mathcal{Q}$ which $B$ intersects are considered as present. Quasi-multiplicativity often generalizes easily to this quantity; for example, if $\mathcal{Q}$ is an $R \times R$ square with two opposite sides being the distinguished arcs, $B$ is a radius $r$ box anywhere in $\mathcal{Q}$, and $x \in B$ is at a distance at least $cr$ from $\partial B$, then

$$\alpha_\Box(x,\mathcal{Q})/\alpha_\Box(B,\mathcal{Q}) \asymp \alpha_4(x,B) \asymp \alpha_4(r)\,, \tag{2.4}$$

with the implied constants depending only on $c$. (A more general version of this will also be proved in Section 5.5.) Here and in the following, when we write $\alpha_\Box(x,\mathcal{Q})$ or $\alpha_4(x,B)$, we are referring to the corresponding 4-arm event from the tile of $x$ to $\partial\mathcal{Q}$ or $\partial B$ (with or without paying attention to any distinguished arms on the boundary, respectively).

Let us also recall what is known about $\alpha_4$ quantitatively. For site percolation on the triangular lattice, by [SW01], we have

$$\alpha_4(r,R) = (r/R)^{5/4+o(1)} \tag{2.5}$$

as $R/r \to \infty$ while $1 \le r \le R$. Similar relations are known for $j \neq 4$ [LSW02, SW01]. For bond percolation on the square grid, we presently have weaker estimates; in particular,

$$C^{-1}\,(r/R)^{2-\epsilon} \le \alpha_4(r,R) \le C\,(r/R)^{1+\epsilon} \tag{2.6}$$

for some fixed constants $C, \epsilon > 0$ and every $1 \le r \le R$. The left inequality can be obtained by combining $\alpha_5(r,R) \asymp (r/R)^2$ (see [KSZ98, Lemma 5] or [SS05, Corollary A.8]), the RSW estimate $\alpha_1(r,R) < O(1)\,(r/R)^\epsilon$, and, finally, the relation $\alpha_1(r,R)\,\alpha_4(r,R) \ge \alpha_5(r,R)$ (which follows from Reimer's inequality [Rei00], or from Proposition 12.1 in our Appendix). The right hand inequality in (2.6) follows from [Kes87]; see also [BKS99, Remark 4.2].

## 2.3 The spectral sample in general

This subsection derives some formulas and estimates for $\mathbb{P}\big[\mathscr{S} \subset A\big]$, for the distribution of $\mathscr{S} \cap A$, and for $\mathbb{P}\big[\mathscr{S} \cap A = \emptyset \neq \mathscr{S} \cap B\big]$. We also briefly present an estimate of the variation distance between the laws of $\mathscr{S}_f$ and of $\mathscr{S}_g$ in terms of $\|f - g\|$. (We generally use $\|\cdot\|$ to denote the $L^2$ norm.) Moreover, the definition of the set of pivotals $\mathscr{P}$ is recalled and some relations between $\mathscr{P}$ and $\mathscr{S}$ are discussed.

As before, let $\Omega := \{-1, 1\}^{\mathcal{I}}$, where $\mathcal{I}$ is finite. Recall that for $f : \Omega \to \mathbb{R}$ with $\|f\| = 1$, we consider the random variable $\mathscr{S}_f$ whose law is given by $\mathbb{P}\big[\mathscr{S} = S\big] = \widehat{f}(S)^2$. More generally, if $\|f\| > 0$, we use the law given by

$$\mathbb{P}\big[\mathscr{S} = S\big] = \widehat{f}(S)^2 / \|f\|^2,$$

but will also consider the un-normalized measure given by

$$\mathbb{Q}\big[\mathscr{S} = S\big] = \widehat{f}(S)^2.$$

(If we wish to indicate the function $f$, we may write $\mathbb{Q}_f$ in place of $\mathbb{Q}$.)

Now suppose that $f, g : \Omega \to \mathbb{R}$. We argue that if $\|f - g\|$ is small, then the law of $\mathscr{S}_f$ is close to the law of $\mathscr{S}_g$, as follows:

$$\begin{aligned} \sum_{S \subset \mathcal{I}} \big|\widehat{f}(S)^2 - \widehat{g}(S)^2\big| &= \sum_S \big|\widehat{f} - \widehat{g}\big| \, \big|\widehat{f} + \widehat{g}\big| \\ &\leq \Big(\sum_S (\widehat{f} - \widehat{g})^2\Big)^{1/2} \Big(\sum_S (\widehat{f} + \widehat{g})^2\Big)^{1/2} \\ &= \|f - g\| \, \|f + g\| , \end{aligned} \tag{2.7}$$

where the inequality is due to Cauchy-Schwarz and the final equality is an application of Parseval's identity.

For $A \subseteq \mathcal{I}$, let $\omega_A$ denote the restriction of $\omega$ to $A$, and let $\mathcal{F}_A$ denote the $\sigma$-field of subsets of $\Omega$ generated by $\omega_A$. We use the notation $A^c := \mathcal{I} \setminus A$ for the complement of $A$. Observe that for $A \subset \mathcal{I}$,

$$\mathbb{E}\big[\chi_S \mid \mathcal{F}_A\big] = \begin{cases} \chi_S & S \subset A\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.8}$$

It follows from this and (1.1) that

$$g := \mathbb{E}\big[f \mid \mathcal{F}_A\big] = \sum_{S \subseteq A} \widehat{f}(S)\, \chi_S\,.$$

Thus $\widehat{g}(S) = \widehat{f}(S)$ for $S \subseteq A$, and $\widehat{g}(S) = 0$ otherwise. Therefore, Parseval's formula implies

$$\mathbb{Q}\big[\mathscr{S} \subseteq A\big] = \mathbb{E}\big[\mathbb{E}[f \mid \mathcal{F}_A]^2\big]. \tag{2.9}$$

In principle, this describes the distribution of $\mathscr{S}$ in terms of $f$, and indeed we will extract information about $\mathscr{S}$ from this formula and its consequences.

Using (1.1) and (2.8), one obtains for $S \subset A \subset \mathcal{I}$

$$\mathbb{E}\big[f\,\chi_S \bigm| \mathcal{F}_{A^c}\big] = \sum_{S' \subset A^c} \widehat{f}(S \cup S')\,\chi'_S\,.$$

This gives

$$\mathbb{E}\Big[\mathbb{E}\big[f\,\chi_S \bigm| \mathcal{F}_{A^c}\big]^2\Big] = \sum_{S' \subset A^c} \widehat{f}(S \cup S')^2 = \mathbb{Q}\big[\mathscr{S} \cap A = S\big]\,,$$

which implies the following Lemma from [LMN93]. Roughly, the lemma says that in order to sample the random variable $\mathscr{S} \cap A$, one can first pick a random sample of $\omega_{A^c}$, and then take a sample from the spectral sample of the function we get by plugging in these values for the bits in $A^c$.

**Lemma 2.1.** *Suppose that $f : \Omega \to \mathbb{R}$, and $A \subset \mathcal{I}$. For $x \in \{-1,1\}^A$ and $y \in \{-1,1\}^{A^c}$, write $g_y(x) := f\big(\omega(x,y)\big)$, where $\omega(x,y)$ is the element of $\Omega$ whose restriction to $A$ is $x$ and whose restriction to $A^c$ is $y$. Then for every $S \subset A$, we have $\mathbb{Q}\big[\mathscr{S}_f \cap A = S\big] = \mathbb{E}\big[\widehat{g_y}(S)^2\big] = \mathbb{E}\big[\mathbb{Q}_{g_y}[\mathscr{S}_{g_y} = S]\big]$.* □

For any $\omega \in \Omega$ and any $A \subset \mathcal{I}$, let $\omega_A^+$ denote the element of $\Omega$ that is equal to 1 in $A$ and equal to $\omega$ outside of $A$. Similarly, let $\omega_A^-$ denote the element of $\Omega$ that is equal to $-1$ in $A$ and equal to $\omega$ outside of $A$. An $i \in \mathcal{I}$ is said to be **pivotal** for $f : \Omega \to \mathbb{R}$ if $f(\omega^+_{\{i\}}) \neq f(\omega^-_{\{i\}})$. Let $\mathscr{P} = \mathscr{P}_f$ denote the (random) set of pivotals.

It is known from [KKL88] that for functions $f : \Omega \to \{-1,1\}$,

$$\mathbb{E}\big[|\mathscr{S}|\big] = \mathbb{E}\big[|\mathscr{P}|\big], \tag{2.10}$$

Gil Kalai (private communication) further observed that also

$$\mathbb{E}\big[|\mathscr{S}|^2\big] = \mathbb{E}\big[|\mathscr{P}|^2\big], \tag{2.11}$$

but this does not hold for higher moments. To prove (2.10) and (2.11), consider some $i, j \in \mathcal{I}$. In Lemma 2.1, if we take $A = \{i\}$, then $g_y$ is a constant function (of $x$) unless $i \in \mathscr{P}$, while $g_y = \pm\chi_{\{i\}}$ if $i \in \mathscr{P}$. Therefore, the lemma gives

$$\mathbb{P}\big[i \in \mathscr{S}\big] = \mathbb{P}\big[\mathscr{S} \cap \{i\} = \{i\}\big] = \mathbb{P}\big[i \in \mathscr{P}\big], \tag{2.12}$$

which sums to give (2.10). Similarly, one can show that

$$\mathbb{P}\big[i, j \in \mathscr{S}\big] = \mathbb{P}\big[i, j \in \mathscr{P}\big] \tag{2.13}$$

by using Lemma 2.1 with $A = \{i, j\}$ to reduce (2.13) to the case where $\Omega = \{-1, 1\}^2$, which easily yields to direct inspection. Now (2.11) follows by summing (2.13) over $i$ and $j$.

We now derive estimates for $\mathbb{Q}\big[\mathscr{S} \cap B \neq \emptyset = \mathscr{S} \cap W\big]$. Define $\Lambda_B = \Lambda_{f,B}$ as the event that $B$ is pivotal for $f$. More precisely, $\Lambda_B$ is the set of $\omega \in \Omega$ such that there is some $\omega' \in \Omega$ that agrees with $\omega$ on $B^c$ while $f(\omega) \neq f(\omega')$. Also define $\lambda_{B,W} := \mathbb{P}\big[\Lambda_B \bigm| W^c\big]$.

**Lemma 2.2.** *Let $\mathscr{S} = \mathscr{S}_f$ be the spectral sample of some $f : \Omega \to \mathbb{R}$, and let $W$ and $B$ be disjoint subsets of $\mathcal{I}$. Then*

$$\mathbb{Q}\big[\mathscr{S} \cap B \neq \emptyset = \mathscr{S} \cap W\big] \leq 4\,\|f\|_\infty^2\,\mathbb{E}\big[\lambda_{B,W}^2\big].$$

**Proof.** From (2.9),

$$\begin{aligned}\mathbb{Q}\big[\mathscr{S} \cap B \neq \emptyset,\ \mathscr{S} \cap W = \emptyset\big] &= \mathbb{Q}\big[\mathscr{S} \subseteq W^c\big] - \mathbb{Q}\big[\mathscr{S} \subseteq (W \cup B)^c\big] \\ &= \mathbb{E}\Big[\mathbb{E}[f \mid \mathcal{F}_{W^c}]^2 - \mathbb{E}[f \mid \mathcal{F}_{(W\cup B)^c}]^2\Big] \\ &= \mathbb{E}\Big[\Big(\mathbb{E}[f \mid \mathcal{F}_{W^c}] - \mathbb{E}[f \mid \mathcal{F}_{(W\cup B)^c}]\Big)^2\Big].\end{aligned} \tag{2.14}$$

On the complement of $\Lambda_B$, we have $f = \mathbb{E}\big[f \bigm| \mathcal{F}_{B^c}\big]$. Therefore,

$$-2\,\|f\|_\infty\,1_{\Lambda_B} \leq f - \mathbb{E}[f \mid B^c] \leq 2\,\|f\|_\infty 1_{\Lambda_B}\,.$$

Taking conditional expecations throughout, we get

$$-2\,\|f\|_\infty\,\lambda_{B,W} \leq \mathbb{E}\big[f \bigm| \mathcal{F}_{W^c}\big] - \mathbb{E}\big[\mathbb{E}[f \mid \mathcal{F}_{B^c}] \bigm| \mathcal{F}_{W^c}\big] \leq 2\,\|f\|_\infty \lambda_{B,W}\,.$$

Note that $\mathbb{E}\big[\mathbb{E}[f \mid \mathcal{F}_{B^c}] \bigm| \mathcal{F}_{W^c}\big] = \mathbb{E}\big[f \bigm| \mathcal{F}_{(B\cup W)^c}\big]$, since our measure on $\Omega$ is i.i.d. Thus, the above gives

$$\Big|\mathbb{E}\big[f \bigm| \mathcal{F}_{W^c}\big] - \mathbb{E}\big[f \bigm| \mathcal{F}_{(B\cup W)^c}\big]\Big| \leq 2\,\|f\|_\infty\,\lambda_{B,W}\,.$$

An appeal to (2.14) now completes the proof. □

What turns out to be important in Section 5 is that, in the context of percolation, the quantity $\mathbb{E}\big[\lambda_{B,W}^2\big]$ can be studied and controlled when $B$ is a box and $W \subset \mathcal{I} \setminus B$ is arbitrary. Likewise, in Section 4, we use a variant of Lemma 2.2 in which we look at the event that $\mathscr{S}$ intersects a collection of boxes and is disjoint from some collection of annuli.

# 3 First percolation spectrum estimates

We will now consider the special case of $\mathscr{S}_f$ when $f = f_{\mathcal{Q}}$ for a quad $\mathcal{Q} \subset \mathbb{R}^2$. As noted in Section 2.1, we will be considering $\mathcal{I}$ and $\mathscr{S} \subset \mathcal{I}$ as subsets of the plane. When $R > 0$, we will use the notation $R\,\mathcal{Q}$ to denote the quad obtained from $\mathcal{Q}$ by scaling by a factor of $R$ about 0.

**Lemma 3.1** (First and second moments)**.** *Let $\mathcal{Q} \subset \mathbb{R}^2$ be some quad and let $U$ be an open set whose closure is contained in the interior of $\mathcal{Q}$. Let $\mathscr{S} := \mathscr{S}_{f_{R\mathcal{Q}}}$ be the spectral sample for the $\pm 1$-indicator function of crossing $R\mathcal{Q}$. There are constants $C, R_0 > 0$, depending only on $\mathcal{Q}$ and $U$, such that for all $R > R_0$*

$$C^{-1}\, R^2\, \alpha_4(R) \le \mathbb{E}\big[|\mathscr{S}_{f_{R\mathcal{Q}}} \cap R\,U|\big] \le C\, R^2\, \alpha_4(R)$$

*and*

$$\mathbb{E}\big[|\mathscr{S}_{f_{R\mathcal{Q}}} \cap R\,U|^2\big] \le C\, \mathbb{E}\big[|\mathscr{S}_{f_{R\mathcal{Q}}} \cap R\,U|\big]^2.$$

The reason for the appearance of $\alpha_4$ in the first moment is the following. We know from (2.12) that $\mathbb{P}\big[x \in \mathscr{S}\big] = \mathbb{P}\big[x \in \mathscr{P}\big]$ for $\mathscr{P} := \mathscr{P}_{f_{R\mathcal{Q}}}$. In order for $x$ to be pivotal for $f_{R\mathcal{Q}}$ it is necessary and sufficient that there are white paths in $R\mathcal{Q}$ from the tile of $x$ to the two distinguished arcs on $R\,\partial\mathcal{Q}$ and two black paths in $R\mathcal{Q}$ from the tile of $x$ to the complementary arcs of $R\partial\mathcal{Q}$. These four paths form the 4-arm event in the annulus between the tile of $x$ and $R\partial\mathcal{Q}$. Moreover, it is well-known (and follows from quasi-multiplicativity arguments) that

$$\forall x \in \mathcal{I} \cap R\,U : \qquad \mathbb{P}\big[x \in \mathscr{P}_{f_{R\mathcal{Q}}}\big] = \alpha_{\square}(x, R\mathcal{Q}) \asymp \alpha_4(R)\,. \tag{3.1}$$

Here and in the proof below, we use the notation $g \asymp g'$ to mean that there is a constant $c > 0$ (which may depend on $U$ and $\mathcal{Q}$), such that $g \le c\, g'$ and $g' \le c\, g$. Likewise, the $O(\cdot)$ notation will involve constants that may depend on $\mathcal{Q}$ and $U$.

**Proof of Lemma 3.1.** From (2.12) and (3.1) we get that

$$\forall x \in \mathcal{I} \cap R\,U: \qquad \mathbb{P}\big[x \in \mathscr{S}\big] \asymp \alpha_4(R)\,. \tag{3.2}$$

The first claim of the lemma is obtained by summing over $x \in \mathcal{I} \cap R\,U$.

Now consider $x, y \in \mathcal{I} \cap R\,U$. Let $a$ be the distance from $\overline{U}$ to $\partial\mathcal{Q}$. Thus $a > 0$. Then by (2.13), we have $\mathbb{P}\big[x, y \in \mathscr{S}\big] = \mathbb{P}\big[x, y \in \mathscr{P}\big]$. Therefore in order for $x, y \in \mathscr{P}$ it is necessary that the 4 arm event occurs from the tile of $x$ to distance $(|x-y|/3) \wedge (a\,R)$ away, and from the tile of $y$ to distance $(|x-y|/3) \wedge (a\,R)$ away, and from the circle of radius $2\,|x-y|$ around $(x+y)/2$ to distance $a\,R$ away (if $2\,|x-y| < a\,R$). By independence on disjoint subsets of $\mathcal{I}$, this (together with the regularity properties of the 4-arm probabilities (2.3)) gives for $R$ sufficiently large

$$\mathbb{P}\big[x, y \in \mathscr{S}\big] \le O(1)\,\alpha_4(|x-y|)^2\,\alpha_4(|x-y|, R)\,.$$

Using the quasi-multiplicativity property of $\alpha_4$, this gives

$$\forall x, y \in \mathcal{I} \cap R\,U: \qquad \mathbb{P}\big[x, y \in \mathscr{S}\big] \le O(1)\,\alpha_4(R)^2/\alpha_4(|x-y|, R)\,. \tag{3.3}$$

The number of pairs $x, y \in \mathcal{I} \cap U$ such that $|x-y| \in [2^n, 2^{n+1})$ is $O(R^2)\,2^{2n}$, and is zero if $|x-y| > R\,\mathrm{diam}(\mathcal{Q})$. Therefore, we get from (3.3) and the regularity property (2.3) that

$$\mathbb{E}\big[|\mathscr{S} \cap RU|^2\big] \le O(R^2)\,\alpha_4(R)^2 \sum_{n=0}^{\log_2(R)+O(1)} \frac{2^{2n}}{\alpha_4(2^n, R)}\,.$$

From (2.6) we get $2^{2n}/\alpha_4(2^n, R) \le O(1)\,R^{2-\epsilon}\,2^{\epsilon n}$. Hence the sum over $n$ is at most $O(R^2)$, and we obtain the desired bound on the second moment. $\square$

Note that $\mathbb{E}\big[|\mathscr{S} \cap R\,U|\big] \to \infty$ as $R \to \infty$, which follows from (2.12), (3.1) and (2.6). Moreover, by the standard Cauchy-Schwarz second-moment argument (also called the Paley-Zygmund inequality), the above lemma implies that for some constant $c > 0$ (which may depend on $\mathcal{Q}$ and $U$),

$$\mathbb{P}\Big[|\mathscr{S} \cap RU| > c\,\mathbb{E}|\mathscr{S} \cap RU|\Big] > c\,.$$

We also note the following lemma.

**Lemma 3.2.** *Let $\mathcal{Q} \subset \mathbb{R}^2$ be a quad, and set $\mathscr{S} = \mathscr{S}_{f_{\mathcal{Q}}}$. Let $B$ be some union of tiles such that $B \cap \mathcal{Q}$ is nonempty and connected. Then*

$$\mathbb{P}[\mathscr{S} \cap B \neq \emptyset] \leq 4\,\alpha_\Box(B, \mathcal{Q})\,, \tag{3.4}$$

*and*

$$\mathbb{P}[\emptyset \neq \mathscr{S} \subseteq B] \leq 4\,\alpha_\Box(B, \mathcal{Q})^2. \tag{3.5}$$

*When $B$ is a single tile, corresponding to $x \in \mathcal{I}$, we have*

$$\mathbb{P}\big[x \in \mathscr{S}\big] = \alpha_\Box(x, \mathcal{Q}) \quad \textit{and} \quad \mathbb{P}\big[\mathscr{S} = \{x\}\big] = \alpha_\Box(x, \mathcal{Q})^2. \tag{3.6}$$

**Proof.** Note that $\mathbb{P}\big[\Lambda_B\big] = \alpha_\Box(B, \mathcal{Q})$, since $\Lambda_B$ holds if an only if the 4-arm event from $B$ to the corresponding arcs on $\partial\mathcal{Q}$ occurs. Since $\lambda_{B,\emptyset} = 1_{\Lambda_B}$, the first claim follows from Lemma 2.2 with $W = \emptyset$. Similarly, (3.5) follows by taking $W = B^c$. The identity $\mathbb{P}\big[x \in \mathscr{S}\big] = \alpha_\Box(x, \mathcal{Q})$ follows from (2.12). Finally, the right hand identity in (3.6) can be derived from (2.14) with $B = \{x\}$ and $W = \mathcal{I} \setminus B$. Alternatively, it also follows from $\mathbb{P}\big[\{x\} = \mathscr{S}\big] = \mathbb{P}\big[x \in \mathscr{S}\big]^2$, which holds for arbitrary monotone $f : \Omega \to \{-1, 1\}$. $\Box$

As we will see in Section 5, both inequalities in Lemma 3.2 are actually approximate equalities when $B \subset \mathcal{Q}$. The main reason for this is a classical arm separation phenomenon, see e.g. [SS05, Appendix], which roughly says that conditioned on having four arms connecting $\partial B$ to the appropriate boundary arcs on $\partial\mathcal{Q}$, with positive conditional probability, these arms are "well-separated" on $\partial B$. On this event, a positive proportion of the $\omega_B$ configurations enable the crossing of $\mathcal{Q}$, while a positive proportion disable all crossings.

Lemma 3.2 has the following immediate consequence. Let $\mathcal{Q}$ be the $R \times R$ square with two opposite sides as distinguished boundary arcs. Then, for a box $B \subseteq \mathcal{Q}$ of radius $r$ and a concentric sub-box $B'$ of radius $r/3$, if $B' \cap \mathcal{I} \neq \emptyset$, then

$$\begin{aligned}\mathbb{E}\Big[|\mathscr{S} \cap B'| \;\Big|\; \mathscr{S} \cap B \neq \emptyset\Big] &= \sum_{x \in B'} \frac{\mathbb{P}[x \in \mathscr{S}]}{\mathbb{P}[\mathscr{S} \cap B \neq \emptyset]} \geq \sum_{x \in B'} \frac{\alpha_\Box(x, \mathcal{Q})}{4\alpha_\Box(B, \mathcal{Q})} \\ &\asymp |B'|\alpha_4(r) \asymp r^2\alpha_4(r),\end{aligned} \tag{3.7}$$

where we used the quasi-multiplicativity result (2.4). This result already suggests that $\mathscr{S}$ has self-similarity properties that a random fractal-like object should have, and it should be unlikely that it is very small. This idea will, in fact, be of key importance to us, and will be developed in Section 5.

# 4 The probability of a very small spectral sample

## 4.1 The statement

In this section, we study the Fourier spectrum of the indicator function $f$ of having a crossing of a square, or more generally, of a quad $\mathcal{Q}$.

Divide the plane into a lattice of $r \times r$ subsquares, that is, $r\,\mathbb{Z}^2$, and for any set of bits $S \subset \mathcal{I}$ define

$$S_r := \{\text{those } r \times r \text{ squares that intersect } S\}.$$

In particular, $\mathscr{S}_r$ is the set of $r$-squares whose intersection with the spectral sample $\mathscr{S}$ of $f$ is nonempty. Following is an estimate for the probability that $\mathscr{S}$ is very small, or, more generally, that $\mathscr{S}_r$ is very small.

**Proposition 4.1.** *Let $\mathscr{S}$ be the spectral sample of $f = f_{[0,R]^2}$, the $\pm 1$ indicator function of the left-right crossing of the square $[0,R]^2$. For $g(k) := 2^{\vartheta \log_2^2(k+2)}$, with $\vartheta > 0$ large enough, and $\gamma_r(R) := (R/r)^2 \alpha_4(r,R)^2$,*

$$\forall k, R, r \in \mathbb{N}_+ \qquad \mathbb{P}\big[|\mathscr{S}_r| = k\big] \leq g(k)\,\gamma_r(R).$$

The square prefactor $(R/r)^2$ in the definition of $\gamma_r$ reflects the two dimensionality of the ambient space — it corresponds to the number of different ways to choose a square of size $r$ inside $[0,R]^2$. The factor $\alpha_4(r,R)^2$ comes from the second inequality in Lemma 3.2. In fact, since that lemma will turn out to be sharp up to a constant factor (when the $r$-square is not close to the boundary of $[0,R]^2$), the $k=1$ case of Proposition 4.1 is also sharp. When we use this proposition, this will be important, as well as the fact that the dependence on $k$ is sub-exponential.

Recall from (2.5) that for critical site percolation on the triangular lattice $\gamma_r(R) = (r/R)^{1/2+o(1)}$, as $R/r \to \infty$. Also, note that for critical bond percolation on $\mathbb{Z}^2$ we have $\gamma_r(R) < O(1)\,(r/R)^\epsilon$ for some $\epsilon > 0$ by (2.6).

Note that the proposition gives $\lim_{R\to\infty} \mathbb{P}\big[0 < |\mathscr{S}| < C(\log R)^a\big] = 0$ for any $C, a > 0$. This is already stronger than the $a=1$ and small $C$ result of [BKS99], whose proof used more analysis but less combinatorics.

In order to demonstrate the main ideas of the proof of the proposition in a slightly simpler setting in which considerations having to do with the boundary of the square do not appear, we will first state and prove a "local" version of this proposition. Then we will see in Subsection 4.3 that the boundary of the square indeed has no significant effect; the main reason for this is that the spectral sample "does not like" to be close to the boundary.

## 4.2 A local result

**Proposition 4.2.** *Consider some quad $\mathcal{Q}$, and let $\mathscr{S}$ be the spectral sample of $f = f_{\mathcal{Q}}$, the $\pm 1$ indicator function for the crossing event in $\mathcal{Q}$. Let $U' \subset U \subset \mathcal{Q}$, let $R$ denote the diameter of $U$, let $a > 0$, and suppose that the distance from $U'$ to the complement of $U$ is at least $a\,R$. Let $\mathcal{S}(r,k)$ be the collection of all sets $S \subset \mathcal{I}$ such that $\big|(S \cap U)_r\big| = k$ and $S \cap (U \setminus U') = \emptyset$. Then for $g$ and $\gamma_r$ as in Proposition 4.1, we have*

$$\forall k, r \in \mathbb{N}_+ \qquad \mathbb{P}\big[\mathscr{S} \in \mathcal{S}(r,k)\big] \leq c_a\, g(k)\, \gamma_r(R)\,,$$

*where $c_a$ is a constant that depends only on $a$.*

We preface the proof of the proposition with a rough sketch of the main ideas. When $S \in \mathcal{S}(r,k)$ and $k$ is small, the set $(S \cap U)_r$ has to consist of one or very few "clusters" of squares that are small and well separated from each other. The probability that $(\mathscr{S} \cap U)_r$ has just one cluster contained in a specific small box $B$ is estimated by (3.5) of Lemma 3.2. One may then sum over an appropriate collection of $B$ to get a reasonable bound for the probability that $\mathrm{diam}(\mathscr{S} \cap U)$ is small while $\mathscr{S} \cap U \neq \emptyset$. To deal with the case where $S \cap U$ has a few different well-separated clusters, we will prove a generalization of (3.5). The more involved part of the proof will be to classify the possible cluster structures of $\mathscr{S}$ and sum up the bounds corresponding to each possibility.

**Proof of Proposition 4.2.** Let $\mathcal{A}$ be a finite collection of disjoint (topological) annuli in the plane; we call this an **annulus structure**. We say that a set $S \subset \mathbb{R}^2$ is **compatible** with $\mathcal{A}$ if it is contained in $\mathbb{R}^2 \setminus \bigcup \mathcal{A}$ and intersects the inner disk of each annulus in $\mathcal{A}$. Define $h(\mathcal{A})$ as the probability that each annulus in $\mathcal{A}$ has the four-arm event. By independence on disjoint sets, we have

$$h(\mathcal{A}) = \prod_{A \in \mathcal{A}} h(A)\,. \tag{4.1}$$

Suppose that $\mathfrak{A}$ is a set of annulus structures $\mathcal{A}$, where each annulus $A \subset \mathcal{A}$ is contained in $\mathcal{Q}$, and $\mathfrak{A}$ has the property that each $S \in \mathcal{S}(r,k)$ must be compatible with at least one $\mathcal{A} \in \mathfrak{A}$. We claim that this set $\mathfrak{A}$ satisfies

$$\mathbb{P}\big[\mathscr{S} \in \mathcal{S}(r,k)\big] \leq \sum_{\mathcal{A}\in\mathfrak{A}} h(\mathcal{A})^2. \tag{4.2}$$

For this, it is clearly sufficient to verify the following lemma, which is a generalization of (3.5) from Lemma 3.2.

**Lemma 4.3.** *For any annulus structure $\mathcal{A}$ with $\bigcup \mathcal{A} \subset \mathcal{Q}$,*

$$\mathbb{P}\big[\mathscr{S} \textit{ is compatible with } \mathcal{A}\big] \leq h(\mathcal{A})^2.$$

**Proof.** We write $f(\theta, \eta)$ for the value of $f$, where $\theta$ is the configuration inside $\bigcup \mathcal{A}$ and $\eta$ is the configuration outside. For each $\theta$, let $F_\theta$ be the function of $\eta$ defined by $F_\theta(\eta) := f(\theta,\eta)$. If $\theta$ is such that there is an annulus $A \in \mathcal{A}$ without the 4-arm event, then the connectivity of points ouside the outer boundary of $A$ does not depend on the configuration inside the inner disk of $A$, and therefore $F_\theta$ does not depend on any of the variables in the inner disk of $A$. Thus, if a subset of variables $S$ is disjoint from $\bigcup \mathcal{A}$ and intersects this inner disk, then the corresponding Fourier coefficient vanishes: $\widehat{F_\theta}(S) = \mathbb{E}[F_\theta \chi_S] = 0$. Therefore, if we let $W$ be the linear space of functions spanned by $\{\chi_S : S \text{ compatible with } \mathcal{A}\}$, and $P_W$ denotes the orthogonal projection onto $W$, then $P_W F_\theta \neq 0$ implies the 4-arm event of $\theta$ in every $A \in \mathcal{A}$.

Now, observe that $P_W f = P_W \mathbb{E}[f \mid \eta]$, because for all $g \in W$ we have $\mathbb{E}\big[\mathbb{E}[f \mid \eta]\, g\big] = \mathbb{E}\big[\mathbb{E}[f\, g \mid \eta]\big] = \mathbb{E}[f\, g]$. Next, note that $\mathbb{E}[f \mid \eta] = \mathbb{E}^\theta[F_\theta]$, where the right hand side is an expectation w.r.t. $\theta$, hence is still a function of $\eta$. Thus, $P_W f = P_W \mathbb{E}^\theta[F_\theta] = \mathbb{E}^\theta[P_W F_\theta]$. Consequently,

$$\mathbb{P}\big[\mathscr{S} \text{ is compatible with } \mathcal{A}\big] = \|P_W f\|^2 = \|\mathbb{E}^\theta[P_W F_\theta]\|^2 \leq \mathbb{E}^\theta\big[\|P_W F_\theta\|\big]^2.$$

For every $\theta$, the function $F_\theta$ is bounded by 1 in absolute value. Therefore, $\|F_\theta\| \leq 1$ for every $\theta$. This implies $\|P_W F_\theta\| \leq 1$ for every $\theta$ (the norm is the $L^2$ norm and $P_W$ is an orthogonal projection). Hence, we get

$$\mathbb{P}\big[\mathscr{S} \text{ is compatible with } \mathcal{A}\big] \leq \mathbb{P}^\theta\big[P_W F_\theta \neq 0\big]^2 \leq h(\mathcal{A})^2.$$

This proves the lemma. □

A trivial but important instance of (4.2) is when $\mathfrak{A} = \{\emptyset\}$: the empty annulus structure $\emptyset$ is compatible with any $S$, while $h(\emptyset)^2 = 1$.

Now, for each set $S \in \mathcal{S}(r,k)$ we construct a compatible annulus structure $\mathcal{A}(S)$, such that the set $\mathfrak{A} = \mathfrak{A}(r,k) = \{\mathcal{A}(S) : S \in \mathcal{S}(r,k)\}$ will be small enough for (4.2) to imply Proposition 4.2. The main idea for this construction is that the spectral sample tends to be clustered together, which can already be foreseen in Lemma 4.3: each additional thick annulus (corresponding to some part of the spectral sample far from all other parts) decreases the weight $h(\mathcal{A})^2$ by quite a lot — more than what can be balanced by the number of essentially different ways that this can happen. (We will make this vague description of clustering more precise after the end of the proof, in Remark 4.5.)

We now prepare to define the annulus structure $\mathcal{A}(S)$ corresponding to an $S \in \mathcal{S}(r,k)$, based on the geometry of $S$. For this definition, we need to first define what we call **clusters** of $S$. Set $V = V_S := (S \cap U)_r$; the set of squares in the $r\,\mathbb{Z}^2$ lattice which meet $S \cap U$. For $j \in \mathbb{N}_+$ let $G_j$ be the graph on $V$, where two squares are joined if their distance is at most $2^j\,r$, and let $G_0$ be the graph on $V$ with no edges. If $j \in \mathbb{N}_+$ and $C \subset V$ is a connected component of $G_j$, but is not connected in $G_{j-1}$, then $C$ is called a level $j$ cluster of $S$. The level 0 clusters are the connected components of $G_0$, that is, the singletons contained in $V$. The level of a cluster $C$ is denoted by $j(C) = j_S(C)$. The diameter of a cluster $C$ is clearly at most $2^{j(C)+2}\,r\,|C|$.

Let $C$ be a cluster in $S$ at some level $j = j(C)$. We now associate to $C$ two "bounding" squares, as follows. Choose a point $z \in C$ in an arbitrary but fixed way (say the lowest among the leftmost points of $C$). Let $z'$ be a point with both coordinates divisible by $2^j\,r$, which is closest to $z$, with ties broken in some arbitrary but fixed manner. Define the **inner bounding square** $B(C)$ as the square with edge-length $|C|\,2^{j+4}\,r$ centered at $z'$ (with edges parallel to the coordinate axes). Note that $C \subset B(C)$ and the distance from $C$ to $\partial B(C)$ is at least $2^{j+3}\,|C|\,r - 2^j\,r - 2^{j+2}\,|C|\,r \geq 2^j\,r$.

If $C \neq V$ is a cluster, then the smallest cluster in $V$ that properly contains $C$ will be called the **parent** of $C$ and denoted by $C^p$. In this case, let the **outer bounding square** $\bar{B}(C) = \bar{B}_S(C)$ of $C$ be the square concentric with $B(C)$ having sidelength $(2^{j(C^p)-4}\,r) \wedge (a\,R/4)$. For the case $C = V$, we let $\bar{B}(V)$ be the square concentric with $B(V)$ having sidelength $a\,R/4$.

It is not necessarily the case that $\bar{B}(C) \supset B(C)$. Also, it may happen that for two disjoint clusters $C, C'$ we have $B(C) \cap B(C') \neq \emptyset$. However, we do have the following important properties of these squares, which are easy to verify:

1. if $C' \subsetneq C$ is a subcluster of $C$, then $\bar B(C') \subset B(C)$;
2. if $C$ and $C'$ are disjoint clusters, then $\bar B(C) \cap \bar B(C') = \emptyset$;
3. $B(C)$ depends only on $C$; and
4. $\bar B(C)$ depends only on $j(C)$, $B(C)$ and $j(C^p)$.

In the case where $r \geq a\,R/4$, Proposition 4.2 is an immediate consequence of (2.3). Assume therefore that $r < a\,R/4$. Then we also have $\bar B(C) \subset U$ for every cluster $C$.

Define $A(C) = A_S(C) := \bar B(C) \setminus B(C)$. Define an annulus structure $\mathcal{A}_1(S)$ associated with $S$ by

$$\mathcal{A}_1(S) := \big\{ A_S(C) : C \text{ is a cluster in } S,\ A_S(C) \neq \emptyset \big\}.$$

By properties 1 and 2 above, the different annuli in $\mathcal{A}_1(S)$ are disjoint. See Figure 4.1. It is also clear that $\mathcal{A}_1(S)$ is compatible with $S$. However, we still need to modify $\mathcal{A}_1(S)$ for it to be useful.

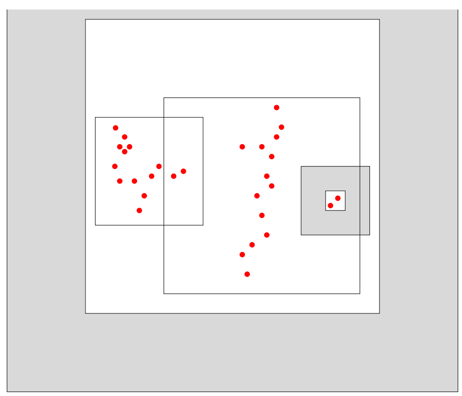


Figure 4.1: A cluster with three child clusters, two of which have empty annuli.

It follows from (2.6) that the function $\gamma_r(2^j\,r)$ is decaying exponentially, in the sense that there are absolute constants $c_0, c_1 > 0$ such that

$$\gamma_r(2^{j'} r) \leq c_0\, 2^{-c_1 (j'-j)}\, \gamma_r(2^j\, r) \qquad \text{if } j' > j \geq 0\,. \tag{4.3}$$

It would be rather convenient in the proof below to have $\gamma_r(2^{j+1}\,r) \leq \gamma_r(2^j\,r)$. However, we do not want to try to prove this. Instead, we use the function

$\bar\gamma_r(\rho) := \inf_{\rho'\in[1,\rho]}\gamma_r(\rho')$ in place of $\gamma$. Clearly, $\bar\gamma$ also satisfies (4.3) and $\bar\gamma_r(\rho) \le \gamma_r(\rho) \le O(1)\,\bar\gamma_r(\rho)$.

A cluster $C$ will be called **overcrowded** if

$$g(|C|)\,\bar\gamma_r(2^{j(C)}\,r) > 1\,,$$

with the constant $\vartheta > 0$ in the definition of $g(k)$ to be determined later. In particular, clusters at level 0 are overcrowded. For each overcrowded cluster, we remove from $\mathcal{A}_1(S)$ all the annuli corresponding to its proper subclusters. The resulting annulus structure will be denoted $\mathcal{A}(S)$, still compatible with $S$. Finally, we set $\mathfrak{A} = \mathfrak{A}(r,k,U,U') := \{\mathcal{A}(S) : S \in \mathcal{S}(r,k)\}$.

We will show that, for some constant $c_0 > 0$,

$$\sum_{\mathcal{A}\in\mathfrak{A}} h(\mathcal{A})^2 \le O(1)\,(k/a)^{c_0}\,g(k)\,\bar\gamma_r(R)\,. \tag{4.4}$$

This and (4.2) together imply Proposition 4.2 (with possibly a different choice for the constant $\vartheta$).

For each $S$, there is a natural tree structure on the clusters. The root is $V$ itself, and the parent $C^p$ of each cluster $C \ne V$ is also its parent in the tree. We will use this tree structure to build the bound (4.4) inductively: we will write each term $h(\mathcal{A})^2$ as a product of weights corresponding to smaller annulus structures, with the trivial annulus structures of overcrowded clusters as the base step.

For a cluster $C$ of $S$ let $\mathcal{A}'(C)$ denote the subset of $\mathcal{A}(S)$ corresponding to the proper subclusters of $C$. Note that $\mathcal{A}'(C)$ depends only on $C$; that is, it is not effected by a modification of $S$, as long as $C$ remains a cluster of $S$. Also note that $\mathcal{A}'(V) = \mathcal{A}(S)\setminus\{A_S(V)\}$, where $V = V_S$.

Fix some $j,k \in \mathbb{N}_+$. If $B = B(C)$, where $|C| = k$ and $j(C) = j$, then the coordinates of $B$ are divisible by $2^j\,r$ and its side-length is $k\,2^{j+4}r$. For such $B$, define

$$\mathfrak{A}(B,k,j) := \Big\{\mathcal{A}'(S) : S \in \mathcal{S}(r,k),\ B(V_S) = B,\ j_S(V_S) = j\Big\},$$

and

$$H(j,k) := \sup_B \sum_{\mathcal{A}\in\mathfrak{A}(B,k,j)} h(\mathcal{A})^2.$$

(Note that the sum on the right may depend on $B$, since it may happen, for example, that $B \not\subset U'$.) Define $J(k) := \max\{j \in \mathbb{N} : g(k)\,\bar\gamma_r(2^j\,r) > 1\} =$

$O(\vartheta)\,\log_2^2(k+2)$. If $j \le J(k)$, then every $S \in \mathcal{S}(r,k)$ with $j_S(V_S) = j$ and $B_S(V_S) = B$ has $V_S$ overcrowded. Hence, in this case, $\mathfrak{A}(B,k,j) = \{\emptyset\}$ or $\mathfrak{A}(B,k,j) = \emptyset$ and thus

$$\forall j \le J(k) \qquad H(j,k) \le 1\,. \tag{4.5}$$

On the other hand, we will show by induction on $j$ that

$$\forall j \in \mathbb{N} \qquad H(j,k) \le \big(g(k)\;\bar\gamma_r(2^j\,r)\big)^{1.99}\,. \tag{4.6}$$

Before proving (4.6), let us demonstrate that it implies (4.4). If $j_S(V_S) = j$ (and $S \in \mathcal{S}(k,r)$), then the number of possible choices for $B(V_S)$ is at most $\big(2R/(2^j\,r)\big)^2$. If $A(V_S) \ne \emptyset$, then the probability of the 4-arm event in $A(V_S)$ is at most $O(1)\,\alpha_4\big(k\,2^j\,r, a\,R\big)$, by (2.3). Therefore,

$$\sum_{\mathcal{A}\in\mathfrak{A}} h(\mathcal{A})^2 \le\; O(1) \sum_{j=0}^{\lceil \log_2(2R/r)\rceil} \left(\frac{R}{2^j r}\right)^2 \alpha_4\big(k\,2^j\,r, a\,R\big)^2\, H(j,k)\,.$$

Now, we use the quasi-multiplicativity of the 4-arm event, (2.3), (4.5) and (4.6) to rewrite this as

$$\begin{aligned}\sum_{\mathcal{A}\in\mathfrak{A}} h(\mathcal{A})^2 \;&\le\; \sum_{j=0}^{\lceil \log_2(2R/r)\rceil} O(1)\,(k/a)^{c_0}\,\frac{\bar\gamma_r(R)}{\bar\gamma_r(2^j r)}\,H(j,k) \qquad (4.7)\\ &\le\; O(1)\,(k/a)^{c_0}\,\bar\gamma_r(R)\Big(\sum_{j\le J(k)} \frac{1}{\bar\gamma_r(2^j r)} + \sum_{j>J(k)} g(k)^{1.99}\,\bar\gamma_r(2^j\,r)^{0.99}\Big),\end{aligned}$$

for some finite constant $c_0$. Because (4.3) holds for $\bar\gamma$, the first sum is dominated by a constant times the summand corresponding to $j = J(k)$ and the second sum is dominated by a constant times the summand corresponding to $j = J(k)+1$. Since $\bar\gamma_r(2^{J(k)}\,r) > 1/g(k)$ and $\bar\gamma_r(2^{J(k)+1}\,r) \le 1/g(k)$, we get the bound claimed in (4.4).

In order to complete the proof of Proposition 4.2, all that remains is to establish (4.6). The proof of this inequality will be inductive and somewhat similar to the proof of (4.4) from (4.6), but there are some important differences. In passing from (4.6) to (4.4), we lost the power 1.99. Such a loss would not be sustainable if it had to be repeated in every inductive step.

What saves us in the following proof of (4.6) is that clusters that are not singletons have more than one child cluster. This results in almost squaring the estimate at each inductive step, which makes the proof work.

We now proceed to prove (4.6) by induction. Since $\bar{\gamma}$ is non-increasing, (4.6) holds for all $j \le J(k)$. We now assume that $j > J(k)$ and that (4.6) holds for all smaller values of $j$. Fix some square $B$ such that $\mathfrak{A}(B,k,j) \neq \emptyset$. Observe that if $B_S(V_S) = B$, then $S$ has some $d = d(S) \ge 2$ children in its tree of clusters (we know $d \neq 0$ because $j > J(k)$). Fix some $d \in \{2,3,\dots\}$, $k_1,\dots,k_d$, $j_1,\dots,j_d$ and sub-squares $B_1,\dots,B_d$ such that there is some $S \in \mathcal{S}(k,r)$ with $j_S(V_S) = j$, $B_S(V_S) = B$, and having cluster children $C^1,\dots,C^d$ with $|C^i| = k_i$, $j(C^i) = j_i$ and $B(C^i) = B_i$. Note that $A_i := A_S(C^i)$ does not depend on the choice of $S$ satisfying the above (by property 4 of squares noted previously).

Let $\mathfrak{A}' = \mathfrak{A}'(d,k_1,\dots,k_d,B_1,\dots,B_d)$ denote the set of all elements of $\mathfrak{A}(B,k,j)$ that arise from such an $S$. Note that every $\mathcal{A} \in \mathfrak{A}'$ is of the form $\{A_1,\dots,A_d\} \cup \bigcup_{i=1}^d \mathcal{A}_i$, where $\mathcal{A}_i \in \mathfrak{A}(B_i,k_i,j_i)$. For such an $\mathcal{A}$, we have by (4.1)

$$h(\mathcal{A}) = \prod_{i=1}^d \big(h(A_i)\,h(\mathcal{A}_i)\big)\,.$$

Hence,

$$\begin{aligned}\sum_{\mathcal{A}\in\mathfrak{A}'} h(\mathcal{A})^2 &\le \sum_{\mathcal{A}_1\in\mathfrak{A}(B_1,k_1,j_1)} \;\sum_{\mathcal{A}_2\in\mathfrak{A}(B_2,k_2,j_2)} \cdots \sum_{\mathcal{A}_d\in\mathfrak{A}(B_d,k_d,j_d)} \prod_{i=1}^d \big(h(A_i)\,h(\mathcal{A}_i)\big)^2 \\ &= \prod_{i=1}^d \Big(h(A_i)^2 \sum_{\mathcal{A}\in\mathfrak{A}(B_i,k_i,j_i)} h(\mathcal{A})^2\Big) \\ &= \prod_{i=1}^d \big(h(A_i)^2\, H(j_i,k_i)\big).\end{aligned}$$

Now, $h(A_i) = O(1)\,\alpha_4(k_i\,2^{j_i}\,r, 2^j\,r)$, where we use the convention that $\alpha_4(\rho,\rho') = 1$ if $\rho \ge \rho'$. Furthermore, given $B$, $d$, $j_1,\dots,j_d$ and $k_1,\dots,k_d$, there are no more than $O(1)\,(k\,2^{j-j_i})^2$ possible choices for $B_i$. Hence, summing the above over all such choices, we get

$$H(j,k) \le \sum_{d=2}^{k} \sum_{\substack{(j_1,\dots,j_d)\\(k_1,\dots,k_d)}} \prod_{i=1}^d \Big(O(k\,2^{j-j_i})^2\,\alpha_4(k_i\,2^{j_i}\,r, 2^j\,r)^2\,H(j_i,k_i)\Big).$$

The inductive hypothesis (4.6) for each of the pairs $(j_i, k_i)$ implies $H(j_i, k_i) \le \bar\gamma_r(2^{j_i} r)\, g(k_i)$ when $j_i > J(k_i)$, and this also obviously holds when $j_i \le J(k_i)$. We now use this, together with the quasi-multiplicativity and the RSW estimate (2.3), as before, to obtain

$$\begin{aligned} H(j,k) &\le \sum_{d=2}^{k} \sum_{\substack{(j_1,\dots,j_d)\\(k_1,\dots,k_d)}} \prod_{i=1}^{d} \Big( O(k)^2\, k_i^{O(1)}\, \frac{\bar\gamma_r(2^j r)}{\bar\gamma_r(2^{j_i} r)}\, \bar\gamma_r(2^{j_i}\, r)\, g(k_i) \Big) \\ &\le \sum_{d=2}^{k} \sum_{\substack{(j_1,\dots,j_d)\\(k_1,\dots,k_d)}} \prod_{i=1}^{d} \Big( O(k)^{O(1)}\, \bar\gamma_r(2^j\, r)\, g(k_i) \Big) \qquad (4.8) \\ &\le \sum_{d=2}^{k} \Big( O(k)^{O(1)}\, j\, \bar\gamma_r(2^j\, r) \Big)^d \sum_{(k_1,\dots,k_d)} \prod_{i=1}^{d} g(k_i)\,. \end{aligned}$$

Since $\log_2^2(x+2)$ is concave on $[0,\infty)$, it follows that when $k_1 + \cdots + k_d = k$, we have $\prod_{i=1}^d g(k_i) \le g(k/d)^d$. Since for a fixed $d$ the number of possible $d$-tuples $(k_1, \dots, k_d)$ is clearly bounded by $k^d$, the above gives

$$H(j,k) \le \sum_{d=2}^{k} \Big( c\, k^c\, j\, \bar\gamma_r(2^j\, r)\, g(k/d) \Big)^d,$$

for some constant $c$. Noting that $g(k/d) \le g(k/2)$, and setting

$$\Phi = \Phi(j,k) := c\, k^c\, j\, \bar\gamma_r(2^j\, r)\, g(k/2),$$

we then get $H(j,k) \le \Phi^2 + \Phi^3 + \Phi^4 + \cdots = \Phi^2/(1-\Phi)$. Consequently, the proof of (4.6) and of Proposition 4.2 are completed by the following lemma. □

**Lemma 4.4.** *For all $\epsilon \in (0, 1/2)$, if $\vartheta$ in the definition of $g$ is chosen sufficiently large (depending only on $\epsilon$), then for all $k \in \mathbb{N}_+$ and all $j > J(k)$,*

$$\Phi(j,k) < \Big( g(k)\, \bar\gamma_r(2^j\, r) \Big)^{1-\epsilon} / 2 \qquad (4.9)$$

*and $\Phi(j,k) < 1/2$.*

**Proof.** The estimate $\Phi < 1/2$ follows from (4.9), since $g(k)\, \bar\gamma_r(2^j\, r) \le 1$. Write

$$\Phi / \Big( g(k)\, \bar\gamma_r(2^j\, r) \Big)^{1-\epsilon} = c\, k^c\, \bar\gamma_r(2^j\, r)^\epsilon\, j\, g(k/2)^\epsilon \Big( g(k/2)/g(k) \Big)^{1-\epsilon}. \qquad (4.10)$$

Since (4.3) holds for $\bar\gamma$ and $\bar\gamma_r(2^{J(k)+1}\,r)\,g(k) \le 1$, we have $\bar\gamma_r(2^j\,r)^\epsilon g(k/2)^\epsilon \le \bar\gamma_r(2^j\,r)^\epsilon g(k)^\epsilon \le O(1)\,2^{-\epsilon\,c_1(j-J(k))}$. Hence, $j\,\bar\gamma_r(2^j\,r)^\epsilon g(k/2)^\epsilon \le O_\epsilon\big(J(k)+1\big)$. As we noted before, $J(k) = O(\vartheta)\,\log_2^2(k+2)$. Hence, (4.10) gives

$$\Phi/\big(g(k)\,\bar\gamma_r(2^j\,r)\big)^{1-\epsilon} \le O_\epsilon(\vartheta)\,\log_2^2(k+2)\,k^c\,\Big(g(k/2)/g(k)\Big)^{1/2}.$$

It is easy to verify that the right hand side tends to 0 as $\vartheta \to \infty$, uniformly in $k \in \mathbb{N}_+$. This completes the proof. □

For later use, let us point out that having an exponent larger than 1 in (4.6) was important when we derived the bound (4.4), but not for the induction. In (4.8), we used only the bound (4.6) with the exponent 1. This was sufficient because of $d \ge 2$.

*Remark* 4.5. Our proof shows a **clustering effect** for spectral samples of very small size. Firstly, a positive proportion of our main upper bound (4.4) comes from annulus structures with a single overcrowded cluster, as it is clear from (4.7). Moreover, the contribution from sets with large diameter is small: for any $d \le \frac{R}{r}$, the calculation in (4.7) implies immediately that if $k \le O(1)\log_2 d$, then

$$\mathbb{P}\Big[\mathscr{S} \in \mathcal{S}(r,k),\ \operatorname{diam}(\mathscr{S}) > rd\Big] \le \gamma_r(R)\,\gamma_r(rd)^{1+o(1)}\,d^{\,o(1)}$$

(of course, the exponent 0.99 in (4.7) can be modified to $1+o(1)$). Recall that (4.3) says $\gamma_r(rd) < O(1)\,d^{-c_1}$ for some $c_1 > 0$. Since we will see in Section 5 that Lemma 3.2 is sharp, and will handle the boundary issues in Subsection 4.3, we easily obtain that $\mathbb{P}\big[1 \le |\mathscr{S}_r| \le k\big] \ge \mathbb{P}\big[|\mathscr{S}_r| = 1\big] \ge O(1)\gamma_r(R)$. Therefore the above bound gives for $k \le O(1)\log_2 d$ that

$$\mathbb{P}\Big[\operatorname{diam}(\mathscr{S}) > rd\,\Big|\,1 \le |\mathscr{S}_r| \le k\Big] \le \gamma_r(rd)^{\,1+o(1)}\,.$$

To illustrate this formula (with $r=1$), if one is looking for spectral samples of size less than $\log R$, then they have small diameter:

$$\mathbb{P}\Big[\operatorname{diam}(\mathscr{S}) > R^\alpha\,\Big|\,1 \le |\mathscr{S}| \le \log R\Big] \le \gamma(R^\alpha)^{1+o(1)},$$

which for the triangular lattice gives $R^{-\alpha/2+o(1)}$.

*Remark* 4.6. Our strategy proving Proposition 4.1 also works for pivotals, showing

$$\mathbb{P}\big[|\mathscr{P}_r| = k\big] \leq g(k)\, \alpha_6(r, R)\, (R/r)^2 \,.$$

The only difference is that we need to replace the factor $\alpha_4(r, R)^2$, coming from Lemma 3.2 and its generalization Lemma 4.3, with $\alpha_6(r, R)$. The reason for this factor is that having pivotals in the inner disk of an annulus but no pivotals in the annulus itself corresponds to the 6-arm event in the annulus. On $\mathbb{Z}^2$ it is known that $\alpha_6(r, R)\,(R/r)^2 \leq O(1)\,(R/r)^\epsilon$ for some $\epsilon > 0$ [SS05, Corollary A.8]; on the triangular lattice, $\alpha_6(r, R)\,(R/r)^2 = (r/R)^{11/12+o(1)}$ [SW01]. Thus the clustering effect for pivotals is expressed here in the following way:

$$\mathbb{P}\Big[\mathrm{diam}(\mathscr{P}) > R^\alpha \,\Big|\, 1 \leq |\mathscr{P}| \leq \log R\Big] \leq R^{2\alpha} \alpha_6(R^\alpha)^{1+o(1)},$$

which for the triangular lattice gives $R^{-\frac{11}{12}\alpha+o(1)}$.

## 4.3 Handling boundary issues

**Proof of Proposition 4.1.** Since the proof is rather similar to that of Proposition 4.2, we will just indicate the necessary modifications.

First of all, we need the **half-plane** and **quarter-plane** $j$**-arm events**. So let $\alpha_j^+(r, R)$ be the probability of having $j$ disjoint arms of alternating colors connecting $\partial B(0, r)$ to $\partial B(0, R)$ inside the half-annulus $(B(0, R) \setminus B(0, r)) \cap (\mathbb{R} \times \mathbb{R}_+)$. Similarly, $\alpha_j^{++}(r, R)$ is the probability of having $j$ arms of alternating colors connecting $\partial B(0, r)$ to $\partial B(0, R)$ inside the quarter-annulus $(B(0, R) \setminus B(0, r)) \cap (\mathbb{R}_+ \times \mathbb{R}_+)$. As before, we let $\alpha_j^+(R) := \alpha_j^+(2j, R)$ and similarly for $\alpha_j^{++}$. The RSW and quasi-multiplicativity bounds (2.1, 2.2) hold for these quantities, as well.

The reason for these definitions is that if $x$ is a point on one of the sides of $[0, R]^2$ such that its distance from the other sides is at least $r'$, then the percolation configuration inside $B(x, r)$ has an effect on the crossing event only if we have the 3-arm event in the half-annulus $(B(x, r') \setminus B(x, r)) \cap [0, R]^2$. Similarly, if $x$ is one of the corners of $[0, R]^2$, and $r' \leq R$, then we need the 2-arm event in the quarter-annulus $(B(x, r') \setminus B(x, r)) \cap [0, R]^2$ in order for the configuration inside $B(x, r)$ to have any effect.

In the annulus structures we are going to build, we will have to consider clusters that are close to a side or even to a corner of $[0, R]^2$. These will be called side and corner clusters (defined precisely below). To understand the contribution of such clusters, we define

$$\gamma_r^+(\rho) := (\rho/r)\,\alpha_3^+(r,\rho)^2 \quad \text{and} \quad \gamma_r^{++}(\rho) := \alpha_2^{++}(r,\rho)^2.$$

The function $\gamma_r^+$ plays a role similar to $\gamma_r$, but in relation to the side clusters. Similarly, $\gamma_r^{++}$ relates to the corner clusters. The motivation for the linear prefactor of $\rho/r$ in the definition of $\gamma_r^+$ is that (up to a constant factor) the number of different ways to choose a square of some fixed size whose center is on a line segment of length $\rho$ and whose position on the line segment is divisible by $\rho'$ is $\rho/\rho'$, when $\rho > \rho' > 0$. Such a prefactor is not necessary in the case of $\gamma_r^{++}$, because it corresponds to a corner, and there is no choice in placing a square of a fixed size into a fixed corner.

As we will see, the key reason for the boundary $\partial[0,R]^2$ to play no significant role in the behavior of the spectral sample is that there is a $\delta > 0$ and a constant $c$ such that when $r \le \rho' \le \rho$

$$\frac{\gamma_r^+(\rho)}{\gamma_r^+(\rho')} \le c\left(\frac{\gamma_r(\rho)}{\gamma_r(\rho')}\right)^{1+\delta} \quad \text{and} \quad \frac{\gamma_r^{++}(\rho)}{\gamma_r^{++}(\rho')} \le c\left(\frac{\gamma_r(\rho)}{\gamma_r(\rho')}\right)^{1+\delta}. \tag{4.11}$$

We now prove these inequalities. Firstly, it is known that

$$\alpha_3^+(\rho',\rho) \asymp (\rho'/\rho)^2, \tag{4.12}$$

see [Wer07, First exercise sheet]. Hence $\gamma_r^+(\rho)/\gamma_r^+(\rho') \asymp (\rho'/\rho)^3$. Secondly, observe that

$$\alpha_2^{++}(\rho',\rho)^2 \le\; \alpha_4^+(\rho',\rho) \le\; \alpha_3^+(\rho',\rho)\,\alpha_1^+(\rho',\rho) \le O(1)\,\alpha_3^+(\rho',\rho)\,(\rho'/\rho)^{\epsilon}, \tag{4.13}$$

with some $\epsilon > 0$, where the second step used Reimer's inequality [Rei00] (or color-switching and the BK inequality), and the third step used RSW for the 1-arm half-plane event. Therefore, $\gamma_r^{++}(\rho)/\gamma_r^{++}(\rho') \le O(1)(\rho'/\rho)^{2+\epsilon}$. On the other hand, (2.6) implies that $\gamma_r(\rho)/\gamma_r(\rho') \ge (\rho'/\rho)^{2-\epsilon'}$ for some $\epsilon' > 0$. Combining these upper and lower bounds we get (4.11).

After these preparations, we define $\mathcal{S}(r,k)$ as the set of all $S \subset \mathcal{I}$ such that $|S_r| = k$. The set $V = V_S$ is defined as $S_r$. As it turns out, we will need to limit the diameters of the clusters. For that purpose, set $\bar{j} = \bar{j}_k := \lfloor \log_2(R/(k\,r)) \rfloor - 5$ and $J := \{0,1,\dots,\bar{j}\}$. Clearly, we may assume without loss of generality that $\bar{j} > 0$. The clusters at level 0 are once again the sets of the type $C = \{x\}$, where $x \in V$. If $j \in J$, then an **interior cluster** at level $j$ is defined as a connected component $C \subset V$ of $G_j$ such that the distance from $C$ to $\partial([0,R]^2)$ is larger than $2^j\,r$ and $C$ is not connected in $G_{j-1}$. The interior clusters at level 0 are just the connected components of $G_0$; that is, the singletons in $V$. Inductively, we define the **side clusters**: a connected component $C \subset V$ of $G_j$ is a side cluster at level $j \in J$ if it is within distance $2^j\,r$ of precisely one of the four boundary edges of $[0,R]^2$ and it is not a side cluster at any level $j' \in \{1,2,\dots,j-1\}$. Likewise, a **corner cluster** at level $j \in J$ is a connected component $C \subset V$ of $G_j$ that is within distance $2^j\,r$ of precisely two adjacent boundary edges of $[0,R]^2$, but is not a corner cluster at any level $j' \in \{1,2,\dots,j-1\}$. Finally, the unique **top cluster** has level $\bar{j}+1$ (by definition) and consists of all of $V$. With these choices, when $j \in J$ every connected component of $G_j$ is either an interior, side, or corner cluster, and this is the reason for setting the upper bound $\bar{j}$.

Note that the top cluster contains at least one cluster (which could be a side cluster or a corner cluster or an interior cluster), a corner cluster contains at least one side or interior cluster, and possibly also a corner cluster, a side cluster contains an interior cluster or at least two side clusters (but no corner clusters), and an interior cluster at level $j > 0$ contains at least two interior clusters (and no side or corner clusters).

The inner and outer bounding squares associated with the clusters are defined as before, except that the squares associated with side clusters are centered at points on the corresponding edge and squares associated with corner clusters are centered at the corresponding corner, which is the meeting point of the two sides of $[0,R]^2$ closest to the cluster. There are no squares associated with the top cluster. The annulus associated with each cluster (other than the top cluster, which does not have its own annulus) is the annulus between its outer square and its inner square, provided that the outer square strictly contains the inner square.

We define $\bar{\gamma}_r^+(\rho) := \inf_{\rho' \in [r,\rho]} \gamma_r^+(\rho)$ and similarly for $\bar{\gamma}_r^{++}$. The exponential decay (4.3) holds for these functions as well. As before, clusters (even side or corner clusters) are considered overcrowded if they satisfy $g(|C|)\,\gamma_r(2^j\,r) > 1$, and the annulus structure $\mathcal{A}(S)$ is defined as above.

There are some modifications necessary in the definition of $h(\mathcal{A})$ in order for (4.2) to still hold in the present setting. For a side annulus $A$ define $h(A)$ as the probability of the 3-arm crossing event within $A \cap [0,R]^2$ between the two boundary components of $A$, and for a corner annulus define $h(A)$ as the probability of the 2-arm crossing event within $A \cap [0,R]^2$ between the two boundary components of $A$. With these modifications, (4.2) still holds, and the proof is essentially the same.

The definition of $H(j,k)$ is similar to the one in the proof of Proposition 4.2, but now the supremum only refers to interior squares. (In the definition of $\mathfrak{A}(B,k,j)$, we presently restrict to such $S$ so that $V_S$ is an interior cluster at level $j$, in addition to being the top cluster, and when we write $B(V_S)$, it is understood as the inner square defined for an interior cluster.) We also define $H^+(j,k)$, $H^{++}(j,k)$, which refer to the supremum over side and corner squares, respectively. We also set

$$H^{\square}(R,k) := \sum_{S\in\mathcal{S}(k,r)} h\big(\mathcal{A}(S)\big)^2\,,$$

and our goal is to show that this quantity is at most $g(k)\,\gamma_r(R)$.

We first prove a similar bound on $H^+(j,k)$. It is convenient to separate the annulus structures $\mathcal{A}'(S)$ where $B_S(V_S)=B$ and $B$ is a side square into those where $V_S$ has a single child cluster and into those where $V_S$ has at least two child clusters. In the case of a single child cluster, that child has to be an interior cluster, and using (4.6), the argument giving (4.4) now gives the bound

$$\sum_{\mathcal{A}} h(\mathcal{A})^2 \le O(1)\,k^{O(1)} g(k)\,\bar\gamma_r(2^j\,r)\,,$$

where the sum extends over such (single interior cluster child) annulus structures. By increasing the constant $\vartheta$ in the definition of $g$, we may then incorporate the factor $O(1)\,k^{O(1)}$ into $g$. The bound on the sum over the annulus structures with at least two child clusters can now be established by induction on $j$, in almost the same way that (4.6) was proved by induction. The main difference here is that the children can fall into two types, which slightly complicates the calculations but adds no significant difficulties. (Indeed, since each square among $B_i$ at level $j_i$ ($i=1,2,\dots,d$) can either correspond to a side cluster or an interior cluster, each factor in the first row of (4.8) presently needs to be replaced by

$$O(k)^2\,k_i^{O(1)}\Big(\frac{\bar\gamma_r(2^j r)}{\bar\gamma_r(2^{j_i} r)}+\frac{\bar\gamma_r^+(2^j r)}{\bar\gamma_r^+(2^{j_i} r)}\Big)\bar\gamma_r(2^{j_i}\,r)\,g(k_i)\,.$$

Thanks to (4.11), the fraction featuring $\bar\gamma^+$ is dominated by a constant times the other fraction and is therefore certainly inconsequential.) A point worth noting is that in the inductive prove of (4.6) we only used the inductive hypothesis with exponent 1 in place of 1.99. In our present situation, this is what we have at our disposal in the base step of the induction, since the induction now has to be started from overcrowded clusters or side clusters with a single child cluster.

Summing up the two types, we conclude (by changing $\vartheta$ again, if necessary) that $H^+(j,k) \le g(k)\,\bar\gamma_r(2^j\,r)$. Of course, in this type of argument we should not change $\vartheta$ at every induction step, for then it may end up depending on $R$. But we have not committed any such offence.

We can show $H^{++}(j,k) \le g(k)\,\bar\gamma_r(2^j\,r)$ similarly. We separate the annulus structures into those with a single child at the top level that is an interior cluster, those with a single child that is a side cluster, and those with several children at the top level. The first type is handled as in the bound for $H^+(j,k)$. The second type is handled similarly, but now we do not have the exponent 1.99 as in (4.6), but only the exponent 1 that we showed for $H^+(j,k)$. So, we use instead the fact that $\delta > 0$ in (4.11). The argument bounding the third type uses induction as in the multi-child case of $H^+(j,k)$.

Finally, the bound for $H^{\square}(R,k)$ follows in the same way, using our previous bounds for $H^+(j,k)$ and $H^{++}(j,k)$ together with (4.11). The last small difference is that the child clusters of the top cluster (at some levels $j_i$) have outer bounding squares of size $\asymp r2^{\bar j}$, but the number of ways to place each of these clusters is $\asymp (R/(r2^{j_i}))^2$, instead of $\asymp (2^{\bar j}/2^{j_i})^2$. But $R/(r2^{\bar j}) \asymp k$, so this discrepancy gives only an $O(k^2)$ factor for each child cluster, which can be absorbed into $g(k)$ in the usual way. This completes the proof. □

## 4.4 The radial case

In this subsection, we will consider the spectral sample $\mathscr{S} = \mathscr{S}_f$, where $f$ is the indicator function (not the $\pm 1$-indicator function) of the $\ell$-arm event in the annulus $[-R,R]^2 \setminus [-\ell,\ell]^2$ and $\ell = 1$ or $\ell \in 2\,\mathbb{N}_+$. Thus, $\mathbb{E}\big[f\big] = \mathbb{E}\big[f^2\big] \asymp \alpha_\ell(R)$. Instead of the probability measure for $\mathscr{S}$ that we have worked with so far, it will be easier notationally to use the un-normalized measure $\mathbb{Q}\big[\mathscr{S} = S\big] := \widehat{f}(S)^2$.

For any $S \subset \mathbb{R}^2$, we let $S^* := S \cup \{0\}$, and define $S_r$ as before. In particular, $\mathscr{S}^*_r$ is the set of $r$-squares whose intersection with $\mathscr{S}^*$ is nonempty. We are going to prove the following analogue of Proposition 4.1:

**Proposition 4.7.** *Let $\ell = 1$ or $\ell \in 2\,\mathbb{N}_+$, and let $\mathscr{S}$ denote the spectral sample of the indicator function of the $\ell$-arm event in the annulus* $[-R,R]^2 \setminus$

$[-\ell, \ell]^2$. *Then there is some* $\vartheta^* = \vartheta^*_\ell > 0$ *such that*

$$\mathbb{Q}\big[|\mathscr{S}_r| = k\big] \le g^*(k)\, \alpha_\ell(r,R)^2\, \alpha_\ell(r).$$

*holds with* $g^*(k) := 2^{\vartheta^* \log_2^2(k+2)}$ *for all* $k \in \mathbb{N}_+$ *and all* $R \ge r \ge \ell$.

**Proof.** The main difference from the square crossing case is that we will use **centered annulus structures**, which have two kinds of annuli: annuli centered at the origin (0), for which we are interested in the $\ell$-arm event, and annuli with outer square disjoint from 0, for which we are interested in the 4-arm event. Each centered annulus structure is required to have an annulus centered at 0 whose inner square does not contain any other annuli. The **inner radius** of the annulus structure is defined as the inner radius of this innermost centered annulus. For a centered annulus structure $\mathcal{A}$, we define $h^*(\mathcal{A})$ to be the probability of having the 4-arm event in the annuli with outer square disjoint from 0 and the $\ell$-arm event in the annuli centered at 0. Now, we have the following analogue of Lemma 4.3.

**Lemma 4.8.** *For any centered annulus structure* $\mathcal{A}$ *with inner radius* $r_{\mathcal{A}}$,

$$\mathbb{Q}\big[\mathscr{S}^* \text{ is compatible with } \mathcal{A}\big] \le \alpha_\ell(r_{\mathcal{A}})\, h^*(\mathcal{A})^2.$$

**Proof.** Similarly to the proof of Lemma 4.3, we divide the set of relevant bits into parts: $\theta$ is the configuration inside $\bigcup \mathcal{A}$, while $\eta_0$ is the configuration inside the inner disk of the smallest centered annulus, and $\eta_1$ is the configuration neither in $\theta$ nor in $\eta_0$. As before, $F_\theta$ is the function defined by $F_\theta(\eta_0, \eta_1) := f(\theta, \eta_0, \eta_1)$; furthermore, $W$ is the linear space of functions spanned by $\{\chi_S : S^* \text{ compatible with } \mathcal{A}\}$, and $P_W$ denotes the orthogonal projection onto $W$. Now, $P_W F_\theta \ne 0$ implies the 4-arm event in every interior non-centered $A \in \mathcal{A}$, the $\ell$-arm event in every centered $A \in \mathcal{A}$, and the 3-arm event in every boundary (or corner) annulus. (Note that this uses the fact that when $\ell \ne 1$ we are considering the alternating arms event. In particular, we are restricted to $\ell \in \{1\} \cup 2\,\mathbb{N}_+$.) Moreover, for any $\theta$, we have $F_\theta(\eta_0, \eta_1) = 0$ if $\eta_0$ does not have the $\ell$-arm event. Thus, $\|P_W F_\theta\|^2 \le \|F_\theta\|^2 = \mathbb{E}[F_\theta^2] \le \alpha_\ell(r_{\mathcal{A}})$. Altogether, similarly to the proof of Lemma 4.3,

$$\begin{aligned}\mathbb{Q}\big[\mathscr{S}^* \text{ is compatible with } \mathcal{A}\big] &= \|P_W f\|^2 \le \mathbb{E}^\theta\big[\|P_W F_\theta\|\big]^2 \\ &\le \mathbb{P}^\theta\big[P_W F_\theta \ne 0\big]^2\, \alpha_\ell(r_{\mathcal{A}}) \le h^*(\mathcal{A})^2\, \alpha_\ell(r_{\mathcal{A}}),\end{aligned}$$

which proves the lemma. □

Note that a centered annulus structure compatible with $\mathscr{S}$ is also compatible with $\mathscr{S}^*$, but not necessarily vice versa, hence the lemma is stronger with $\mathscr{S}^*$ than it would be with $\mathscr{S}$; this strengthening will be crucial for us in the sequel.

Analogously to Subsection 4.2, for each $S \subset [-R,R]^2$ with $|S_r| = k$ we will build a centered annulus structure $\mathcal{A}(S)$ that is compatible with $S^*$ (but not necessarily with $S$ itself!) and that has $r_{\mathcal{A}(S)} \geq r$. Furthermore, the collection $\mathfrak{A}^*(R,k)$ of all these centered annulus structures will be small enough to have

$$\sum_{\mathcal{A}\in\mathfrak{A}^*(R,k)} h^*(\mathcal{A})^2 \leq g^*(k)\;\alpha_\ell(r,R)^2. \tag{4.14}$$

The combination of (4.14) with Lemma 4.8 proves Proposition 4.7.

To construct the annulus structure $\mathcal{A}(S)$, we take $V = V_S$ to be $S_r^*$, set $\bar{j} := \lfloor \log_2(R/(kr)) \rfloor - 5$, and define the clusters exactly as in Subsection 4.3. A cluster is called centered if it contains 0. In constructing the inner and outer bounding squares, we use the additional rule that for centered clusters $C$ the inner bounding square must be centered at 0. (Note that this is just a special case of the "arbitrary but fixed way" of choosing a vertex $z \in C$.)

The centered analogue of $\gamma_r(\rho)$ is now $\gamma_r^*(\rho) := \alpha_\ell(r,\rho)^2$. We again let $\bar{\gamma}_r^*(\rho) := \inf_{\rho'\in[r,\rho]} \gamma_r^*(\rho')$, and we note the exponential decay (4.3) for $\bar{\gamma}_r^*(r2^j)$ in $j$. A non-centered cluster $C$ is called overcrowded if $g(|C|)\,\bar{\gamma}_r(r2^{j(C)}) > 1$, with the function $g(k)$ of Proposition 4.1. A centered cluster is overcrowded if $g^*(|C|)\,\bar{\gamma}_r^*(r2^{j(C)}) > 1$. We define $J(k)$ as before, using $g(k)$, and similarly define $J^*(k)$, using $g^*(k)$. In particular, $\bar{\gamma}_r^*(r2^{J^*(k)})\,g^*(k) \asymp 1$.

Using these notions of overcrowded, we get our annulus structure $\mathcal{A}(S)$. The collection of them for all $S \subset \mathcal{I}$ with $|S| \leq k$ is $\mathfrak{A}^*(R,k)$.

We define $H(j,k)$ similarly as before, but now only for interior inner bounding squares that do not contain 0. We similarly define the quantities $H^+(j,k)$ and $H^{++}(j,k)$ for side and corner squares not containing 0. Finally, we let $H^*(j,k)$ be the analogous quantity where the inner bounding square is required to be centered. We will show that there is some constant $\delta > 0$, depending only on $\ell$, such that, for $j \in \{J^*(k),\ldots,\bar{j}\}$,

$$H^*(j,k) \leq \left(g^*(k)\;\bar{\gamma}_r^*(r2^j)\right)^{1+\delta}. \tag{4.15}$$

This implies (4.14) (with a possibly larger constant $\vartheta^*$) in exactly the same way as in Subsection 4.2 the bound (4.6) implied (4.4) (with the small additional care regarding the cutoff $\bar{j}$ that we have seen in Subsection 4.3).

As usual, we prove (4.15) by induction on $j$. We may assume that $j > J^*(k)$. Recall that $H^*(j,k)$ is defined as a sum, where each summand corresponds to a centered annulus structure. Suppose that $\mathcal{A}$ is a centered annulus structure contributing to the sum, where $\mathcal{A} = \mathcal{A}(S)$ for some $S \subset \mathcal{I}$ with $|S_r| = k$. Let $j_*$ be $j(C^*)$, where $C^*$ is the largest proper centered subcluster of $S_r^*$, and let $k_* = |C^* \cap S_r|$. Every such $\mathcal{A}$ can be formed as a union of the topmost (centered) annulus in $\mathcal{A}$, a centered annulus structure $\mathcal{A}^*$ for $(j_*, k_*)$, and the annulus structure $\mathcal{A}'$ formed by dropping from $\mathcal{A}$ all the centered annuli. Moreover,

$$h^*(\mathcal{A})^2 \asymp (k+2)^{O(1)}\, h^*(\mathcal{A}^*)^2\, h(\mathcal{A}')^2\, \alpha_\ell(r\,2^{j_*}, r\,2^j)^2.$$

The sum over such $\mathcal{A}$ with $j_*$ and $k_*$ fixed is bounded by

$$(k+2)^{O(1)}\, \alpha_\ell(r\,2^{j_*}, r\,2^j)^2 \Big(\sum_{\mathcal{A}^*} h^*(\mathcal{A}^*)^2\Big)\Big(\sum_{\mathcal{A}'} h(\mathcal{A}')^2\Big),$$

where the sums run over the appropriate collections of annulus structures. The first sum is bounded by $H^*(j_*, k_*)$, and the proof of (4.4), possibly incorporating boundary clusters, shows that the second factor is bounded by $g(k-k_*)\,\bar{\gamma}_r(r\,2^j)$, with possibly a different choice of the constant $\vartheta$ implicit in $g$. (Note that the annulus structure $\mathcal{A}'$ may have just one annulus whose outer square is roughly at the scale corresponding to $j$. For that reason, it is of the type estimated in (4.4), rather than the type estimated in (4.6).) The induction hypothesis therefore gives

$$\begin{aligned} H^*(j,k) &\le (k+2)^{O(1)} \sum_{k_*, j_*} \alpha_\ell(r2^{j_*}, r2^j)^2\, g^*(k_*)\, \bar{\gamma}_r^*(r2^{j_*})\, g(k-k_*)\, \bar{\gamma}_r(r2^j) \\ &\le (k+2)^{O(1)}\, g(k)\, g^*(k)\, j\, \bar{\gamma}_r(r2^j)\, \bar{\gamma}_r^*(r2^j)\,. \end{aligned}$$

If $\delta > 0$ is small enough, then $j\,\bar{\gamma}_r(r2^j) \le O(1)\,\bar{\gamma}_r^*(r2^j)^\delta$. Given this $\delta$, if $\vartheta^*$ is large enough, we also have $O(1)\,(k+2)^{O(1)} g(k) \le g^*(k)^\delta$. Thus, our last upper bound is at most $\big(g^*(k)\,\bar{\gamma}_r^*(r2^j)\big)^{1+\delta}$, so (4.15) is proved, and our proof of Proposition 4.7 is complete. ∎

# 5 Partial independence in spectral sample

## 5.1 Setup and main statement

Let $\mathscr{S}$ denote the spectral sample of the $\pm 1$ indicator function of having a percolation left-right crossing in $[0,R]^2$ (in either of our two favorite lattices). In order to prove that $|\mathscr{S}|$ is rarely much smaller than its mean it would be useful to have some independence of the following kind: if $B_1, B_2$ are two distant squares, then we would expect that

$$\mathbb{P}\big[\mathscr{S}\cap B_1 = S_1 \bigm| \mathscr{S}\cap B_1 \neq \emptyset,\, \mathscr{S}\cap B_2 = S_2\big] \asymp \mathbb{P}\big[\mathscr{S}\cap B_1 = S_1 \bigm| \mathscr{S}\cap B_1 \neq \emptyset\big].$$

It turns out that it is hard to control such correlations. Nevertheless, we will prove a weaker independence result that will be enough for our purposes.

Consider some box $B$ of radius $r$ inside $[0,R]^2$. (Recall from Section 2.1 that a box $B(x,r)$ of radius $r$ is the union of tiles whose centers are in $x+[-r,r)^2$.) We want to understand the behaviour of $\mathscr{S}$ in $B$. Because of boundary issues, we will actually look at $\mathscr{S}$ in a smaller concentric box $B'$, of radius $r/3$.

We saw in (3.7) that $O(1)\,\mathbb{E}\big[|\mathscr{S}\cap B'| \bigm| \mathscr{S}\cap B\neq\emptyset\big] \geq r^2\,\alpha_4(r)$. In this section, we will strengthen this by proving that $|\mathscr{S}\cap B'|$ is at least of this size with a uniform positive probability, moreover, this remains true when we add $\mathscr{S}\cap W=\emptyset$ to the conditioning, where $W$ is an arbitrary set in the complement of $B$:

$$\mathbb{P}\big[|\mathscr{S}\cap B'|\geq c\,r^2\alpha_4(r) \bigm| \mathscr{S}\cap B\neq\emptyset,\ \mathscr{S}\cap W=\emptyset\big] > a\,, \tag{5.1}$$

with some fixed constants $c, a>0$. However, the following stronger statement is closer to what we actually need.

**Proposition 5.1.** *Let $\mathscr{S}$ be the spectral sample of the $\pm 1$-indicator function of the left-right crossing event in $\mathcal{Q}=[0,R]^2$. Let $B$ be a box of some radius $r$. Let $B'$ be the concentric box with radius $r/3$, and assume that $B'\subset\mathcal{Q}$. We also assume that $r\geq\bar r$, where $\bar r>0$ is some universal constant. Fix any set $W\subset\mathbb{R}^2\setminus B$, and let $\mathcal{Z}$ be a random subset of $\mathcal{I}$ that is independent from $\mathscr{S}$, where each element of $\mathcal{I}$ is in $\mathcal{Z}$ with probability $1/(\alpha_4(r)\,r^2)$ independently. (By (2.6), $\alpha_4(r)\,r^2\geq 1$ if $\bar r$ is sufficiently large.) Then*

$$\mathbb{P}\big[\mathscr{S}\cap B'\cap\mathcal{Z}\neq\emptyset \bigm| \mathscr{S}\cap B\neq\emptyset,\ \mathscr{S}\cap W=\emptyset\big] > a\,,$$

*where $a > 0$ is a universal constant.*

The estimate (5.1) follows immediately from the proposition, since

$$\mathbb{P}\big[\mathscr{S} \cap B' \cap \mathcal{Z} \neq \emptyset \bigm| \mathscr{S} \cap B',\ \mathscr{S} \cap B \neq \emptyset,\ \mathscr{S} \cap W = \emptyset\big] \\ = 1 - \big(1 - r^{-2}\,\alpha_4(r)^{-1}\big)^{|\mathscr{S} \cap B'|}.$$

It is important to note that in the proposition the constant $a > 0$ is independent of the position of the box $B$ relative to the square $[0, R]^2$. Such a uniform control over the domain is harder to achieve in the case of general quads. Still, after proving this uniform result for the square we will prove a local version (Proposition 5.11) for general quads. We will also prove a radial version (Proposition 5.12), which will be important for the application to exceptional times of dynamical percolation.

The proof of the proposition is straightforward once we have the following bounds on the first and second moments. Recall the definition of $\lambda_{B,W}$ right before Lemma 2.2.

**Proposition 5.2** (First moment)**.** *Assume the setup of Proposition 5.1. There is an absolute constant $c_1 > 0$ such that for any $x \in B' \cap \mathcal{I}$,*

$$\mathbb{P}\big[x \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] \geq c_1\,\mathbb{E}\big[\lambda_{B,W}^2\big]\,\alpha_4(r). \tag{5.2}$$

**Proposition 5.3** (Second moment)**.** *Let $\mathscr{S}$ be the spectral sample of $f = f_{\mathcal{Q}}$, where $\mathcal{Q} \subset \mathbb{R}^2$ is some arbitrary quad. Let $z \in \mathcal{Q}$ and $r > 0$. Set $B := B(z, r)$ and $B' := B(z, r/3)$. Suppose that $B(z, r/2) \subset \mathcal{Q}$. Then for every $x, y \in B' \cap \mathcal{I}$ we have*

$$\mathbb{P}\big[x, y \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] \leq c_2\,\mathbb{E}\big[\lambda_{B,W}^2\big]\,\alpha_4(|x - y|)\,\alpha_4(r)\,, \tag{5.3}$$

*where $c_2 < \infty$ is an absolute constant.*

**Proof of Proposition 5.1** (assuming the first and second moment estimates)**.** Consider the random variable

$$Y := |\mathscr{S} \cap B' \cap \mathcal{Z}|\,1_{\{\mathscr{S} \cap W = \emptyset\}}.$$

Since $\mathcal{Z}$ is independent from $\mathscr{S}$ and $\mathbb{P}[x \in \mathcal{Z}] = 1/(\alpha_4(r)\,r^2)$, we obtain by summing (5.2) over all $x \in B' \cap \mathcal{I}$ that $O(1)\,\mathbb{E}[Y] \geq \mathbb{E}\big[\lambda_{B,W}^2\big]$. On the other hand, summing (5.3) over all $x, y \in B' \cap \mathcal{I}$, similarly to the second moment estimate in Lemma 3.1, gives

$$\begin{aligned}\mathbb{E}[Y^2] &\le \overbrace{O(1)\,\mathbb{E}\big[\lambda_{B,W}^2\big]\,\alpha_4(r)\,r^2\,\mathbb{P}\big[x \in \mathcal{Z}\big]}^{\text{diagonal term}} + \overbrace{O(1)\,\mathbb{E}\big[\lambda_{B,W}^2\big]\,\alpha_4(r)^2\,r^4\,\mathbb{P}\big[x \in \mathcal{Z}\big]^2}^{\text{off-diagonal term}} \\ &\le O(1)\,\mathbb{E}\big[\lambda_{B,W}^2\big],\end{aligned}$$

by our choice of $\mathbb{P}\big[x \in \mathcal{Z}\big]$. Note that this choice is of the smallest possible order that does not make the diagonal term the leading contribution. Now, by Cauchy-Schwarz,

$$\mathbb{P}\big[Y > 0\big] \ge \frac{\mathbb{E}[Y]^2}{\mathbb{E}\big[Y^2\big]} \ge \frac{\mathbb{E}[\lambda_{B,W}^2]^2}{O(1)\,\mathbb{E}[\lambda_{B,W}^2]} = \frac{\mathbb{E}\big[\lambda_{B,W}^2\big]}{O(1)}. \tag{5.4}$$

The proposition now follows from Lemma 2.2. □

*Remark* 5.4. $\mathbb{P}\big[\mathscr{S} \cap B \neq \emptyset,\ \mathscr{S} \cap W = \emptyset\big]$ is obviously not smaller than the left hand side of (5.4). Therefore, (5.4) and Lemma 2.2 imply that in the present setting

$$\mathbb{P}\big[\mathscr{S} \cap B \neq \emptyset,\ \mathscr{S} \cap W = \emptyset\big] \asymp \mathbb{E}\big[\lambda_{B,W}^2\big]. \tag{5.5}$$

The definition of $\lambda_{B,W}$ easily gives

$$\mathbb{E}\big[\lambda_{B,\emptyset}^2\big] = \alpha_\square(B, \mathcal{Q}) \quad \text{and} \quad \lambda_{B,B^c} = \alpha_\square(B, \mathcal{Q})\,. \tag{5.6}$$

Combining these with (5.5), we get that for $B$ as above, approximate equalities hold in Lemma 3.2, i.e.,

$$\mathbb{P}\big[\mathscr{S} \cap B \neq \emptyset\big] \asymp \alpha_\square(B, \mathcal{Q}) \quad \text{and} \quad \mathbb{P}\big[\emptyset \neq \mathscr{S} \subseteq B\big] \asymp \alpha_\square(B, \mathcal{Q})^2\,. \tag{5.7}$$

## 5.2 Bounding the second moment

Due to the way in which $\lambda_{B,W}$ was defined, it is generally easier to obtain $\lambda_{B,W}$ as an upper bound up to constants, than as a lower bound up to constants. Consequently, the second moment estimate is easier to prove, and for this reason we start with that.

**Proof of Proposition 5.3.** Let $\theta$ denote the restriction of $\omega$ to the complement of $W \cup \{x,y\}$. Then Lemma 2.1 gives

$$\begin{aligned}\mathbb{P}\big[x,y \in \mathscr{S}, \mathscr{S} \cap W = \emptyset\big] &= \mathbb{P}\big[\mathscr{S} \cap (W \cup \{x,y\}) = \{x,y\}\big] \\ &= \mathbb{E}\Big[\mathbb{E}\big[\chi_{\{x,y\}}(\omega)\, f(\omega) \;\big|\; \theta\big]^2\Big]\,.\end{aligned} \tag{5.8}$$

Set

$$g(\theta) := \mathbb{E}\big[\chi_{\{x,y\}}(\omega)\, f(\omega) \;\big|\; \theta\big].$$

Then $\mathbb{E}\big[g^2\big]$ is the quantity that we need to estimate. Since $B \cap W = \emptyset$, the information in $\theta$ includes the configuration in $B \setminus \{x,y\}$. If $\omega$ does not have the 4 arm event from the tile of $x$ to distance $|x-y|/4$, then flipping $\omega_x$ does not effect $f(\omega)$, and hence $g(\theta)=0$. A similar statement holds for $y$. Also, if the box $\tilde{B}$ of radius $2\,|x-y|$ centered at $(x+y)/2$ does not intersect $\partial B$, then $g(\theta)=0$ unless $\omega$ has the 4 arm event in the corresponding annulus $B \setminus \tilde{B}$. Let $A_x$, $A_y$ and $A_{x,y}$ denote the indicator functions for the 4-arm event in the corresponding 3 annuli, where we take $A_{x,y}=1$ if $\tilde{B} \cap \partial B \neq \emptyset$. Then we have $g(\theta)=0$ if $A_x\, A_y\, A_{x,y} = 0$.

We now argue that $|g(\theta)| \le \lambda_{B,W}$. For this purpose, write

$$g = \mathbb{E}\big[\chi_{\{x,y\}} f \;\big|\; \mathcal{F}_{(W \cup \{x,y\})^c}\big] = \mathbb{E}\big[\mathbb{E}[\chi_{\{x,y\}} f \mid \mathcal{F}_{\{x,y\}^c}] \;\big|\; \mathcal{F}_{W^c}\big].$$

Clearly, $\big|\mathbb{E}[\chi_{\{x,y\}} f \mid \mathcal{F}_{\{x,y\}^c}]\big| \le 1_{\Lambda_{\{x,y\}}} \le 1_{\Lambda_B}$, where $\Lambda_\cdot$ is as defined above Lemma 2.2. Taking conditional expectation given $\mathcal{F}_{W^c}$ then gives

$$\bigg|\mathbb{E}\Big[\mathbb{E}[\chi_{\{x,y\}} f \mid \mathcal{F}_{\{x,y\}^c}] \;\Big|\; \mathcal{F}_{W^c}\Big]\bigg| \le \mathbb{E}\Big[\big|\mathbb{E}[\chi_{\{x,y\}} f \mid \mathcal{F}_{\{x,y\}^c}]\big| \;\Big|\; \mathcal{F}_{W^c}\Big] \le \lambda_{B,W}\,.$$

Since the left hand side is $|g|$, we get $|g| \le \lambda_{B,W}$.

Putting together the above, we arrive at $|g(\theta)| \le A_x\, A_y\, A_{x,y}\, \lambda_{B,W}$. Thus, $g(\theta)^2 \le A_x\, A_y\, A_{x,y}\, \lambda_{B,W}^2$. Independence on disjoint sets then gives

$$\mathbb{E}\big[g^2\big] \le \alpha_4(|x-y|/4)^2\, \alpha_4(2\,|x-y|, r/3)\, \mathbb{E}\big[\lambda_{B,W}^2\big]\,.$$

The proposition now follows from the familiar properties of $\alpha_4$. ☐

## 5.3 Reformulation of first moment estimate

Before proving the first moment estimate (Proposition 5.2), we explain how it can be reformulated as a quasi-multiplicativity property analogous to the quasi-multiplicativity property of the $j$-arm events (2.2). Recall that

$$\mathbb{E}\big[\lambda_{B,W}^2\big] = \mathbb{E}\Big[\mathbb{P}\big[\Lambda_B \mid \mathcal{F}_{W^c}\big]^2\Big].$$

It is not apriori clear how to work with $\mathbb{E}\big[\lambda_{B,W}^2\big]$, but here is a useful observation about this quantity. Let $\omega'$ and $\omega''$ be two critical percolation configurations which coincide on $W^c$ but are independent on $W$. Let $\mathscr{A}_\square(B,\mathcal{Q})$ denote the set of percolation configurations $\omega$ for which the 4-arm event occurs in the annulus $\mathcal{Q} \setminus B$ with the appropriately colored arms terminating on the correct boundary arcs of $\mathcal{Q}$; that is, the primal (white) arms terminating on the two distinguished arcs of $\partial\mathcal{Q}$ and the dual (black) arms terminating on the two complementary arcs. Then

$$\mathbb{E}\big[\lambda_{B,W}^2\big] = \mathbb{P}\big[\omega', \omega'' \in \Lambda_B\big] = \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square(B,\mathcal{Q})\big];$$

that is, $\mathbb{E}\big[\lambda_{B,W}^2\big]$ is just the probability that the corresponding 4 arm event occurs in both $\omega'$ and $\omega''$. Lemma 2.1 gives

$$\mathbb{P}\big[x \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] = \mathbb{E}\Big[\mathbb{E}[f\,\chi_x \mid \mathcal{F}_{(W\cup\{x\})^c}]^2\Big].$$

Now, if $f$ is a monotone increasing function taking values in $\{-1,1\}$, then

$$\mathbb{E}[f\,\chi_x \mid \mathcal{F}_{\{x\}^c}] = 1_{\Lambda_{\{x\}}}, \tag{5.9}$$

and

$$\mathbb{E}[f\,\chi_x \mid \mathcal{F}_{(W\cup\{x\})^c}] = \mathbb{E}\Big[\mathbb{E}[f\,\chi_x \mid \mathcal{F}_{\{x\}^c}] \;\Big|\; \mathcal{F}_{W^c}\Big] = \mathbb{E}\big[1_{\Lambda_{\{x\}}} \mid \mathcal{F}_{W^c}\big] = \lambda_{x,W}\,.$$

Hence,

$$\mathbb{P}\big[x \in \mathscr{S},\ \mathscr{S} \cap W = \emptyset\big] = \mathbb{E}\big[\lambda_{x,W}^2\big] = \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square(x,\mathcal{Q})\big],$$

where $\mathscr{A}_\square(x,\mathcal{Q})$ has the obvious meaning. Likewise, since $W \cap B = \emptyset$, we have $\omega' = \omega''$ in $B$, and so,

$$\alpha_4(r) \asymp \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4(x,B)\big],$$

where $\mathscr{A}_4(x,B)$ is the 4-arm event (which does not pay attention to any distinguished arcs on $\partial B$). Hence (5.2) can be rewritten as

$$\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square(x,\mathcal{Q})\big] \ge c_1\,\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4(x,B)\big]\,\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square(B,\mathcal{Q})\big]. \tag{5.10}$$

To see that this is indeed a quasi-multiplicativity property, observe that if we take $W = \emptyset$ and replace the events with $\mathscr{A}_\square$ by the corresponding events with $\mathscr{A}_4$, then this essentially the same as the case $j = 4$ in the left inequality of (2.2).

It turns out that with a few extra twists, a proof which gives the quasi-multiplicativity estimates (2.2) generalizes to give (5.10). This will be explained in the next subsections.

*Remark* 5.5. Proposition 5.1 generalizes to the radial setting, in which we consider the event of a crossing from the origin to a large distance away. However, at present it does not generalize to the radial 2-arm event where a vacant crossing and an occupied crossing occur simultaneously. The only argument in the proof that does not generalize to the 2-arm event is (5.9), which is not true for non-monotone functions. Instead, we have

$$\mathbb{E}[f\,\chi_x \mid \mathcal{F}_{\{x\}^c}] = 1_{M_x^+} - 1_{M_x^-}, \tag{5.11}$$

where $M_x^+$ is the event that $x$ is monotonically pivotal (i.e., $f(\omega^+_{\{x\}}) = 1 = -f(\omega^-_{\{x\}})$) and $M_x^-$ is the event that $x$ is anti-monotonically pivotal. The problem with such functions is that for the first moment we would need to bound from below $\mathbb{E}\big[\mathbb{E}[1_{M_x^+} - 1_{M_x^-} \mid \mathcal{F}_{W^c}]^2\big]$. This expression is easily controlled from above by $\mathbb{E}\big[\lambda_{x,W}^2\big]$, but not from below due to cancellations between $M_x^+$ and $M_x^-$. These cancellations are far from being negligible, thus there is no hope to get $O(1)\mathbb{E}\big[\mathbb{E}[1_{M_x^+} - 1_{M_x^-} \mid \mathcal{F}_{W^c}]^2\big] \geq \mathbb{E}\big[\lambda_{x,W}^2\big]$ for general $W$. For instance, if $W = \{x\}^c$ and $f$ is an even function ($f(-\omega) = f(\omega)$) like the 2-arm indicator function for site percolation on the triangular grid, then $\mathbb{P}\big[\mathscr{S} = \{x\}\big] = \mathbb{E}\big[\mathbb{E}\big[1_{M_x^+} - 1_{M_x^-}\big]^2\big] = 0$. This “unfortunate” cancellation between events $M_x^+$ and $M_x^-$ is the reason of the breakdown of our methods for such events.

## 5.4 Quasi-multiplicativity for coupled configurations

Rather than proving specifically the inequality (5.10), we first address a related statement which is somewhat cleaner. In the following, $W$ is any fixed subset of $\mathcal{I}$, and $\omega'$, $\omega''$ are the above coupled configurations, which are independent in $W$ and agree on $\mathcal{I} \setminus W$. The annulus $B(0,R) \setminus B(0,r)$ will be denoted by $A(r,R)$. Let $j \in \mathbb{N}_+$ be either 1 or an even number and let $\mathscr{A}_j(r,R)$ denote the set of configurations $\omega$ that satisfy the alternating $j$-arm event in the annulus $A(r,R)$. Set

$$\beta_j^W(r,R) := \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_j(r,R)\big].$$

We will prove the following quasi-multiplicativity result:

**Proposition 5.6** (Quasi-multiplicativity)**.** *Let $j \in \mathbb{N}_+$ be either one or an even integer, and let $W \subset \mathcal{I}$. Then*

$$\beta_j^W(r_1,r_2)\,\beta_j^W(r_2,r_3) \le C_j\,\beta_j^W(r_1,r_3)$$

*holds for every $0 < r_1 < r_2 < r_3$ satisfying $r_2 \ge \bar{r}_j$, where $C_j$ and $\bar{r}_j$ are finite constants depending only on $j$ (and in particular, not on $W$).*

Note that the opposite inequality with $C_j = 1$ holds by independence on disjoint sets.

To prepare for the proof of the proposition, we first need to prove a few lemmas. The first observation is the following monotonicity property:

$$\beta_j^{W_2}(r,R) \le \beta_j^{W_1}(r,R) \qquad \text{if } W_1 \subset W_2\,. \tag{5.12}$$

Indeed, since

$$\beta_j^W(r,R) = \mathbb{E}\Big[\mathbb{P}[\omega \in \mathscr{A}_j(r,R) \mid \mathcal{F}_{W^c}]^2\Big],$$

the claimed monotonicity follows by the orthogonality property of martingale increments.

The case $j = 1$ in Proposition 5.6 easily follows from the Russo-Seymour-Welsh theorem and from the Harris-FKG inequality. In the following, we will restrict to the case $j = 4$, since the other even values of $j$ are essentially the same.

Let $\delta$ be some small positive constant, and let $r_0 > 0$. We say that $r > r_0$ is $\delta$-**good** if $\beta_4^W(r_0, 2\,r) \ge \delta \beta_4^W(r_0, r)$. Of course, this notion of good depends on $W, \delta$ and $r_0$.

**Lemma 5.7.** *Fix $r_0, \delta > 0$ and $W \subset \mathcal{I}$. Then there is a constant $\bar{r} = \bar{r}(\delta) > 0$, which depends only on $\delta$, such that if $r > r_0$ is $\delta$-good and $r \ge \bar{r}$, then for every $r' > r$*

$$\beta_4^{W \cup A(r,r')}(r_0, r') \ge c\,\beta_4^W(r_0, r)\,(r/r')^d,$$

*where $c = c(\delta) > 0$ depends only on $\delta$ and $d$ is a universal constant.*

The proof of this lemma will rely on Lemmas A.2 and A.3 from [SS05].

**Proof.** Assume that $r$ is $\delta$-good. Then $\beta_4^W(r_0, 2\,r) \geq \delta\,\beta_4^W(r_0, r)$. Set

$$X' := \mathbb{P}\big[\omega' \in \mathscr{A}_4(r_0, 2\,r) \bigm| \omega'_{B(0,r)}\big],$$
$$X'' := \mathbb{P}\big[\omega'' \in \mathscr{A}_4(r_0, 2\,r) \bigm| \omega''_{B(0,r)}\big].$$

Then

$$\begin{aligned}\beta_4(r_0, 2\,r) &= \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4(r_0, 2\,r)\big] \\ &= \mathbb{E}\Big[\mathbb{P}[\omega', \omega'' \in \mathscr{A}_4(r_0, 2\,r) \mid \omega'_{B(0,r)}, \omega''_{B(0,r)}]\Big].\end{aligned} \tag{5.13}$$

Now, since

$$\begin{aligned}&\mathbb{P}[\omega', \omega'' \in \mathscr{A}_4(r_0, 2\,r) \mid \omega'_{B(0,r)}, \omega''_{B(0,r)}] \\ &\qquad \leq \mathbb{P}[\omega' \in \mathscr{A}_4(r_0, 2\,r) \mid \omega'_{B(0,r)}, \omega''_{B(0,r)}] = X',\end{aligned}$$

and a similar relation holds with $X''$, we have

$$\mathbb{P}[\omega', \omega'' \in \mathscr{A}_4(r_0, 2\,r) \mid \omega'_{B(0,r)}, \omega''_{B(0,r)}] \leq X' \wedge X'' =: \tilde{X},$$

where $\tilde{X} = X' \wedge X''$ denotes the minimum of $X'$ and $X''$. Because $r$ is $\delta$-good, (5.13) now gives $\mathbb{E}\big[\tilde{X}\big] \geq \delta\,\beta_4^W(r_0, r)$. Since $\{\tilde{X} > 0\} \subset \{\omega', \omega'' \in \mathscr{A}_4(r_0, r)\}$, and the latter event has probability $\beta_4^W(r_0, r)$, this gives

$$\mathbb{E}\big[\tilde{X} \bigm| \omega', \omega'' \in \mathscr{A}_4(r_0, r)\big] \geq \delta\,. \tag{5.14}$$

Now let $\tilde{\omega}'$ and $\tilde{\omega}''$ be two percolation configurations that have the same law as $\omega$ that are independent of each other outside of $B(0, r)$ and inside $B(0, r)$ they satisfy $\tilde{\omega}' = \omega'$ and $\tilde{\omega}'' = \omega''$. Let $s'$ be the least distance between the endpoints on $\partial B(0, 2\,r)$ of any pair of disjoint interfaces of $\tilde{\omega}'$ that cross the annulus $A(r, 2\,r)$. (Take $s' = \infty$ if there is at most one such interface.) We claim that $r/s'$ is tight, in the following sense: for every $\epsilon > 0$ there is a constant $M = M_\epsilon$, depending only on $\epsilon$, such that $\mathbb{P}\big[r/s' > M\big] < \epsilon$. This is proved, for example, in [SS05, Lemma A.2]. We use this with $\epsilon = \delta/2$. Thus, we have

$$\mathbb{P}\big[s' < r/M\big] < \delta/2\,. \tag{5.15}$$

This property will be referred to below as the "separation of arms" phenomenon.

Assume now that $r \geq 100\,M =: \bar{r}$. Then when $s' \geq r/M$, we know that $s'$ is substantially larger than the lattice mesh. Observe that the distance between the endpoints on $\partial B(0, 2\,r)$ of any two disjoint interfaces of $\tilde\omega'$ that cross $A(r_0, 2\,r)$ is at least $s'$ (since every such interface also crosses $A(r, 2\,r)$), and if $\omega' \in \mathscr{A}_4(r_0, 2\,r)$ then there exist at least four such interfaces. Let $L_k$ denote the sector $\{\rho\, e^{i\theta} : \rho > 0, \theta \in [\pi\, k/4, \pi\,(k+1)/4]\}$. Let $\mathscr{Z}'$ denote the event that in $\omega'$ for each $k \in \{0, 2, 4, 6\}$ there is a crossing from $\partial B(0, r_0)$ to $\partial B(0, 8\,r)$ in $L_k \cup A(r_0, 4\,r)$, which is white when $k \in \{0, 4\}$ and black when $k \in \{2, 6\}$. By the proof of [SS05, Lemma A.3], we know (see Figure 5.1) that there is a constant $c_0 = c_0(M) > 0$ such that

$$\mathbb{P}\big[\mathscr{Z}' \;\big|\; \tilde\omega'_{B(0,r)},\, \tilde\omega' \in \mathscr{A}_4(r_0, 2\,r),\, s' \ge r/M\big] \ge c_0\,. \tag{5.16}$$

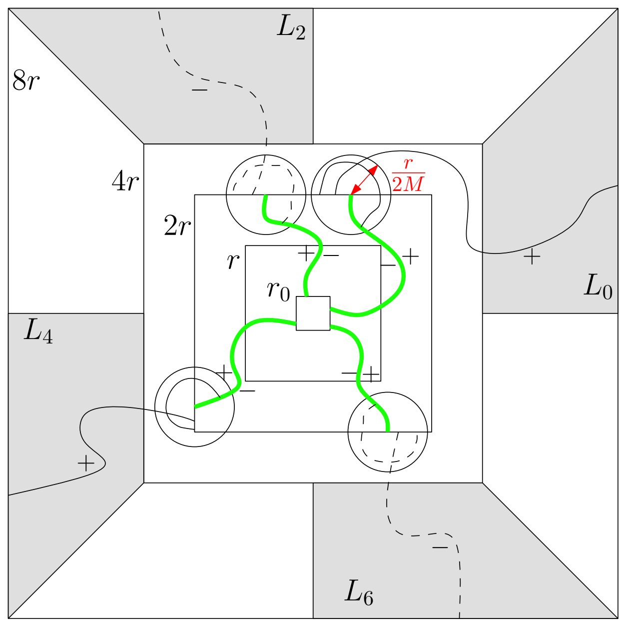



Figure 5.1: How to use "separation of arms" in order to get equation (5.16).

Note that $s'$ is independent from $\tilde\omega'_{B(0,r)} = \omega'_{B(0,r)}$. Therefore, (5.15) gives

$$\begin{aligned}
&\mathbb{P}\big[s' \ge r/M,\, \tilde\omega' \in \mathscr{A}_4(r_0, 2\,r) \;\big|\; \omega'_{B(0,r)}\big] \\
&\qquad \ge \mathbb{P}\big[\tilde\omega' \in \mathscr{A}_4(r_0, 2\,r) \;\big|\; \omega'_{B(0,r)}\big] - \mathbb{P}\big[s' < r/M \;\big|\; \omega'_{B(0,r)}\big] \\
&\qquad \ge \tilde X - \delta/2\,.
\end{aligned}$$

Together with (5.16), this shows that

$$\mathbb{P}\big[\mathscr{Z}' \;\big|\; \omega'_{B(0,r)}\big] \ge c_0\,(\tilde X - \delta/2)\,.$$

Now let $\mathscr{Z}''$ be defined as $\mathscr{Z}'$, but with $\tilde{\omega}''$ replacing $\tilde{\omega}'$. Since $\tilde{\omega}'$ and $\tilde{\omega}''$ are conditionally independent given $(\omega'_{B(0,r)}, \omega''_{B(0,r)})$, we get

$$\mathbb{P}\big[\mathscr{Z}',\, \mathscr{Z}'' \bigm| \omega'_{B(0,r)},\, \omega''_{B(0,r)}\big] \geq c_0^2 \left((\tilde{X} - \delta/2)_+\right)^2, \tag{5.17}$$

where $(x)_+$ denotes $x \vee 0$. Since $\left((\tilde{X} - \delta/2)_+\right)^2$ is a convex function of $\tilde{X}$, we get from Jensen's inequality and (5.14)

$$\mathbb{E}\big[\left((\tilde{X} - \delta/2)_+\right)^2 \bigm| \omega', \omega'' \in \mathscr{A}_4(r_0, r)\big] \geq \delta^2/4\,.$$

Thus, taking the expectation of both sides of (5.17) gives

$$\mathbb{P}\big[\mathscr{Z}',\, \mathscr{Z}''\big] \geq c_0^2\, (\delta^2/4)\, \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4(r_0, r)\big] = c_0^2\, (\delta^2/4)\, \beta_4^W(r_0, r)\,. \tag{5.18}$$

This clearly implies the statement of the lemma in the case $r' \leq 8\,r$. Assume therefore that $r' > 8\,r$. Note that $\mathscr{Z}' \cap \mathscr{Z}''$ is monotone increasing inside $(L_0 \cup L_4) \setminus B(0, 6\,r)$ and monotone decreasing in $(L_2 \cup L_6) \setminus B(0, 6\,r)$. Hence, it is positively correlated with the event $\tilde{\mathscr{Z}}$ that for each of $\tilde{\omega}'$ and $\tilde{\omega}''$ there are white paths separating $\partial B(0, 6\,r)$ from $\partial B(0, 8\,r)$ in each of $L_0$ and $L_4$, and similar black paths in $L_2$ and $L_6$, and moreover, there are black paths in each of $L_2$ and $L_6$ joining $\partial B(0, 6\,r)$ and $\partial B(0, r')$ and white paths in each of $L_0$ and $L_4$ joining $\partial B(0, 6\,r)$ and $\partial B(0, r')$. By the Russo-Seymour-Welsh theorem (see Figure 5.2), $\mathbb{P}\big[\tilde{\mathscr{Z}}\big] \geq c_1\, (r/r')^d$ for some absolute constants $c_1 > 0$ and $d < \infty$.

Taking $c := c_1\, c_0^2\, \delta^2/4$, we obtain $\mathbb{P}\big[\mathscr{Z}',\, \mathscr{Z}'',\, \tilde{\mathscr{Z}}\big] \geq c\, \beta_4^W(r_0, r)\, (r/r')^d$. The lemma follows, since $\mathscr{Z}' \cap \mathscr{Z}'' \cap \tilde{\mathscr{Z}} \subset \{\tilde{\omega}', \tilde{\omega}'' \in \mathscr{A}_4(r_0, r')\}$. □

**Lemma 5.8.** *There are absolute constants $\delta_0 > 0$ and $\bar{R} > 0$ such that*

$$\beta_4^W(r_0, 2\,\rho) \geq \delta_0\, \beta_4^W(r_0, \rho) \tag{5.19}$$

*holds if $0 < r_0 \leq \rho$ and $\rho \geq \bar{R}$ and*

$$\beta_4^W(\rho/2, r_0) \geq \delta_0\, \beta_4^W(\rho, r_0) \tag{5.20}$$

*holds if $\bar{R} \leq \rho < r_0$.*

**Proof.** We start by proving the first claim. Let $\bar{r}, d$ and $c(\delta)$ be as in Lemma 5.7. Let $\delta \in (0, 2^{-d-1})$. Let $r \geq \bar{r} \vee r_0$, and assume for now that $r$ is $\delta$-good. Then by Lemma 5.7 and the monotonicity property (5.12), we have

$$\beta_4^W(r_0, r') \geq c(\delta)\, \beta_4^W(r_0, r)\, (r/r')^d, \tag{5.21}$$

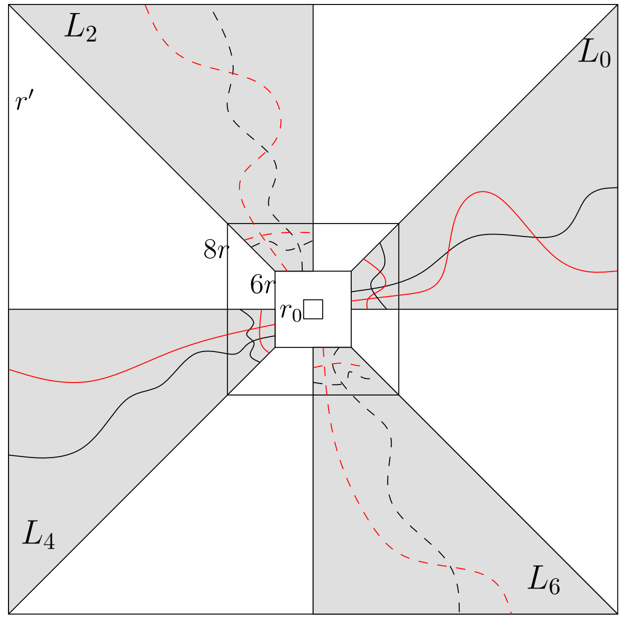



Figure 5.2: A realization of the event $\tilde{\mathcal{Z}}$. The black color corresponds to arms in $\tilde{\omega}'$, while the red color corresponds to arms in $\tilde{\omega}''$.

for every $r' \geq r$. Set $\rho_k := 2^k\, r$, and let $\bar{k} := \inf\{k \in \mathbb{N}_+ : \rho_k \text{ is } \delta\text{-good}\}$, with $\bar{k} := \infty$ if this set is empty. If $m \in \mathbb{N}_+$ and $m \leq \bar{k}$, then by the definition of $\bar{k}$ and by (5.21) with $r' := \rho_m$, we have

$$\begin{aligned}\beta_4^W(r_0, \rho_1)\, \delta^{m-1} &> \beta_4^W(r_0, \rho_m)\\ &\geq c(\delta)\, \beta_4^W(r_0, r)\, 2^{-dm}\\ &\geq c(\delta)\, \beta_4^W(r_0, \rho_1)\, 2^{-dm}.\end{aligned}$$

Now, since $\delta < 2^{-d-1}$, the above gives $2^{-(d+1)(m-1)} \geq c(\delta)\, 2^{-dm}$, which implies $2^{d+1} \geq c(\delta)\, 2^m$. Hence, $\bar{k}$ is bounded by some finite constant depending only on $\delta$ (recall that $d$ is a universal constant). We may conclude that $\delta$-good radii appear in scales with bounded gaps, since the same argument may be applied with $r$ replaced by $\rho_{\bar{k}}$. If $\rho$ is in the range $(r, \rho_{\bar{k}})$, then we have the estimate

$$\frac{\beta_4^W(r_0, 2\,\rho)}{\beta_4^W(r_0, \rho)} \geq \frac{\beta_4^W(r_0, 2\,\rho_{\bar{k}})}{\beta_4^W(r_0, r)} \overset{(5.21)}{\geq} c(\delta)\, 2^{-d(\bar{k}+1)},$$

which means that $\rho$ is $\delta_0$-good with $\delta_0 := c(\delta)\, 2^{-d(\bar{k}+1)}$. The same statement would apply to any $\rho \geq r$, as above $r$ the $\delta$-good radii appear in scales with bounded gaps.

The proof of (5.19) is nearly complete. It only remains to be shown that there are constants $\delta, \bar{R} > 0$ that do not depend on $r_0$ such that $\bar{R} \vee r_0$ is $\delta$-good (recall that the definition of $\delta$-good in Lemma 5.7 above depends on $r_0$). (Then we can start the above argument with $r := \bar{R} \vee r_0$.) Clearly, $\beta_4^W(r, 2\,r)$ is bounded from below once $r \geq R_0$, where $R_0$ is some absolute constant. We take $\bar{R} > R_0$, and hence the case $r_0 \geq \bar{R}$ is covered. It is easy to take care of the case $r_0 < \bar{R}$ by choosing $\delta$ sufficiently small, since this reduces to finitely many possible annuli, due to the lattice discretization.

To prove (5.20), we follow a similar argument but with annuli growing towards 0 rather than towards infinity. For this, a corresponding analogue of Lemma 5.7 is needed. Since the proofs in this case are essentially the same, they are omitted. ☐

**Proof of Proposition 5.6.** As remarked above, we only prove the case $j = 4$, since $j = 1$ is very easy and the proof for the case $j = 4$ applies with no essential changes to all even $j$.

If $r_2 \leq 4\,r_1$ or $r_3 \leq 4\,r_2$, then the claim follows from Lemma 5.8. Hence, assume that $r_1 < r_2/4$ and $4\,r_2 < r_3$. By the monotonicity property (5.12), it suffices to prove

$$O(1)\,\beta_4^{W \cup A(r_2/4, 4\,r_2)}(r_1, r_3) \geq \beta_4^W(r_1, r_2/4)\,\beta_4^W(4\,r_2, r_3)\,.$$

This follows from the proof of Lemma 5.7: we just need to apply the same argument twice, once going outwards from 0 and using (5.19) with $r_0 := r_1$ and $\rho := r_2/4$ to verify that $r_2/4$ is $\delta_0$-good, and once going inwards towards 0 and using (5.20) with $r_0 := r_3$ and $\rho := 4\,r_2$. The easy details are left to the reader. ☐

Although this will not be needed in the following, we note that the following generalization of Proposition 5.6 to arbitrary sequences of crossings holds. (This can be proved by combining the above arguments with the proof of [SS05, Proposition A.5].)

**Proposition 5.9.** *Let $j \in \mathbb{N}_+$ and fix a color sequence $X \in \{\text{black}, \text{white}\}^j$. For any set $W \subset \mathcal{I}$, the probabilities for the existence in both coupled configurations $\omega'$ and $\omega''$, of $j$ crossings whose colors match this sequence in counterclockwise order satisfy the inequalities in Proposition 5.6.*

## 5.5 Proof of first moment estimate

**Proof of Proposition 5.2.** As remarked in Subsection 5.3, the proof of the first moment estimate reduces to proving (5.10). We will now explain how the proof of Proposition 5.6 needs to be adapted to give (5.10).

Fix some $x \in \mathcal{I}$ that is relevant for the left-right crossing in $\mathcal{Q} = [0, R]^2$. Let $x^+$ be the closest point to $x$ on $\partial\mathcal{Q}$ and let $x^{++}$ be the closest point to $x$ among the four corners of $\mathcal{Q}$. Set $R^+ := \|x - x^+\|_\infty$ and $R^{++} := \|x - x^{++}\|$. We now define

$$B_r := \begin{cases} B(x, r), & \bar{r} \le r \le R^+/8, \\ B(x^+, r), & 8\,R^+ \le r \le R^{++}/8, \\ B(x^{++}, r), & 8\,R^{++} \le r \le R/8, \end{cases}$$

where $\bar{r} > 1$ is some fixed constant, and let $\mathcal{R}$ denote the set of $r$ for which $B_r$ is defined; that is, $\mathcal{R} := [\bar{r}, R^+/8] \cup [8\,R^+, R^{++}/8] \cup [8\,R^{++}, R/8]$. Given any $r \in \mathcal{R}$, define $\tilde{r} := \inf(\mathcal{R} \cap [2\,r, R])$. Then we say that $r$ is $\delta$-good if $\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4(x, B_{\tilde{r}})\big] \ge \delta\,\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4(x, B_r)\big]$. The proof that there is a universal constant $\delta$ such that every $r \in \mathcal{R}$ satisfying $2\,r \le \sup \mathcal{R}$ is $\delta$-good proceeds like the proof of (5.19) with a few minor changes. The fact that some of the boxes considered are not concentric with each other is of no consequence. The only significant modification needed is that in the argument corresponding to Lemma 5.7, if $B_r \cap \partial\mathcal{Q} \ne \emptyset$, then the interfaces considered are in the intersection of the corresponding annulus and $\mathcal{Q}$ and the definition of $s'$ needs to be modified. In the adapted proof, $s'$ is defined as the least distance between any two distinct points that are either endpoints on $\partial B_{\tilde{r}}$ of the interfaces or points in the intersection $\partial B_{\tilde{r}} \cap \partial\mathcal{Q}$. The remaining details are left to the reader, as is the similar proof that $\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\Box(B_{\check{r}}, \mathcal{Q})\big] \ge \delta\,\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\Box(B_r, \mathcal{Q})\big]$ when $\check{r} = \sup(\mathcal{R} \cap [\bar{r}, r/2]) \ge \bar{r}$. The proof of (5.10) then follows as in the proof of Proposition 5.6. □

*Remark* 5.10. In the applications, we will need to apply Proposition 5.1 to a set of boxes $B$ that form a grid covering $\mathcal{Q}$. Since we need each $B'$ to be contained in $\mathcal{Q}$, there is some care needed in placing the grid of boxes. In fact, for some radii $r$, this is actually impossible. There are several alternative solutions to this problem. The easiest solution is to restrict $r$ to the set of radii that admit grids of boxes that cover $\mathcal{Q}$ well. This happens, for example, when $r$ divides $R$. However, this solution has the drawback of not being easily adaptable to other settings, for example, to the setting in which $\mathcal{Q}$ is a rectangle or some smooth perturbation of a rectangle. For this reason, we now describe a somewhat different solution. Let $V$ be a maximal set of points in $\mathcal{Q}$ such that the distance between any pair of distinct points in $V$ is at least $r$ and the distance between any $v \in V$ to the closest point on $\partial\mathcal{Q}$ is at least $r$. Consider the intrinsic metric $d_{\mathcal{Q}}$ on $\mathcal{Q}$, where $d_{\mathcal{Q}}(x, x')$ is the infimum length of any curve in $\mathcal{Q}$ connecting $x$ and $x'$. Let $(T_v : v \in V)$ denote the Voronoi tiling associated with $V$ and with this metric, and let $B_v$ denote the union of the lattice tiles meeting $T_v$. If we assume that $\mathcal{Q}$ is "reasonably nice", then the maximal $d_{\mathcal{Q}}$-diameter of any $B_v$ is $O(r)$. (Of course, we assume $r > 1$.) This will be the case, for example, when $\mathcal{Q}$ is a rectangle whose smaller sidelength is larger than $2\,r$, or more generally, if $\mathcal{Q} = R\mathcal{Q}_0$ for a piecewise smooth quad $\mathcal{Q}_0$ and $R > c(\mathcal{Q}_0)r$. Now observe that the disk of radius $r/4$ around each $v \in V$ is contained in the interior of $T_v$ and is bounded away from $\partial\mathcal{Q}$. We may define $B'_v$ as the union of the lattice tiles that meet this disk. The statement and proof of Proposition 5.1 hold with $B_v$ and $B'_v$ replacing $B$ and $B'$, though the constants will depend on the upper bound we have for $\operatorname{diam}(B_v)/r$.

## 5.6 A local result for general quads

In this subsection, we prove the following local result, which is a key step in estimates for noise sensitivity in the case of general quads.

**Proposition 5.11.** *Let $\mathcal{Q} \subset \mathbb{R}^2$ be some quad, and let $U$ be an open set whose closure is contained in the interior of $\mathcal{Q}$. For $R > 0$, let $\mathscr{S} := \mathscr{S}_{f_{R\mathcal{Q}}}$ be the spectral sample of $f_{R\mathcal{Q}}$, the $\pm 1$ indicator function for the crossing event in $R\mathcal{Q}$. Then, there is a constant $\bar{r} = \bar{r}(U, \mathcal{Q})$ such that for any box $B \subset RU$ of radius $r \in [\bar{r}, R\operatorname{diam}(U)]$ and any set $W$ with $W \cap B = \emptyset$, we have*

$$\mathbb{P}\big[\mathscr{S}_{f_{R\mathcal{Q}}} \cap B' \cap \mathcal{Z} \neq \emptyset \;\big|\; \mathscr{S}_{f_{R\mathcal{Q}}} \cap B \neq \emptyset,\ \mathscr{S}_{f_{R\mathcal{Q}}} \cap W = \emptyset\big] \geq a(U, \mathcal{Q})\,,$$

*where $B'$ is concentric with $B$ and has radius $r/3$, the random set $\mathcal{Z}$ is defined as in Proposition 5.1, and $a(U, \mathcal{Q}) > 0$ is a constant that depends only on $U$ and $\mathcal{Q}$.*

**Proof.** Here, the main new issue to deal with is that the quad $\mathcal{Q}$ is general; but, in contrast to the situation in Proposition 5.2, the box $B$ is bounded away from $\partial\mathcal{Q}$, which simplifies parts of the proof.

The second moment estimate (Proposition 5.3) applies in the present setup. We now prove the corresponding first moment estimate.

In the following discussion, the constants are allowed to depend on $U$ and $\mathcal{Q}$. We start by proving the analogue of (5.20). Let $K$ be a compact set contained in the interior of $\mathcal{Q}$ and containing the closure of $U$ in its interior. Let $M$ denote the set of all squares $S \subset K$ that intersect $\overline{U}$ while the concentric square of twice the radius is not contained in the interior of $K$. Then $M$ is a compact set of squares in the natural topology, and the radius of the squares in $M$ is bounded away from zero. Fix some $S \in M$. Let $B(R\,S)$ denote the union of the lattice tiles that meet $R\,S$. A simple application of the Russo-Seymour-Welsh theorem shows that $\liminf_{R\to\infty} \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square\big(B(R\,S), R\,\mathcal{Q}\big)\big] > 0$. Moreover, the same estimate holds in a neighborhood of $S$; that is, there is a set $V \subset M$ that contains $S$ and is open in the topology of $M$, and there is a constant $R_0 = R_0(V, U, \mathcal{Q}) > 0$ such that

$$\inf_{R\geq R_0} \inf_{S'\in V} \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square\big(B(R\,S'), R\,\mathcal{Q}\big)\big] > 0\,.$$

Since $M$ is compact, this cover of $M$ by open subsets $V$ has a finite subcover, and therefore there is some constant $R_1 = R_1(U, \mathcal{Q})$ such that

$$\inf_{R\geq R_1} \inf_{S\in M} \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square\big(B(R\,S), R\,\mathcal{Q}\big)\big] > 0\,.$$

It is clear that there is some constant $b > 1$ such that for every $S \in M$ the concentric square whose radius is $b$ times the radius of $S$ is still contained in $\mathcal{Q}$. The above then shows that there is a constant $\delta > 0$ such that for all $R \geq R_1$ and all $S \in M$,

$$\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square\big(B(R\,S), R\,\mathcal{Q}\big)\big] \geq \delta\, \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_\square\big(B(R\,S_b), R\,\mathcal{Q}\big)\big]\,, \tag{5.22}$$

where $S_b$ denotes the square concentric with $S$ whose radius is $b$ times the radius of $S$. Let $\hat{M}$ denote the set of squares that are contained in and concentric with some square in $M$. Once we have (5.22) for all $S \in M$, we can conclude as in the proof of (5.20) in Lemma 5.8 that the same holds with possibly a different constant $\delta$ for every $S \in \hat{M}$ such that $\operatorname{diam}(R\,S) \geq \bar{r}$, for some constant $\bar{r} > 0$. For this, the powers of 2 that were used in the proof of (5.19) and (5.20) (for example, for the definition of the notion of "good") need to be replaced by powers of $b$, but this is of little consequence. We also need here a version of Lemma 5.7 for the events $\mathscr{A}_\square(B(RS_b), R\mathcal{Q})$, but that can be proved the same way as the original version, using [SS05, Lemmas A.2, A.3] and (5.22). Finally, the restriction that $R \geq R_1$ may be avoided by taking $\bar{r}$ sufficiently large. Thus, the analogue of (5.20) is established.

Based on (5.19) and the above analogue of (5.20), we obtain the analogue of (5.10) for the current setup, yielding the first moment estimate. (Note that we do not need to adapt the outward bound (5.19) to this local result.) The proof of the current proposition from the first and second moment estimates follows as in the proof of Proposition 5.1. □

## 5.7 The radial case

For the study of the set of exceptional times for dynamical percolation, we will need some concentration for the spectral samples of the "radial" indicator function. For this purpose, the following analog of Proposition 5.1 for the radial setting will be useful.

**Proposition 5.12.** *Let $f = f_R$ be the* 0-1 *indicator function of the existence of a white crossing between the two boundary components of the annulus $[-R,R]^2 \setminus [-1,1]^2$, and let $\mathscr{S} = \mathscr{S}_f$ be its spectral sample with law $\mathbb{P}\big[\mathscr{S} = S\big] = \widehat{f}(S)^2/\|f\|^2$. Also let $W \subset \mathcal{I}$. Let $B$ be a box of some radius $r$ that does not intersect $W$ and let $B'$ be the concentric box with radius $r/3$. Suppose that $B' \subset [-R,R]^2$ and $B \cap [-4\,r, 4\,r]^2 = \emptyset$. We also assume that $r \geq \bar{r}$, where $\bar{r} > 0$ is some universal constant. Then*

$$\mathbb{P}\big[\mathscr{S} \cap B' \cap \mathcal{Z} \neq \emptyset \;\big|\; \mathscr{S} \cap B \neq \emptyset,\, \mathscr{S} \cap W = \emptyset\big] > a,$$

*where $\mathcal{Z}$ is as in Proposition 5.1 and $a > 0$ is a universal constant.*

**Proof.** The proof will be similar to the above proofs. For this reason, we will be brief and leave many details to the reader. Let $z$ be the center of the box $B$, and set $r_1 := |z| > 4r$. Assume first that $r_1 < R/3$. We then consider the three annuli $B(0, r_1/3) \setminus B(0,1)$, $B(0,R) \setminus B(0, 3\,r_1)$ and $B(z, r_1/3) \setminus B$. In order for $\Lambda_B$ to hold, it is necessary that the 1-arm event occurs in the first two annuli and that the 4-arm event occurs in the third annulus. Thus,

$$\begin{aligned}\mathbb{P}\big[B \cap \mathscr{S} \neq \emptyset = \mathscr{S} \cap W\big] &\leq 4\,\mathbb{E}\big[\lambda_{B,W}^2\big]\\ &\leq 4\,\mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_1\big(B(0,1), B(0, r_1/3)\big)\big] \times\\ &\quad\ \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_1\big(B(0, 2\,r_1), B(0,R)\big)\big] \times\\ &\quad\ \mathbb{P}\big[\omega', \omega'' \in \mathscr{A}_4\big(B, B(z, r_1/3)\big)\big]. \end{aligned} \tag{5.23}$$

Now, using quasi-multiplicativity for coupled configurations and the separation of arms as before, for $x \in B'$ we have the first moment estimate

$$\begin{aligned}\mathbb{P}\big[x \in \mathscr{S}, \mathscr{S} \cap W = \emptyset\big] &\geq c_1\, \beta_1^W(1, r_1/3)\, \beta_4^W(x, r_1/3)\, \beta_1^W(2r_1, R) \\ &\geq c_2\, \beta_1^W(1, r_1/3)\, \alpha_4(r)\, \beta_4^W(r, r_1/3) \beta_1^W(2r_1, R) \\ &\geq c_3\, \alpha_4(r)\, \mathbb{E}\big[\lambda_{B,W}^2\big], \end{aligned} \tag{5.24}$$

where (5.23) is used for the last step. One can easily prove the analogous second moment estimate, and the claim now follows as in the proof of Proposition 5.1.

Suppose now that $r_1 > 2\,R/3$. In this case, we need to consider a different system of annuli. Let $d$ denote the distance from $B$ to $\partial B(0, R)$ and let $z'$ denote a closest point to $B$ on $\partial B(0, R)$. In the annulus $B(0, R/3) \setminus B(0, 1)$ we consider the 1-arm event, in the annulus $B(z, r + d/2) \setminus B$ we consider the 4-arm event, and in the intersection of $B(0, R)$ with $B(z', R/3) \setminus B(z', 5\,r + d)$ (assuming that this is nonempty), we consider the 3-arm event between $\partial B(z', R/3)$ and $\partial B(z', 5\,r + d)$. Again, the claim follows.

In the intermediate case $R/3 \leq r_1 \leq 2\,R/3$, we need to consider the 1-arm event in $B(0, R/6) \setminus B(0, 1)$ and the 4-arm event in $B(z, R/6) \setminus B$, and the claim likewise follows. □

# 6 A large deviation result

In order to deduce Theorem 1.1 from the results of Sections 5 and 4, we will need the following general result.

**Proposition 6.1.** *Let $n \in \mathbb{N}_+$, let $x$ and $y$ be random variables in $\{0, 1\}^n$, and set $X := \sum_{j=1}^n x_j$ and $Y := \sum_{j=1}^n y_j$. Suppose that a.s. $y_i \leq x_i$ for each $i \in [n]$ and that there is a constant $a \in (0, 1]$ such that for each $j \in [n]$ and every $I \subset [n] \setminus \{j\}$ we have*

$$\mathbb{P}\big[y_j = 1 \;\big|\; y_i = 0\, \forall i \in I\big] \geq a\, \mathbb{P}\big[x_j = 1 \;\big|\; y_i = 0\, \forall i \in I\big]\,. \tag{6.1}$$

*Then*

$$\mathbb{P}\big[Y = 0 \;\big|\; X > 0\big] \leq a^{-1}\, \mathbb{E}\big[e^{-aX/e} \;\big|\; X > 0\big]. \tag{6.2}$$

For completeness, we will also show below that

$$\mathbb{P}\big[Y \leq t\big] \leq \mathbb{P}\big[X < (e/a)\,(t + s)\big] + \big(e^{t-1}/s\big)\, \mathbb{E}\big[e^{-a\,X/e}\big] \tag{6.3}$$

holds for every $t \geq 0$ and $s > 0$. However, we do not have an application for this inequality.

**Proof.** We may write our assumption (6.1) as follows

$$\mathbb{P}\big[y_j = 1,\, y_i = 0\,\forall i \in I\big] \geq a\,\mathbb{P}\big[x_j = 1,\, y_i = 0\,\forall i \in I,\, j \notin I\big], \tag{6.4}$$

where the restriction $j \notin I$ is no longer necessary. This gives us many inequalities, which we will average out in a useful manner. Fix $\lambda \in (0,1)$. Now multiply (6.4) by $\lambda^{n-|I|}\,(1-\lambda)^{|I|}$ and sum over all choices of $j \in [n]$ and $I \subset [n]$, to get

$$\mathbb{E}\big[Y\,\lambda^Y\big] \geq a\,\mathbb{E}\big[X\,\lambda^{Y+1}\big].$$

This may be rewritten as $\mathbb{E}[Z] \geq 0$, where $Z := (Y - a\,\lambda\,X)\,\lambda^Y$. At this point, we choose $\lambda := e^{-1}$. In order to bound $Z$ from above by a function of $X$ only, we maximize $Z$ over $Y$, and get the bound $Z \leq \exp(-1 - a\,X/e)$. On $X = 0$, we also have $Y = 0$ and $Z = 0$, while on $Y = 0 < X$, $Z \leq -a\,e^{-1}$ holds. Therefore, $\mathbb{E}[Z] \geq 0$ gives

$$a\,e^{-1}\,\mathbb{P}\big[Y = 0 < X\big] \leq \mathbb{E}\big[1_{X>0}\,\exp(-1 - a\,X/e)\big].$$

Dividing by $a\,e^{-1}\,\mathbb{P}\big[X > 0\big]$, we obtain (6.2). □

We now prove (6.3). Set $r := (e/a)\,(t+s)$, $Z_+ := \max(Z,0)$ and $Z_- := Z - Z_+$. Note that on the event $\{X \geq r, Y \leq t\}$ we have $Z \leq -s\,e^{-t}$. Hence,

$$\mathbb{E}\big[Z_-\big] \leq -s\,e^{-t}\,\mathbb{P}\big[X \geq r,\, Y \leq t\big] \leq -s\,e^{-t}\,\big(\mathbb{P}[Y \leq t] - \mathbb{P}[X < r]\big).$$

On the other hand, $\mathbb{E}\big[Z_+\big] \leq \mathbb{E}\big[\exp(-1 - a\,X/e)\big]$. Since $0 \leq \mathbb{E}[Z] = \mathbb{E}\big[Z_+\big] + \mathbb{E}\big[Z_-\big]$, (6.3) follows.

# 7 The lower tail of the spectrum

In this section, we prove Theorem 1.1 and a few related results.

## 7.1 Local version

As in Section 4, we start with a version which avoids the issues involving the boundary.

**Theorem 7.1.** *Consider some quad $\mathcal{Q}$, and let $\mathscr{S} = \mathscr{S}_{f_{R\mathcal{Q}}}$ be the spectral sample of $f_{R\mathcal{Q}}$, the $\pm 1$ indicator function for the crossing event in $R\mathcal{Q}$. Let*

*$U \subset \mathcal{Q}$ be open, and let $U' \subset \overline{U'} \subset U$. Then, for some constants $\bar r = \bar r(U', U, \mathcal{Q}) > 0$ and $q(U', U, \mathcal{Q}) > 0$, for any $r \in [\bar r, R\,\mathrm{diam}(U)]$,*

$$\begin{aligned}\mathbb{P}\big[0 < |\mathscr{S}_{f_{R\mathcal{Q}}} \cap RU| \le r^2\,\alpha_4(r),\, \mathscr{S}_{f_{R\mathcal{Q}}} \cap RU \subset RU'\big]& \\ \le q(U', U, \mathcal{Q})\,\frac{R^2\,\alpha_4(R)^2}{r^2\,\alpha_4(r)^2}\,.& \qquad (7.1)\end{aligned}$$

**Proof.** Let the distance between $U'$ and the complement of $U$ be $\delta > 0$. Then, with no loss of generality, we may assume that $r \le \delta R/10$. (Otherwise, $r/R$ remains bounded away from 0, so, by choosing $q(U', U, \mathcal{Q})$ large enough compared to $\delta$, the upper bound in (7.1) becomes larger than 1, and we are done.) Consider the tiling of the plane by $r \times r$ squares given by the grid $r\,\mathbb{Z}^2$, recall from the end of Section 2.1 that each square gives a box that together form a tiling of the plane, and let $\{B_1, B_2, \dots, B_n\}$ be the set of those boxes that intersect $RU'$. Let $U'' \subset U$ such that $RU'' \supset \bigcup_{j=1}^n B_j$ but the distance of $U''$ to the complement of $U$ is at least $\delta/2$. Let $\mathcal{Z}$ be a subset of $\mathcal{I} \cap RU''$, where each bit $i \in RU \cap \mathcal{I}$ is in $\mathcal{Z}$ with probability $1/(r^2\alpha_4(r))$, independently from each other and from $\mathscr{S}$. Let $y_j$ be the indicator function of the event

$$\mathscr{S} \cap B_j \cap \mathcal{Z} \ne \emptyset\,,$$

and let $x_j$ be the indicator function of the event $\mathscr{S} \cap B_j \ne \emptyset$. Let $\tilde{\mathbb{P}}$ denote the law of $\mathscr{S}$ conditioned on the event $\mathscr{S} \cap (RU \setminus RU'') = \emptyset$, and let $\tilde{\mathbb{E}}$ denote the corresponding expectation operator. For every $I \subset \{1, \dots, n\}$ and every $j \in \{1, \dots, n\} \setminus I$, we get by Proposition 5.11 that

$$\tilde{\mathbb{P}}\big[y_j = 1 \,\big|\, y_i = 0\,\forall i \in I\big] \ge a\,\tilde{\mathbb{P}}\big[x_j = 1 \,\big|\, y_i = 0\,\forall i \in I\big]\,,$$

for some constant $a = a(U'', \mathcal{Q}) > 0$. Therefore, the large deviation result (Proposition 6.1) gives

$$\tilde{\mathbb{P}}\big[\mathscr{S} \cap \mathcal{Z} = \emptyset \ne \mathscr{S} \cap RU\big] \le a^{-1}\,\tilde{\mathbb{E}}\big[e^{-aX/e}\,1_{X>0}\big],$$

where $X := \big|\{j : \mathscr{S} \cap B_j \ne \emptyset\}\big|$. We may rewrite this as

$$\begin{aligned}\mathbb{P}\big[\mathscr{S} \cap \mathcal{Z} = \emptyset \ne \mathscr{S} \cap RU \subset RU''\big]& \\ \le a^{-1}\sum_{k=1}^{\infty} e^{-ak/e}\,\mathbb{P}\big[X = k,\, \mathscr{S} \cap RU \subset RU''\big].&\end{aligned}$$

We estimate the terms $\mathbb{P}\big[X = k,\, \mathscr{S} \cap RU \subset RU''\big]$ using Proposition 4.2 and get the bound

$$\begin{aligned}\mathbb{P}\big[\mathscr{S} \cap \mathscr{Z} = \emptyset \neq \mathscr{S} \cap RU \subset RU''\big] &\le O(1) \sum_{k=1}^{\infty} e^{-ak/e}\, g(k)\, \gamma_r(R) \\ &= O(1)\, \gamma_r(R)\,,\end{aligned} \tag{7.2}$$

where $g$ is as defined in the proposition and the constants implied by the $O(1)$ terms may depend on $(U', U, \mathscr{Q})$.

Now, by the choice of $\mathscr{Z}$, for $|\mathscr{S} \cap RU''| \le r^2 \alpha_4(r)$ we have

$$\mathbb{P}\big[\mathscr{S} \cap \mathscr{Z} = \emptyset \bigm| \mathscr{S}\big] = \left(1 - \frac{1}{r^2 \alpha_4(r)}\right)^{|\mathscr{S} \cap RU''|} \ge c$$

for some absolute constant $c > 0$, and hence

$$\begin{aligned}c\, \mathbb{P}\big[|\mathscr{S} \cap RU''| \le r^2 \alpha_4(r),\, \emptyset \neq \mathscr{S} \cap RU \subset RU''\big] \\ \le \mathbb{P}\big[\mathscr{S} \cap \mathscr{Z} = \emptyset \neq \mathscr{S} \cap RU \subset RU''\big] \\ \le O(1)\, \gamma_r(R) \qquad \text{by (7.2).}\end{aligned}$$

Since $RU' \subset RU''$, this implies (7.1). □

## 7.2 Square version

We prepare for the proof of Theorem 1.1 by first showing that

$$\mathbb{E}|\mathscr{S}_{f_R}| \asymp R^2\, \alpha_4(R)\,. \tag{7.3}$$

Note that this also implies (1.5) for the triangular lattice. First, the lower bound on $\mathbb{E}|\mathscr{S}_{f_R}|$ follows immediately from Lemma 3.1. For the upper bound, we will need to consider the half-plane 3-arm events and the quarter-plane 2-arm events that were discussed in Section 4.3. Let $\mathscr{Q} = [0, R]^2$, $f = f_{\mathscr{Q}}$ and $\mathscr{S} = \mathscr{S}_f$. Let $x \in \mathcal{I}$ be an input bit of $f$. If $x$ is at distance $r_0$ from the closest edge of $[0, R]^2$, and at distance $r_1$ from the closest corner, then by (3.6) and quasi-multiplicativity, we have

$$\mathbb{P}\big[x \in \mathscr{S}\big] = \alpha_4(x, \mathscr{Q}) \asymp \alpha_4(r_0)\, \alpha_3^+(r_0, r_1)\, \alpha_2^{++}(r_1, R)\,.$$

Now observe that $\alpha_3^+(r_0, r_1) \le O(1)\, \alpha_4(r_0, r_1)$ follows from (4.12), and (2.6). Thus, $\alpha_4(r_0)\, \alpha_3^+(r_0, r_1) \le O(1)\, \alpha_4(r_1)$. Moreover, $\alpha_2^{++}(r_1, R) \le O(r_1/R)$, by (4.13) and (4.12). Since the number of $x \in \mathcal{I}$ with $r_1 \in [2^j, 2^{j+1})$ is $O(2^{2j})$, we get

$$\begin{aligned}\mathbb{E}|\mathscr{S}| &= \sum_{x\in\mathcal{I}} \mathbb{P}\big[x \in \mathscr{S}\big] \le \sum_{j=0}^{\lceil \log_2 R\rceil} O(2^{2j})\,\alpha_4(2^j)\,\frac{2^j}{R} \\ &= \frac{\alpha_4(R)}{R}\sum_{j=0}^{\lceil \log_2 R\rceil} \frac{O(2^{3j})}{\alpha_4(2^j,R)} \overset{(2.6)}{\le} \frac{\alpha_4(R)}{R}\sum_{j=0}^{\lceil \log_2 R\rceil} \frac{O(2^{3j})}{(2^j/R)^2} = O(R^2)\,\alpha_4(R)\,.\end{aligned}$$

Thus, we get (7.3).

**Proof of Theorem 1.1.** The proof of

$$\mathbb{P}\big[0 < |\mathscr{S}| < r^2\,\alpha_4(r)\big] \le O(1)\,\gamma_r(R)\,, \tag{7.4}$$

for $1 \le r \le R$, is very similar to the proof of Theorem 7.1, with some small modifications, which we now discuss. Note that the set of $x \in \mathcal{I}$ that are relevant for $f$ are all within distance at most 2 from $[0,R]^2$. Note that we may assume, without loss of generality, that $r \ge \bar{r}$ for some absolute constant $\bar{r}$, and that $(R+4)/r \in \mathbb{N}_+$. Let $\{B_1, B_2, \dots, B_n\}$ be the set of boxes corresponding to the tiling of $[-2, R+2]^2$ by $r \times r$ squares. In the proof of Theorem 7.1 we are now allowed to take $RU = RU' = RU'' = [-2, R+2]^2$, by replacing the appeal to Propositions 5.11 and 4.2 with an appeal to Propositions 5.1 and 4.1, respectively. This gives (7.4).

We now show that the inequality in (7.4) is actually an equality up to constants. Let $\mathcal{N}$ be the set of indices $i$ such that the $r$-box $B_i$ is at least at distance $R/10$ from the boundary $\partial[0,R]^2$. Consider the events

$$\begin{aligned} V_i &:= \big\{|\mathscr{S} \cap B_i| \ge C\,r^2\alpha_4(r)\big\}\,, \\ W_i &:= \big\{\mathscr{S} \cap B_i \ne \emptyset,\ \mathscr{S} \subseteq B_i\big\}\,, \end{aligned}$$

for $i \in \mathcal{N}$. As we will see in a moment, we may take the constant $C$ large enough so that $\mathbb{P}\big[V_i | W_i\big] \le 1/2$. This will follow from Markov's inequality, once we know that

$$\mathbb{E}\Big[|\mathscr{S} \cap B_i| \;\Big|\; W_i\Big] \le O(1)\,r^2\alpha_4(r). \tag{7.5}$$

Firstly, for each $i \in \mathcal{N}$,

$$\mathbb{P}\big[W_i\big] \overset{(5.7)}{\asymp} \alpha_\square(B_i, [0,R]^2)^2 \overset{(2.4)}{\asymp} \alpha_4(r,R)^2\,.$$

Secondly, we need a good upper bound on $\mathbb{P}\big[x \in \mathscr{S},\ \mathscr{S} \subseteq B_i\big]$. We know this equals $\mathbb{E}\big[\lambda^2_{x,B_i^c}\big]$, see e.g. Subsection 5.3, and, similarly to Lemma 3.2, one can easily show that it is at most $\alpha_4(x, B_i)\, \alpha_\square(B_i, [0,R]^2)^2$. Summing up for all $x \in B_i$, and using the above estimate on $\mathbb{P}[W_i]$, we get (7.5).

So, we have $\mathbb{P}\big[V_i^c \bigm| W_i\big] \geq 1/2$. Note that the events $V_i^c \cap W_i$ for different $i$'s are disjoint, hence

$$\mathbb{P}\Big[0 < |\mathscr{S}| \leq C\, r^2 \alpha_4(r)\Big] \geq \sum_{i \in \mathcal{N}} \mathbb{P}\big[V_i^c \cap W_i\big] \geq c\, (R/r)^2\, \alpha_4(r, R)^2\,,$$

for some $c > 0$, and the lower bound is proved. □

*Remark* 7.2. For the triangular lattice, the following variant of (1.6) may also be established:

$$\begin{aligned} \limsup_{R\to\infty} \mathbb{P}\Big[0 < |\mathscr{S}_{f_R}| \leq \lambda\, \mathbb{E}|\mathscr{S}_{f_R}|\Big] &\asymp \lambda^{2/3}, \qquad \text{and} \\ \liminf_{R\to\infty} \mathbb{P}\Big[0 < |\mathscr{S}_{f_R}| \leq \lambda\, \mathbb{E}|\mathscr{S}_{f_R}|\Big] &\asymp \lambda^{2/3}, \end{aligned} \tag{7.6}$$

holds for every $\lambda \in (0,1]$, where the implied constants do not depend on $\lambda$. In view of Theorem 1.1, this follows from the fact that

$$\lim_{R\to\infty} \alpha_4(t\, R, R) \asymp t^{5/4}, \qquad t \in (0,1]\,,$$

which holds since the limit of critical percolation is described by $\mathrm{SLE}_6$ (this is explained in [SW01]), and the probabilities for the corresponding events for SLE are determined up to constant factors [LSW01a] and have no lower order corrections to the power law.

## 7.3 Radial version

We also have the following radial version, where $\mathscr{S}$ is the spectral sample of the 0-1 indicator function $f$ of the crossing event from $\partial[-1,1]^2$ to $\partial[-R,R]^2$, so that $\mathbb{E}\big[f^2\big] \asymp \alpha_1(R)$. Recall that we have the measures $\mathbb{P}\big[\mathscr{S} = S\big] = \mathbb{Q}\big[\mathscr{S} = S\big]/\mathbb{E}\big[f^2\big] = \widehat{f}(S)^2/\mathbb{E}\big[f^2\big]$.

**Theorem 7.3.** *Let $\mathscr{S}$ be as above, and let $r \in [1, R]$. Then*

$$\mathbb{Q}\big[|\mathscr{S}| < \alpha_4(r)\, r^2\big] \leq O(1)\, \frac{\alpha_1(R)^2}{\alpha_1(r)}\,, \qquad \mathbb{P}\big[|\mathscr{S}| < \alpha_4(r)\, r^2\big] \leq O(1)\, \frac{\alpha_1(R)}{\alpha_1(r)}\,.$$

**Proof.** Again, the bits relevant for $f$ are contained in $[-R', R']^2$, where $R' = R + 2$. We may assume that $r$ is such that $R'/r \in \mathbb{N}_+$ and $r \in [\bar{r}, R/8]$ for some fixed constant $\bar{r} > 0$, which guarantees that $k := \alpha_4(r)\, r^2 > 1$. Take a subdivision of $[-R', R']^2$ into boxes $\{B_j\}$ of side-length $r$, and let $K$ denote the union of the boxes that intersect $[-4r, 4r]^2$. We now let $\mathscr{Z}$ be the random set in $\mathcal{I} \cap [-R', R']^2 \setminus K$ where each bit is in $\mathscr{Z}$ with probability $1/k$, and $\mathscr{Z}$ is independent from $\mathscr{S}$. We also let $X := |\{j : \mathscr{S} \cap B_j \neq \emptyset,\, B_j \not\subset K\}| = |(\mathscr{S} \setminus K)_r|$. Note that $|\mathscr{S}_r| - X$ is bounded by the number of boxes in $K$, which is bounded by a constant.

Exactly as before, Propositions 5.12 and 6.1 give that

$$\mathbb{P}\big[\mathscr{S} \cap \mathscr{Z} = \emptyset \neq \mathscr{S} \setminus K\big] \leq a^{-1}\mathbb{E}\big[e^{-aX/e}\, 1_{X>0}\big],$$

with some absolute constant $a > 0$. We can use Proposition 4.7 to bound each $\mathbb{P}\big[X = n\big]$, $n \in \mathbb{N}_+$, and the argument that finished the proof of Theorem 7.1 above now gives

$$\mathbb{P}\big[0 < |\mathscr{S} \setminus K| < k\big] \leq O(1)\, \mathbb{P}\big[\mathscr{S} \cap \mathscr{Z} = \emptyset \neq \mathscr{S} \setminus K\big] \leq O(1)\, \alpha_1(r, R)\,.$$

Finally, observe that

$$\begin{aligned}
\mathbb{P}\big[|\mathscr{S}| < k\big] &\leq \mathbb{P}\big[\mathscr{S} \subset K\big] + \mathbb{P}\big[0 < |\mathscr{S} \setminus K| < k\big] \\
&\leq \mathbb{P}\Big[\mathscr{S}^* \text{ is compatible with } [-R', R']^2 \setminus K\Big] + O(1)\, \alpha_1(r, R) \\
&\leq O(1)\, \alpha_1(r, R) + O(1)\, \alpha_1(r, R) \qquad \text{by Lemma 4.8}\,,
\end{aligned}$$

and the theorem is proved. □

## 7.4 Concentration of the quad spectral sample

This section will be devoted to the following analog of (1.3) in the setting of more general quads, showing that the spectral sample is concentrated.

**Proposition 7.4.** *Let $\mathcal{Q} \subset \mathbb{R}^2$ be a quad and for $R > 0$ let $\mathscr{S}_{f_{R\mathcal{Q}}}$ denote the spectral sample of $f_{R\mathcal{Q}}$, the $\pm 1$-indicator function of the crossing event of $R\mathcal{Q}$. Then*

$$\lim_{t \to \infty} \inf_{R > 1} \mathbb{P}\Big[|\mathscr{S}_{f_{R\mathcal{Q}}}| \in \big[t^{-1} R^2\, \alpha_4(R), t\, R^2\, \alpha_4(R)\big] \cup \{0\}\Big] = 1\,.$$

We do not presently prove that $\mathbb{E}|\mathscr{S}_{f_{R\mathcal{Q}}}| \asymp R^2\,\alpha_4(R)$ as $R \to \infty$, though we tend to believe that this holds.

The main technical difficulty in the case of a general quad $\mathcal{Q}$ compared to a square is the boundary: our explicit computations in Subsection 4.3 do not apply to a general quad (even if it has piecewise smooth boundary). Thus, the proof will begin by showing that even in a general quad the spectral sample is unlikely to be very close to $\partial\mathcal{Q}$.

**Proof.** For every fixed $\delta > 0$ we can find a quad $\mathcal{Q}'$ that is contained in the interior of $\mathcal{Q}$ and such that

$$\limsup_{R\to\infty} \mathbb{P}\big[f_{R\mathcal{Q}} \neq f_{R\mathcal{Q}'}\big] < \delta\,. \tag{7.7}$$

This is easy to see, and also worked out in detail in [SS]. Let $U', U \subset \mathcal{Q}$ be open sets satisfying $\mathcal{Q}' \subset U' \subset \overline{U'} \subset U$.

Now, (7.7) and (2.7) imply that for all large enough $R$, the laws of the spectral samples $\mathscr{S}_{f_{R\mathcal{Q}}}$ and $\mathscr{S}_{f_{R\mathcal{Q}'}}$ have a total variation distance at most $4\sqrt{\delta}$, and

$$\mathbb{P}\big[\mathscr{S}_{f_{R\mathcal{Q}}} \subseteq RU'\big] \geq 1 - 4\sqrt{\delta}\,. \tag{7.8}$$

Theorem 7.1 can now be invoked to get

$$\lim_{t\to\infty}\,\limsup_{R\to\infty} \mathbb{P}\big[\mathscr{S}_{f_{R\mathcal{Q}}} \subseteq RU',\, 0 < |\mathscr{S}_{f_{R\mathcal{Q}}}| < t^{-1}\,R^2\,\alpha_4(R)\big] = 0\,.$$

In conjunction with (7.8), this gives

$$\limsup_{t\to\infty}\,\limsup_{R\to\infty} \mathbb{P}\big[0 < |\mathscr{S}_{f_{R\mathcal{Q}}}| < t^{-1}\,R^2\,\alpha_4(R)\big] \leq 4\sqrt{\delta}\,,$$

and since $\delta$ was an arbitrary positive number,

$$\lim_{t\to\infty}\,\limsup_{R\to\infty} \mathbb{P}\big[0 < |\mathscr{S}_{f_{R\mathcal{Q}}}| < t^{-1}\,R^2\,\alpha_4(R)\big] = 0\,. \tag{7.9}$$

In the other direction, it is easy to see that $\mathbb{E}|\mathscr{S}_{f_{R\mathcal{Q}}} \cap R\mathcal{Q}'| = O(R^2)\,\alpha_4(R)$, as $R \to \infty$. Therefore, Markov's inequality and (7.8) imply that for all sufficiently large $R$,

$$\mathbb{P}\big[|\mathscr{S}_{f_{R\mathcal{Q}}}| > t\,R^2\,\alpha_4(R)\big] \leq 4\sqrt{\delta} + O(1/t)\,,$$

where the implied constant may depend on $\delta$ but not on $R$. Thus,

$$\lim_{t\to\infty}\,\limsup_{R\to\infty} \mathbb{P}\big[|\mathscr{S}_{f_{R\mathcal{Q}}}| > t\,R^2\,\alpha_4(R)\big] = 0\,. \tag{7.10}$$

Since for every $R_0 \in (1,\infty)$ we obviously have

$$\lim_{t\to\infty} \sup_{R\in[1,R_0]} \mathbb{P}\Big[|\mathscr{S}_{f_{RQ}}| \notin \big[t^{-1} R^2 \alpha_4(R), t\, R^2 \alpha_4(R)\big] \cup \{0\}\Big] = 0\,,$$

the proposition follows immediately from (7.9) and (7.10). □

# 8 Applications to noise sensitivity

We are ready to prove Corollary 1.2 and Theorem 1.3, together with some generalizations.

## 8.1 Noise sensitivity in a square and a quad

**Proof of Corollary 1.2.** Let $\mathcal{I}_R$ denote the set of bits on which $f_R$ depends, and write $\epsilon = \epsilon_R$. Recall from (1.2) that if $y$ is an $\epsilon$-noisy version of $x \in \{-1,+1\}^{\mathcal{I}_R}$, then

$$\Psi_R := \mathbb{E}\big[f_R(y) f_R(x)\big] - \mathbb{E}[f_R(x)]^2 = \sum_{k=1}^{|\mathcal{I}_R|} (1-\epsilon)^k\, \mathbb{P}\big[|\mathscr{S}_{f_R}| = k\big]\,. \tag{8.1}$$

Breaking the sum over $k$ in (8.1) into parts $(j-1)/\epsilon < k \le j/\epsilon$, with $j = 1, 2, \dots$, we get

$$\begin{aligned}\Psi_R &\le \sum_{j=1}^{\infty} (1-\epsilon)^{(j-1)/\epsilon}\, \mathbb{P}\big[(j-1)/\epsilon < |\mathscr{S}_{f_R}| \le j/\epsilon\big] \\ &\le \sum_{j=1}^{\infty} e^{1-j}\, \mathbb{P}\big[0 < |\mathscr{S}_{f_R}| \le j/\epsilon\big]\,.\end{aligned} \tag{8.2}$$

Recall that $r^2\,\alpha_4(r) \to \infty$ as $r \to \infty$, by (2.6). For $s \ge 1$, let $\rho(s)$ be the least $r \in \mathbb{N}_+$ such that $r^2\,\alpha_4(r) \ge s$, and let $\gamma(r) := r^2\,\alpha_4(r)^2$. The properties of $\alpha_4$, namely (2.3) and (2.6), imply that

$$s \le \rho(s)^2\,\alpha_4\big(\rho(s)\big) \le O(s) \qquad \text{for } s \ge 1, \tag{8.3}$$

and then

$$O(1)\,\gamma\big(\rho(s')\big)/\gamma\big(\rho(s)\big) \ge (s/s')^{O(1)} \qquad \text{for } s' \ge s \ge 1. \tag{8.4}$$

Now set $\rho_j := \rho(j/\epsilon)$. Then, for $j \in \mathbb{N}_+$,

$$\begin{aligned}\mathbb{P}\big[0 < |\mathscr{S}_{f_R}| \le j/\epsilon\big] &\le \mathbb{P}\big[0 < |\mathscr{S}_{f_R}| \le \rho_j^2\, \alpha_4(\rho_j)\big] \\ &\le O(1)\, \gamma(R)/\gamma(\rho_j) \qquad \text{by (7.4)} \\ &\le O(1)\, \gamma(R)\, j^{O(1)}/\gamma(\rho_1) \qquad \text{by (8.4).}\end{aligned}$$

Therefore (8.2) gives

$$\Psi_R \le O(1)\, \gamma(R)/\gamma(\rho_1)\,. \tag{8.5}$$

If $\lim_{R\to\infty} \mathbb{E}|\mathscr{S}_{f_R}|\, \epsilon_R = \infty$, then by (7.3) and the usual properties of $\alpha_4$ we have $R/\rho(1/\epsilon) \to \infty$ as well as $\gamma(R)/\gamma(\rho_1) = \gamma(R)/\gamma\big(\rho(1/\epsilon)\big) \to 0$, which together with (8.5) proves (1.7).

Now assume that $\epsilon_R\, \mathbb{E}|\mathscr{S}_{f_R}| \to 0$. Applying (1.2) and Jensen's inequality, we get

$$\mathbb{E}\big[f_R(x)\, f_R(y)\big] = \mathbb{E}\Big[(1-\epsilon)^{|\mathscr{S}_{f_R}|}\Big] \ge (1-\epsilon)^{\mathbb{E}|\mathscr{S}_{f_R}|} \to 1\,,$$

as $R \to \infty$. Since $f_R(x)\, f_R(y) \le 1 = f_R(x)^2$, (1.8) follows. $\square$

Suppose that we are in the setting of the triangular grid, and $\epsilon = t/\mathbb{E}|\mathscr{S}_{f_R}|$, where $t > 1$. Then with the above notations, we have by (7.6) and (8.2) that $\limsup_{R\to\infty} \Psi_R \le O(1)\, t^{-2/3}$. Using the fact that we also have lower bounds in Theorem 1.1, it is easy to see that

$$\limsup_{R\to\infty} \Psi_R \asymp t^{-2/3} \asymp \liminf_{R\to\infty} \Psi_R\,. \tag{8.6}$$

In a forthcoming paper we plan to use this to show that in the appropriate scaling limit of critical dynamical percolation, the crossing events in the unit square at time 0 and at time $t$ have correlations that decay like $t^{-2/3}$ as $t \to \infty$.

We also have the following generalization for the $\pm 1$-indicator function of the left-right crossing in scaled versions of an arbitrary fixed quad $\mathcal{Q}$.

**Corollary 8.1.** *Assume that $\epsilon_R \in (0,1)$ is such that $\epsilon_R\, R^2 \alpha_4(R) \to \infty$ as $R \to \infty$, and $y$ is an $\epsilon_R$-noisy version of $x$. Then*

$$\mathbb{E}\big[f_{R\mathcal{Q}}(y)\, f_{R\mathcal{Q}}(x)\big] - \mathbb{E}\big[f_{R\mathcal{Q}}(x)\big]\, \mathbb{E}\big[f_{R\mathcal{Q}}(y)\big] \to 0\,.$$

*On the other hand, if $\epsilon_R\, R^2 \alpha_4(R) \to 0$, then*

$$\mathbb{E}\big[f_{R\mathcal{Q}}(y)\, f_{R\mathcal{Q}}(x)\big] - \mathbb{E}\big[f_{R\mathcal{Q}}(x)^2\big] \to 0\,.$$

**Proof.** First assume $\epsilon_R R^2\alpha_4(R) \to \infty$. Given $\delta > 0$, by Proposition 7.4 we have

$$\inf_{R>1} \mathbb{P}\big[0 < |\mathscr{S}_{f_{R\mathcal{Q}}}| < t^{-1} R^2 \alpha_4(R)\big] < \delta$$

if $t \geq t_1(\delta)$ is large enough. Now let $R$ be large enough so that $\epsilon_R R^2\alpha_4(R) > t^2$. Then, by (8.1),

$$\begin{aligned}\Psi_R \leq \delta + &\sum_{|S|\geq R^2\alpha_4(R)/t} \widehat{f}(S)^2 (1-\epsilon_R)^{|S|} \\ &\leq \delta + \left(1 - \frac{t^2}{R^2\,\alpha_4(R)}\right)^{R^2\alpha_4(R)/t} \leq \delta + O(1)\exp(-t)\,,\end{aligned}$$

which is at most $2\delta$ if $t \geq t_2(\delta)$. We can choose $R$ large enough with respect to this new $t$, and hence $\Psi_R \to 0$ is proved.

Now assume $\epsilon_R R^2\alpha_4(R) \to 0$. Given $\delta > 0$, by Proposition 7.4 we have

$$\inf_{R>1} \mathbb{P}\big[|\mathscr{S}_{f_{R\mathcal{Q}}}| > t\, R^2\, \alpha_4(R)\big] < \delta$$

if $t \geq t_1(\delta)$ is large enough. Now let $R$ be so large that $\epsilon_R R^2\alpha_4(R) < t^{-2}$. Then, by (1.2), for $\mathscr{S} = \mathscr{S}_{f_{R\mathcal{Q}}}$,

$$\begin{aligned}\mathbb{E}\big[f_{R\mathcal{Q}}(y)\, f_{R\mathcal{Q}}(x)\big] &\geq \mathbb{E}\Big[(1-\epsilon_R)^{|\mathscr{S}|} \;\Big|\; |\mathscr{S}| < tR^2\alpha_4(R)\Big]\,\mathbb{P}\big[|\mathscr{S}| < tR^2\alpha_4(R)\big] \\ &\geq \left(1 - \frac{t^{-2}}{R^2\,\alpha_4(R)}\right)^{tR^2\alpha_4(R)} (1-\delta) \\ &\geq \exp(-O(1)/t)\,(1-\delta)\,.\end{aligned}$$

This is arbitrarily close to $\mathbb{E}\big[f_{R\mathcal{Q}}(x)^2\big] = 1$ for $t$ large; hence we are done. □

## 8.2 Resampling a fixed set of bits

We now prove a general version of Theorem 1.3. If $y \in \Omega = \{-1,1\}^{\mathcal{I}}$ is a noisy version of $x$, then $y_j = x_j$, except on a small random set of $j \in \mathcal{I}$. We may consider a variation of this situation, where we have some fixed deterministic set $\mathcal{U} \subset \mathcal{I}$, and we take $y_j = x_j$ for $j \in \mathcal{U}$ and take the restriction of $y$ to $\mathcal{U}^c := \mathcal{I} \setminus \mathcal{U}$ be independent from $x$ (and $y$ is uniform in $\Omega$). Although this setup was mentioned in [BKS99], the techniques developed there and in [SS05] fell short of being able to handle this variation. Now, we can analyse this variation without difficulty, and prove the following proposition:

**Proposition 8.2.** *Let $\mathcal{Q} \subset \mathbb{R}^2$ be some quad and for $R > 1$ let $f_R = f_{R\mathcal{Q}}$ be the $\pm 1$-indicator function of the crossing event in $R\mathcal{Q}$ (either in $\mathbb{Z}^2$ or in the triangular lattice). For every $R > 1$, let $\mathcal{U}_R \subset \mathcal{I}$ be some set of bits, and let $r_R$ be the maximal radius of any disk contained in $R\,\mathcal{Q}$ that is disjoint from $\mathcal{U}_R^c := \mathcal{I} \setminus \mathcal{U}_R$. If*

$$\lim_{R\to\infty} \frac{r_R}{\sqrt{R^2\,\alpha_4(R)}} = 0\,, \tag{8.7}$$

*then the family $(\mathcal{U}_R)_{R>0}$ is* **asymptotically clueless** *in the sense that*

$$\lim_{R\to\infty} \big\| \mathbb{E}[f_R \mid \mathcal{F}_{\mathcal{U}_R}] - \mathbb{E}[f_R] \big\| = 0\,.$$

*On the other hand, if*

$$\lim_{R\to\infty} \big|\mathcal{U}_R^c\big|\,\alpha_4(R) = 0\,, \tag{8.8}$$

*then $(\mathcal{U}_R)_{R>0}$ is* **asymptotically decisive** *in the sense that*

$$\lim_{R\to\infty} \big\| \mathbb{E}[f_R \mid \mathcal{F}_{\mathcal{U}_R}] - f_R \big\| = 0\,,$$

*which means that there is asymptotically no loss of information about the crossing $f_R$.*

Notice that even though the convergence of $\mathbb{E}\big[f_{R\mathcal{Q}}\big]$ is not known in $\mathbb{Z}^2$ for general quads or even rectangles other than squares, our definitions of being asymptotically clueless or decisive still make perfect sense.

Note that $O(1)\,|\mathcal{U}_R^c| \geq (R/r_R)^2$ and there are examples where $|\mathcal{U}_R^c| \asymp (R/r_R)^2$. Thus, in some sense the conditions (8.7) and (8.8) are nearly complementary. However, the following two examples are not covered. Suppose that $\mathcal{Q}$ is the unit square, and for each $R$ we take $\mathcal{U}_R$ to be the set of bits contained in the left half of the square $R\,\mathcal{Q}$. It is left to the reader to verify that in this case $\mathcal{U}_R$ is neither asymptotically decisive, nor asymptotically clueless.

In the second example, we take $\mathcal{Q}$ to be the unit square again, and let $\mathcal{U}_R$ be the set of bits outside of the disk of radius $\rho_R$ centered at the center of the square $R\,\mathcal{Q}$. Then $\mathcal{U}_R$ is asymptotically decisive as long as $\rho_R/R \to 0$, but this does not follow from the proposition. However, Remark 8.5 below does give a general statement which covers this example.

*Remark* 8.3. When $(\mathcal{U}_R)_{R>1}$ is asymptotically clueless, it is immediate to see that if $x_R$ and $y_R$ are two coupled percolation configurations which coincide on $\mathcal{U}_R$, but are independent elsewhere, then

$$\lim_{R\to\infty} \mathbb{E}\big[f_R(x_R) f_R(y_R)\big] - \mathbb{E}\big[f_R\big]^2 = 0.$$

On the other hand, if $(\mathcal{U}_R)_{R>0}$ is asymptotically decisive, then

$$\lim_{R\to\infty} \mathbb{E}\big[f_R(x_R) f_R(y_R)\big] - \mathbb{E}\big[f_R^2\big] = 0.$$

**Proof of Proposition 8.2.** By (2.9) and orthogonality of martingale differences, we have

$$\begin{aligned}\mathbb{P}\big[\emptyset \neq \mathscr{S}_{f_R} \subset \mathcal{U}_R\big] &= \mathbb{E}\Big[\mathbb{E}[f_R \mid \mathcal{F}_{\mathcal{U}_R}]^2 - \mathbb{E}[f_R]^2\Big] \\ &= \mathbb{E}\Big[\big(\mathbb{E}[f_R \mid \mathcal{F}_{\mathcal{U}_R}] - \mathbb{E}[f_R]\big)^2\Big].\end{aligned}$$

Thus, $(\mathcal{U}_R)_{R>0}$ is asymptotically clueless if and only if the spectral sample of $f_R$ satisfies $\mathbb{P}\big[\emptyset \neq \mathscr{S}_{f_R} \subset \mathcal{U}_R\big] \to 0$. Similarly,

$$\mathbb{P}\big[\mathscr{S}_{f_R} \not\subset \mathcal{U}_R\big] = \mathbb{E}\Big[f_R^2 - \mathbb{E}[f_R \mid \mathcal{F}_{\mathcal{U}_R}]^2\Big] = \mathbb{E}\Big[\big(\mathbb{E}[f_R \mid \mathcal{F}_{\mathcal{U}_R}] - f_R\big)^2\Big].$$

Hence, a necessary and sufficient condition for $(\mathcal{U}_R)$ to be asymptotically decisive is that $\mathbb{P}\big[\mathscr{S}_{f_R} \subset \mathcal{U}_R\big] \to 1$.

We now consider the simpler case in which $\mathcal{Q} = [0,1]^2$, and assume (8.7). Since the proof is rather similar to the proof of Theorem 1.1, we will be brief here, and only indicate some of the essential points and the arguments where a more substantial modification is necessary. As in Section 7.2, subdivide $[-2, R+2]^2$ into boxes $B_1, B_2, \ldots, B_{m^2}$ of radius $(R+4)/(2\,m)$, where $m = m_R \in \mathbb{N}$ tends to infinity as $R \to \infty$, but very slowly. As above let $B_j'$ denote the box concentric with $B_j$ whose radius is a third of the radius of $B_j$. Let $H_R \subset \mathcal{U}_R^c$ be a maximal subset of $\mathcal{U}_R^c$ with the property that the distance between any two distinct elements in $H_R$ is at least $r_R$. Then for some constant $C$ every disk of radius $C\,r_R$ in $R\,\mathcal{Q}$ contains a point of $H_R$, but a disk of radius smaller than $r_R/2$ contains at most one point of $H_R$. Let $x_j$ be the indicator function of the event $\mathscr{S}_{f_R} \cap B_j \neq \emptyset$ and let $y_j$ be the indicator function of the event

$$\mathscr{S}_{f_R} \cap B_j' \cap H_R \neq \emptyset\,.$$

Our goal is to prove that for each $I \subset \{1,\dots,m^2\}$ and every $j \in \{1,\dots,m^2\} \setminus I$,

$$\mathbb{P}\big[y_j = 1 \bigm| y_i = 0\, \forall i \in I\big] \ge a\, \mathbb{P}\big[x_j = 1 \bigm| y_i = 0\, \forall i \in I\big], \tag{8.9}$$

holds with some absolute constant $a > 0$. We mimic the proof of Proposition 5.1. Fix such $j$ and $I$, and set $n := |B'_j \cap H_R|$, $W := H_R \cap \bigcup_{i\in I} B'_i$ and $Y := \big|\mathscr{S}_{f_R} \cap B'_j \cap H_R\big|$. Using Proposition 5.2, we get

$$O(1)\, \mathbb{E}\big[Y\, 1_{\mathscr{S}_{f_R} \cap W = \emptyset}\big] \ge \mathbb{E}\big[\lambda^2_{B,W}\big]\, \alpha_4(R/m)\, n\,,$$

and Proposition 5.3 can be used to obtain

$$\mathbb{E}\big[Y^2\, 1_{\mathscr{S}_{f_R} \cap W = \emptyset}\big] \le O(1)\, \mathbb{E}\big[\lambda^2_{B,W}\big]\, \alpha_4(R/m)^2\, n^2\,,$$

provided that $\liminf_{R\to\infty} \alpha_4(R/m)\, n > 0$ and hence the diagonal term is dominated by a constant times the off-diagonal term. (Intuitively, we need that the "density" of the set $H_R$ inside the box $B_j$ of radius $R/m$ is good enough to see pivotals once the box has any of them.) Since $n \asymp R^2\,(m\, r_R)^{-2}$, this follows from (8.7), provided that $m_R$ tends to $\infty$ sufficiently slowly. Then, (8.9) follows from the Cauchy-Schwarz second moment bound.

Using Propositions 4.1 and 6.1, and following the proof of the concentration Theorem 1.1, we have for $R$ large enough that

$$\mathbb{P}\big[\emptyset \ne \mathscr{S}_{f_R} \subset \mathcal{U}_R\big] \le O(1)\, \gamma_{R/m}(R) \xrightarrow[R\to\infty]{(2.6)} 0\,.$$

By the above, it follows that $(\mathcal{U}_R)$ is asymptotically clueless.

For the case of a general quad, we can do the same trick as in Section 7.4 and in the proof of Corollary 8.1. For any $\delta > 0$, there is a quad $\mathcal{Q}'$ contained in the interior of $\mathcal{Q}$ such that for $R$ large enough

$$\mathbb{P}\big[\mathscr{S}_{f_R} \subseteq R\mathcal{Q}'\big] > 1 - \delta\,. \tag{8.10}$$

We may assume with no loss of generality that $\mathcal{Q}'$ is smooth. The above arguments (for the square) can easily be adapted to show that

$$\limsup_{R\to\infty} \mathbb{P}\big[\emptyset \ne \mathscr{S}_{f_R} \subset \mathcal{U}_R \cap R\,\mathcal{Q}'\big] = 0\,,$$

because the distance from $R\,\mathcal{Q}'$ to the complement of $R\,\mathcal{Q}$ is bounded below by a positive constant times $R$. In combination with (8.10), this gives

$$\limsup_{R\to\infty} \mathbb{P}\big[\emptyset \neq \mathscr{S}_{f_R} \subset \mathcal{U}_R\big] \leq \delta\,.$$

Since the left hand side does not depend on $\delta$, we may let $\delta$ tend to 0 and deduce the first claim of the proposition.

For the other direction, let $\mathcal{Q}'$ and $\delta$ satisfy (8.10), as before. Let $\rho_0 > 0$ denote the distance from $\partial\mathcal{Q}'$ to $\partial\mathcal{Q}$. Then for $x \in \mathcal{I} \cap (R\,\mathcal{Q}')$, we have $\mathbb{P}\big[x \in \mathscr{S}_{f_R}\big] \leq O(1)\,\alpha_4(\rho_0\,R)$. Thus,

$$\mathbb{E}\big[|\mathscr{S}_{f_R} \cap \mathcal{U}^c \cap (R\,\mathcal{Q}')|\big] \leq |\mathcal{U}^c|\,\alpha_4(\rho_0\,R)\,.$$

If we assume (8.8), then this tends to zero as $R \to \infty$. Thus,

$$\lim_{R\to\infty} \mathbb{P}\big[\mathscr{S}_{f_R} \cap \mathcal{U}^c \cap (R\,\mathcal{Q}') \neq \emptyset\big] = 0\,,$$

and (8.10) gives

$$\limsup_{R\to\infty} \mathbb{P}\big[\mathscr{S}_{f_R} \cap \mathcal{U}^c \neq \emptyset\big] \leq \delta\,.$$

Once again, since $\delta > 0$ is arbitrary, this completes the proof. □

**Proof of Theorem 1.3.** The Theorem follows immediately from Proposition 8.2 and Remark 8.3. □

*Remark* 8.4. Recall that we used the random set $\mathcal{Z}$ in Sections 5 and 7, with $\mathbb{P}[x \in \mathcal{Z}] = 1/(\alpha_4(r)\,r^2)$, just as a tool to measure the size of $\mathscr{S}$. However, in the spirit of our above proof, in place of $\mathcal{U}_r^c$, we can think of $\mathcal{Z}$ as the actual set of bits being resampled.

*Remark* 8.5. It may be concluded from a slight variation on the proof of Proposition 8.2 that in the setting of the triangular grid if the Hausdorff limit $F := \lim_{R\to\infty} \mathcal{U}_R^c/R$ exists and has Hausdorff dimension strictly less than $5/4$, then $\mathcal{U}_R$ is asymptotically decisive. Indeed, assuming that $s := \dim(F) < 5/4$, for every $\epsilon > 0$ may find a countable collection of points $z_j$ and radii $\rho_j$, such that $\sum_j \rho_j^{s+\epsilon} < \epsilon$ and the union of the disks with these centers and radii contains a neighborhood of $F$. The probability that $\mathscr{S}_{f_R}$ comes within distance $O(1)\,\rho_j\,R$ of $R\,z_j$ is bounded by $O(1)\,\rho_j^{5/4-\epsilon}$, if $z_j$ is not too close to the boundary of $\mathcal{Q}$ and $R$ is sufficiently large. A sum bound and the above argument for dealing with a neighborhood of the boundary of $\mathcal{Q}$ complete the proof.

*Remark* 8.6. We obtain here a somewhat sharp result for "sensitivity to selective noise", though it would be even more satisfying to have a necessary and sufficient condition for a family $(\mathcal{U}_R)_{R>1}$ to be asymptotically clueless. We believe that $(\mathcal{U}_R)$ is asymptotically clueless if and only if $\mathbb{P}\big[\emptyset \neq \mathscr{P}_R \subset \mathcal{U}_R\big] \to 0$, where $\mathscr{P}_R$ is the set of pivotals. Similarly, $(\mathcal{U}_R)$ should be asymptotically decisive if and only if $\mathbb{P}\big[\mathscr{P}_R \subset \mathcal{U}_R\big] \to 1$. In other words, even though $\mathscr{P}_R$ and $\mathscr{S}_R$ are asymptotically quite different (compare, e.g., Remark 4.6 with Proposition 4.1), they should have the same polar sets.

*Remark* 8.7. Tsirelson [Tsi04] distinguishes two types of noise sensitivities: micro and block sensitivities, where the latter is stronger than the micro sensitivity we have been considering so far. He gives the following illustrative examples. Consider the two functions on $\{-1,1\}^n$: $f_1 = \frac{1}{\sqrt{n}}\sum_{i=1}^n x_i$ and $f_2 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n-n^{1/2}}\prod_{k=i}^{i+n^{1/2}} x_k$. Both correspond to renormalized random walks which converge to Brownian motion, but the first is stable while the second is noise-sensitive. Block sensitivity is defined as follows: instead of resampling the bits one by one, each with probability $\epsilon$, we resample simultaneously blocks of bits. For $\delta > 0$, divide the $n$ bits into about $\delta^{-1}$ blocks of about $\delta n$ bits: $B_i := \mathbb{N} \cap \big[i\,\delta\, n, (i+1)\,\delta\, n\big)$. Each block is now resampled (i.e., all bits within the block) with probability $\epsilon$. A sequence of functions is block sensitive if for any fixed $\epsilon > 0$, the limsup as $n$ goes to infinity of the correlation in this block procedure is bounded by a function of $\delta$ which goes to 0 when $\delta$ goes to 0. It is easy to see that the sequence of functions $f_2$ $(n = 1, 2, \dots)$ is not block-sensitive. This is related to the fact that the sensitivity of $f_2$ is "localized", in the sense that its spectral sample $\mathscr{S}_{f_2}$, when rescaled by $1/n$, converges in law to a finite (random) set of points. As we will see in Section 10, this is not at all the case with the spectral sample of percolation. It is easy to check that percolation crossing events are indeed block sensitive.

# 9 Applications to dynamical percolation

In this section, we prove Theorems 1.4 and 1.5.

As in Subsection 7.3, we consider the 0-1 indicator function $f = f_R$ of the percolation crossing event from $\partial([-1,1]^2)$ to $\partial([-R,R]^2)$. Then $\mathbb{E}\big[f\big] = \mathbb{E}\big[f^2\big] \asymp \alpha_1(R)$. We let $\omega_t$ be the dynamical percolation configuration at time $t$, started at the stationary distribution at $t = 0$. Recall that we have

$$\mathbb{E}\big[f(\omega_0)\, f(\omega_t)\big] = \sum_{k=0}^{\infty} e^{-kt} \sum_{|S|=k} \widehat{f}(S)^2, \qquad t > 0\,. \tag{9.1}$$

As in Subsection 8.1, for $s \geq 1$ define $\rho(s)$ as the least $r \in \mathbb{N}_+$ such that $r^2\,\alpha_4(r) \geq s$, and break the sum over $k$ in (9.1) into parts according to the $j \in \mathbb{N}$ satisfying $j/t \leq k < (j+1)/t$. Then use Theorem 7.3 and the estimates (2.6) and (2.3), to get

$$\mathbb{E}\big[f(\omega_0)\, f(\omega_t)\big] \leq O(1)\,\frac{\alpha_1(R)^2}{\alpha_1(\rho(1/t))} = O(1)\,\frac{\mathbb{E}[f(\omega_0)^2]}{\alpha_1(\rho(1/t))}\,. \tag{9.2}$$

Let $\mathscr{E}$ denote the set of exceptional times $t \in [0,\infty)$ for the event that the origin is in an infinite open cluster. To give a lower bound on the Hausdorff dimension of $\mathscr{E}$, a well-known technique is Frostman's criterion, see e.g. [Per01, Theorem 6.6] or [Mat95, Theorem 8.9]. Combined with a compactness argument, it gives the following; see [SS05, Theorem 6.1]. For any $\gamma > 0$, let

$$M_\gamma(R) := \int_0^1 \int_0^1 \frac{\mathbb{E}\big[f_R(\omega_0)\, f_R(\omega_t)\big]}{\mathbb{E}\big[f_R(\omega_0)\big]^2\, |t-s|^\gamma}\, dt\, ds\,. \tag{9.3}$$

If $\sup_R M_\gamma(R) < \infty$, then $\mathscr{E} \cap [0,1]$ is nonempty with positive probability, and on this event a.s. its dimension is at least $\gamma$. It is easy to see that there is a constant $d$ such that $\dim_H(\mathscr{E}) = d$ a.s. Therefore, $\sup_R M_\gamma(R) < \infty$ also implies $\dim_H(\mathscr{E}) \geq \gamma$ almost surely.

**Proof of Theorem 1.4.** To start with, we have $\rho(s) = s^{4/3+o(1)}$, by (2.5). Secondly,

$$\alpha_1(r) = r^{-5/48+o(1)}$$

by [LSW02]. Thus, translation invariance and (9.2) give

$$\mathbb{E}\big[f(\omega_s)\, f(\omega_t)\big]/\mathbb{E}\big[f(\omega)\big]^2 \leq O(1)\,|t-s|^{-(4/3)(5/48)+o(1)},$$

as $|t-s| \to 0$. Therefore, as long as $\gamma < 1 - (4/3)\,(5/48)$, we have $\sup_R M_\gamma(R) < \infty$. The above discussion therefore gives $\dim_H(\mathscr{E}) \geq 31/36$ a.s. The matching upper bound is given by Theorem 1.9 of [SS05]. This implies statements 1 of the Theorem.

The proof of Statement 2 is similar. For the 0-1 indicator function $f^+$ of the crossing event between radius 1 and $R$ in a half plane one gets the following analog of Theorem 7.3: if $k \in \mathbb{N}_+$ satisfies $k \le \alpha_4(r)\, r^2$, then

$$\mathbb{Q}\big[|\mathscr{S}_{f^+}| < k\big] \le O(1)\,\frac{\alpha_1^+(R)^2}{\alpha_1^+(r)}\,.$$

The proof is similar to the proof of Theorem 7.3, and is left to the reader. The bound corresponding to (9.2) is then

$$\frac{\mathbb{E}\big[f^+(\omega_0) f^+(\omega_t)\big]}{\mathbb{E}\big[f^+(\omega_0)^2\big]} = O(1)\,\alpha_1^+\big(\rho(1/t)\big)^{-1}. \tag{9.4}$$

By [SW01, Theorem 3], we have $\alpha_1^+(r) = r^{-\xi_1^+ + o(1)}$, with $\xi_1^+ = 1/3$. The proof of the lower bound of $1 - (4/3)\,\xi_1^+$ on the Hausdorff dimension then proceeds as above. For the upper bound, we refer to [SS05, Theorem 1.13]. This proves part 2.

For the proof of the third part, we now let $f$ be the indicator function of the event that there is a white crossing in the upper half plane and a black crossing in a translation of the lower half plane from some fixed radius $r_0$ to radius $R$. The translation is chosen so that the percolation configurations in the two half planes are independent, and $r_0$ is chosen so that the event has positive probability for all $R > r_0$. Then by independence (and since the choice of $r_0$ is insignificant), we get from (9.4)

$$\frac{\mathbb{E}\big[f(\omega_0) f(\omega_t)\big]}{\mathbb{E}\big[f(\omega_0)^2\big]} = O(1)\,\alpha_1^+\big(\rho(1/t)\big)^{-2}.$$

Thus, in this case we get the lower bound of 1/9 for the corresponding Hausdorff dimension, which completes the proof. ☐

**Proof of Theorem 1.5.** Let $f = f_R$ be the indicator function for the existence of an open crossing from 0 to $\partial([-R,R]^2)$. We will apply the relation between $\alpha_1$, $\alpha_4$ and $\alpha_5$ that comes from the $k = 2$ case of Proposition 12.1 in our Appendix. Since $\alpha_5(r) \asymp r^{-2}$ (see [KSZ98, Lemma 5] or [SS05, Corollary A.8]), the estimate (12.1) says that there are some constants $c_1, \epsilon > 0$ such that

$$\alpha_1(r)\,\alpha_4(r) > c_1 r^{\epsilon - 2}, \tag{9.5}$$

holds for all $r > 1$. Thus,

$$\begin{aligned} \frac{\mathbb{E}\big[f(\omega_0)\,f(\omega_t)\big]}{\mathbb{E}\big[f(\omega_0)\big]^2} &\le \frac{O(1)}{\alpha_1\big(\rho(1/t)\big)} && \text{by (9.2)} \\ &\le O(1)\,\rho(1/t)^{2-\epsilon}\,\alpha_4\big(\rho(1/t)\big) && \text{by (9.5)} \\ &\le \frac{O(1)}{t}\,\rho(1/t)^{-\epsilon} && \text{by (8.3)} \\ &\le O(1)\,t^{\epsilon/2-1}, \end{aligned}$$

where the last inequality follows from the definition of $\rho$. Therefore, if we take $\gamma \in (0, \epsilon/2)$, then $\sup_R M_\gamma(R) < \infty$, and the set of exceptional times for having an infinite cluster almost surely has a positive Hausdorff dimension. □

Finally, note that if there were exceptional times with two distinct infinite white clusters with positive probability, then there would also be times with the 4-arm event from the origin to infinity. It was shown in [SS05] that this does not happen on the triangular lattice, and that there are no exceptional times on $\mathbb{Z}^2$ with three infinite white clusters. However, one can also easily prove the stronger result for $\mathbb{Z}^2$. Recall that (2.6) implies that $\alpha_4(r)^2\,r^2 < O(1)\,r^{-\epsilon}$ for some $\epsilon > 0$. This implies that the expected number of pivotals for the 4-arm event between radius 4 and $R$ tends to zero as $R \to \infty$. (One should also take into account the sites near the outer boundary and near the inner boundary. Indeed, the total expected number of pivotals is $O(1)\,R^2\,\alpha_4(R)^2$.) Hence [SS05, Theorem 8.1] says that there are a.s. no exceptional times for the 4-arm event even on $\mathbb{Z}^2$.

## 10 Scaling limit of the spectral sample

Given $\eta > 0$ let $\mu_\eta$ denote the law of Bernoulli(1/2) site percolation on the triangular grid $T_\eta$ of mesh $\eta$. Let $\omega$ denote a sample from $\mu_\eta$. Given a quad $\mathcal{Q} \subset \mathbb{C}$, we can consider the event that $\mathcal{Q}$ is crossed by $\omega$. To make this precise in the case where $\mathcal{Q}$ is not adapted to the grid, we may consider the white and black coloring of the hexagonal grid dual to $T_\eta$, as in Subsection 2.1. We let $f_\mathcal{Q}$ denote the $\pm 1$ indicator function of the crossing event. Let $\widehat{\mu}_\eta^\mathcal{Q}$ denote the law of the spectral sample of $f_\mathcal{Q}$, that is, if $\mathcal{X}$ is a collection of subsets of the vertices of $T_\eta$, then

$$\widehat{\mu}_\eta^{\mathcal{Q}}(\mathcal{X}) = \sum_{S \in \mathcal{X}} \widehat{f}_{\mathcal{Q}}(S)^2.$$

Let $d_0$ denote the spherical metric on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (with diameter $\pi$). If $S_1, S_2 \subset \hat{\mathbb{C}}$ are closed and nonempty, let $d_H(S_1, S_2)$ be the Hausdorff distance between $S_1$ and $S_2$ with respect to the underlying metric $d_0$. If $S \neq \emptyset$, define $d_H(\emptyset, S) = d_H(S, \emptyset) := \pi$, and set $d_H(\emptyset, \emptyset) = 0$. Then $d_H$ is a metric on the set $\mathfrak{S}$ of closed subsets of $\hat{\mathbb{C}}$. Since $(\mathfrak{S} \setminus \{\emptyset\}, d_H)$ is compact, the same holds for $(\mathfrak{S}, d_H)$. We may consider the probability measure $\widehat{\mu}_\eta^{\mathcal{Q}}$ as a Borel measure on $(\mathfrak{S}, d_H)$.

**Theorem 10.1.** *Let $\mathcal{Q}$ be a piecewise smooth quad in $\mathbb{C}$. Then the weak limit $\widehat{\mu}^{\mathcal{Q}} := \lim_{\eta \searrow 0} \widehat{\mu}_\eta^{\mathcal{Q}}$ (with respect to the metric $d_H$) exists. Moreover, it is conformally invariant, in the sense that if $\phi$ is conformal in a neighborhood of $\mathcal{Q}$ and $\mathcal{Q}' := \phi(\mathcal{Q})$, then $\widehat{\mu}^{\mathcal{Q}} = \widehat{\mu}^{\mathcal{Q}'} \circ \phi$.*

As mentioned in the introduction, the existence of the limit follows from Tsirelson's theory and [SS]. Nevertheless, we believe that our exposition below might be helpful.

*Remark* 10.2. The proof of the existence of the limit also works for subsequential scaling limits of critical bond percolation on $\mathbb{Z}^2$: if $\eta_j \searrow 0$ is a sequence along which bond percolation on $\eta_j \mathbb{Z}^2$ has a limit (in the sense of [SS], say), then the corresponding spectral sample measures also have a limit. (The existence of such sequences $\{\eta_j\}$ follows from compactness.)

In the proof of Theorem 10.1, we will use the following result.

**Proposition 10.3** ([SS])**.** *Let $\mathcal{Q}$ be a piecewise smooth quad in $\mathbb{C}$. Suppose that $\alpha \subset \mathbb{C}$ is a finite union of finite length paths, and that $\alpha \cap \partial \mathcal{Q}$ is finite. Then for every $\epsilon > 0$ there is a finite collection of piecewise smooth quads $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n \subset \mathbb{C} \setminus \alpha$ and a function $g : \{-1, 1\}^n \to \{-1, 1\}$ such that*

$$\lim_{\eta \searrow 0} \mu_\eta \Big[ f_{\mathcal{Q}}(\omega) \neq g\big(f_{\mathcal{Q}_1}(\omega), f_{\mathcal{Q}_2}(\omega), \dots, f_{\mathcal{Q}_n}(\omega)\big) \Big] < \epsilon . \quad \square$$

Another result from [SS] that we will need is that for any finite sequence $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n$ of piecewise smooth quads in $\mathbb{C}$, the law of the vector $\big(f_{\mathcal{Q}_1}(\omega), f_{\mathcal{Q}_2}(\omega), \dots, f_{\mathcal{Q}_n}(\omega)\big)$ under $\mu_\eta$ has a limit as $\eta \searrow 0$, and that this limiting joint law is conformally invariant.

**Proof of Theorem 10.1.** Let $U \subset \mathbb{C}$ be an open set such that $\partial U$ is a disjoint finite union of smooth simple closed paths and $\partial U \cap \partial \mathcal{Q}$ is finite. In order to establish the existence of the limit $\widehat{\mu}^{\mathcal{Q}}$, it is clearly enough to show that for every such $U$ the limit

$$W(\mathcal{Q}, U) := \lim_{\eta \searrow 0} \widehat{\mu}_\eta^{\mathcal{Q}}\big(\mathscr{S} \subset U\big)$$

exists, and for the conformal invariance statement, it suffices to show that $W\big(\mathcal{Q}', \phi(U)\big) = W(\mathcal{Q}, U)$.

Fix some $\epsilon > 0$ arbitrarily small. In Proposition 10.3, take $\alpha := \partial U$ and let $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ and $g$ be as guaranteed there. Let $J := \big\{j \in \{1, 2, \dots, n\} : \mathcal{Q}_j \subset U\big\}$ and $J' := \{1, \dots, n\} \setminus J$. Then $\mathcal{Q}_j \cap U = \emptyset$ when $j \in J'$. Set $x := \big(f_{\mathcal{Q}_1}(\omega), \dots, f_{\mathcal{Q}_n}(\omega)\big) \in \{-1, 1\}^n$, and let $x_J$ and $x_{J'}$ denote the restrictions of $x$ to $J$ and $J'$, respectively. Then for all $\eta$ sufficiently small $x_{J'}$ is independent from $\omega_U$ and $x_J$ is determined by $\omega_U$.

Let $G = G(\omega) := g(x)$, let $\nu_\eta$ be the law of $x$ under $\mu_\eta$, and let $\nu := \lim_{\eta \searrow 0} \nu_\eta$. By (2.9), we have for all $\eta$ sufficiently small

$$W_\eta(G, U) := \sum_{S \subset U} \widehat{G}(S)^2 = \mathbb{E}\Big[\mathbb{E}\big[G(\omega) \bigm| \omega_U\big]^2\Big].$$

We may write $g(x)$ as a sum

$$g(x) = \sum_{y \in \{-1,1\}^J} 1_{x_J = y}\, g_y(x_{J'})$$

with some functions $g_y : \{-1, 1\}^{J'} \to \{-1, 1\}$. Then

$$W_\eta(G, U) = \sum_y \nu_\eta(x_J = y)\, \nu_\eta[g_y]^2 \quad \underset{\eta \searrow 0}{\longrightarrow} \quad \sum_y \nu(x_J = y)\, \nu[g_y]^2.$$

(Here, $\nu[g_y]$ denotes the expectation of $g_y$ with respect to $\nu$, and similarly for $\nu_\eta$.) Hence, $W(G, U) := \lim_{\eta \searrow 0} W_\eta(G, U)$ exists.

By (2.7), we have

$$\big|\widehat{\mu}_\eta^{\mathcal{Q}}(\mathscr{S} \subset U) - W_\eta(G, U)\big| \le 4\, \mu_\eta(G \ne f_{\mathcal{Q}})^{1/2}\,.$$

For $\eta$ sufficiently small, the right hand side is smaller than $4\sqrt{\epsilon}$, by our choice of $g$. Since $W(G, U) = \lim_{\eta \searrow 0} W_\eta(G, U)$, we conclude that

$$\big|\widehat{\mu}_\eta^{\mathcal{Q}}(\mathscr{S} \subset U) - W(G, U)\big| < 5\,\sqrt{\epsilon}$$

for all $\eta$ sufficiently small. (But we cannot say that $\lim_{\eta\searrow 0}\widehat{\mu}^{\mathcal{Q}}_\eta(\mathscr{S}\subset U) = W(G,U)$, since $G$ depends on $\epsilon$.) This implies

$$\limsup_{\eta\searrow 0}\widehat{\mu}^{\mathcal{Q}}_\eta(\mathscr{S}\subset U) - \liminf_{\eta\searrow 0}\widehat{\mu}^{\mathcal{Q}}_\eta(\mathscr{S}\subset U) \le 10\sqrt{\epsilon}\,.$$

Since $\epsilon$ is an arbitrary positive number, this establishes the existence of the limit $W(\mathcal{Q},U)$. The proof of conformal invariance is similar, and left to the reader. □

We now describe some a.s. properties of the limiting law.

**Theorem 10.4.** *If $\mathscr{S}$ is a sample from $\widehat{\mu}^{\mathcal{Q}}$, then a.s. $\mathscr{S}$ is contained in the interior of $\mathcal{Q}$ and for every open $U\subset\mathbb{C}$, if $U\cap\mathscr{S}\neq\emptyset$, then $U\cap\mathscr{S}$ has Hausdorff dimension $3/4$. In particular, $\mathscr{S}$ is a.s. homeomorphic to a Cantor set, unless it is empty.*

**Proof.** It follows from (7.8) from Subsection 7.4 that $\mathscr{S}$ is $\widehat{\mu}^{\mathcal{Q}}$-a.s. contained in the interior of $\mathcal{Q}$. Now fix some open $U$ whose closure is contained in the interior of $\mathcal{Q}$. Fix $\eta>0$ and let $\mathscr{S}_\eta$ denote a sample from $\widehat{\mu}^{\mathcal{Q}}_\eta$. Let $\lambda_\eta$ denote the counting measure on $\mathscr{S}_\eta\cap U$ divided by $\eta^{-2}\,\alpha_4(1,1/\eta)$. We may consider $\lambda_\eta$ as a random point in the metric space of Borel measures on $\mathcal{Q}$ with the Prokhorov metric. By the estimate (3.6), we have $\limsup_{\eta\searrow 0}\mathbb{E}\big[\lambda_\eta(U)\big]<\infty$. Therefore, the law of $\lambda_\eta$ is tight as $\eta\searrow 0$. Likewise, the law of the pair $(\mathscr{S}_\eta,\lambda_\eta)$ is tight. Hence, there is a sequence $\eta_j\to 0$ such that the law of the pair $(\mathscr{S}_{\eta_j},\lambda_{\eta_j})$ converges weakly as $j\to\infty$. Let $(\mathscr{S},\lambda)$ denote a sample from the weak limit. Then $\lambda$ is a.s. a measure whose support is contained in $\mathscr{S}$.

Now let $B\subset U$ be a closed disk. Let $B'\subset B$ be a concentric open disk with smaller radius, and let $\delta$ denote the distance from $\partial B'$ to $\partial B$. Theorem 7.1 with $B'$ and $B$ in place of $U'$ and $U$ implies that $\lambda(B)>0$ a.s. on the event $\emptyset\neq\mathscr{S}\cap B\subset B'$. (Note that $\emptyset\neq\mathscr{S}\cap B\subset B'$ is an open condition on $\mathscr{S}$, since it is equivalent to having $\emptyset\neq\mathscr{S}\cap B'$ and $\mathscr{S}\cap(B\setminus B')=\emptyset$.) But $B\setminus B'$ can be covered by $O(\delta^{-1})$ disks of radius $\delta$. By (3.4) and (2.5), for each of these radius $\delta$ disks, the probability that $\mathscr{S}$ intersects it is $O(\delta^{5/4+o(1)})$. Therefore, $\mathbb{P}\big[\mathscr{S}\cap B\not\subset B'\big]=O(\delta^{1/4+o(1)})$. In particular, we have $\mathbb{P}\big[\mathscr{S}\cap B\neq\emptyset,\ \lambda(B)=0\big]=o(1)$ as $\delta\searrow 0$; that is, $\mathbb{P}\big[\mathscr{S}\cap B\neq\emptyset,\ \lambda(B)=0\big]=0$. By considering a countable collection of disks covering $U$ it follows that on $U\cap\mathscr{S}\neq\emptyset$ we have $\lambda(U)>0$ a.s. The correlation estimate (3.3) and the asymptotics $\alpha_4(r)=r^{-5/4+o(1)}$ from (2.5) imply that for $\eta > 0$ and every $s > -3/4$ we have

$$\mathbb{E}\Big[\int_U \int_U \big(|x-y| \vee \eta\big)^s \, d\lambda_\eta(x)\, d\lambda_\eta(y)\Big] = O(1)\,.$$

This implies that a.s.

$$\int_U \int_U |x-y|^s \, d\lambda(x)\, d\lambda(y) < \infty\,.$$

Therefore, Frostman's criterion implies that the Hausdorff dimension of $\mathscr{S}$ is a.s. at least $3/4$ on the event $\mathscr{S} \cap U \neq \emptyset$. By Lemma 3.2, the expected number of disks of radius $r$ needed to cover $\mathscr{S} \cap U$ is bounded by $r^{3/4+o(1)}$. Hence, the Hausdorff dimension of $U \cap \mathscr{S}$ is a.s. at most $3/4$ on the event $U \cap \mathscr{S} \neq \emptyset$. This proves the claim for any fixed $U$. The assertion for every $U$ then follows by considering a countable basis for the topology (i.e., disks having rational radius and centers with rational coordinates). □

*Remark* 10.5. It would be interesting to prove the weak convergence of the law of $(\mathscr{S}_\eta, \lambda_\eta)$ as $\eta \searrow 0$.

Note that the proof above shows that for any subsequential scaling limit $(\mathscr{S}, \lambda)$ of $(\mathscr{S}_\eta, \lambda_\eta)$, the support of the measure $\lambda$ a.s is the whole $\mathscr{S}$.

**Proof of Theorem 1.6.** Since $[0,R]^2$ is a square, we have $\mathbb{E}\big[f_R\big] \to 0$ a.s. Therefore $\mathbb{P}\big[\mathscr{S}_{f_R} = \emptyset\big] \to 0$. Consequently, the claims follow from Theorems 10.1 and 10.4. □

# 11 Some open problems

Here is a list of some questions and open problems:

1. For any Boolean function $f : \{-1,1\}^n \to \{-1,1\}$, define its **spectral entropy** $\mathrm{Ent}(f)$ to be

$$\mathrm{Ent}(f) = \sum_{S \subset \{1,\dots,n\}} \widehat{f}(S)^2 \log \frac{1}{\widehat{f}(S)^2}\,.$$

Friedgut and Kalai conjectured in [FK96] that there is some absolute constant $C > 0$ such that for any Boolean function $f$,

$$\mathrm{Ent}(f) \le C \sum_{S \subset \{1,\dots,n\}} \widehat{f}(S)^2 |S| = \mathbb{E}\big[|\mathscr{S}_f|\big];$$

in other words, that the spectral entropy is controlled by the total influence. As was pointed out to us by Gil Kalai, it is natural to test the conjecture in the setting of percolation: if $f_R$ is the $\pm 1$-indicator function of the left-right crossing in the square $[0, R]^2$, is it true that $\mathrm{Ent}(f_R) = O(R^2 \alpha_4(R))$?

2. Our paper deals with noise sensitivity of percolation and its applications to dynamical percolation. One could ask similar questions about the Ising model, for which a natural dynamics is the Glauber dynamics. For instance, Broman and Steif ask in [BS06, Question 1.8] if there exist exceptional times for the Ising model on $\mathbb{Z}^2$ at $\beta = \beta_c$ for which there is an infinite up-spin cluster. Since $SLE_3$ (which is supposedly the scaling limit of critical Ising interfaces, see Smirnov's recent breakthrough [Smi06]) does not have double points, there should be very few pivotals, and thus such exceptional times should not exist, but the missing argument is a quasi-multiplicativity property for the probabilities of the alternating 4-arm events in the Ising model. Similar questions can be asked for the FK model, Potts models, etc.

3. Prove the weak convergence of the law of $(\mathscr{S}_\eta, \lambda_\eta)$; see Remark 10.5. In a forthcoming paper [GPS], we will prove the weak convergence of the law of $(\mathscr{P}_\eta, \tilde{\lambda}_\eta)$, where $\tilde{\lambda}_\eta$ is the counting measure on the set of pivotals renormalized by $\eta^{-2}\alpha_4(1, 1/\eta)$.

4. Prove that $\mathscr{P}_R$ and $\mathscr{S}_R$ asymptotically have the same "polar sets". See Remark 8.6 for a more precise description.

5. Prove that the laws of $\mathscr{P}_R$ and $\mathscr{S}_R$ are asymptotically mutually singular, or that their scaling limits are singular. Remark 4.6 suggests that this should be the case, since both these sets should be statistically self similar, in some sense.

6. Do we have $\mathbb{E}\big[|\mathscr{S}_{R\mathcal{Q}}|\big] = \mathbb{E}\big[|\mathscr{P}_{R\mathcal{Q}}|\big] \asymp R^2\alpha_4(R)$ for any quad $\mathcal{Q} \subset \mathbb{C}$, as $R$ goes to infinity? (See Proposition 7.4).

7. In the same fashion, prove that the sharp concentration as in Theorem 1.1 still holds for general quads. With our techniques, this would require a uniform control over the domain on the constants involved in Proposition 5.11, as well as a statement analogous to Proposition 4.1 for the case of general quads.

8. Prove that the main statement in Section 5 (Proposition 5.1) still holds for non-monotone functions such as the $\ell$-arm annulus crossing events. (See Subsection 5.3 for an explanation why we needed the monotonicity assumption for the first moment.) If such a generalization was proved, then it would imply in particular that for the triangular lattice the set of exceptional times with both infinite black and white clusters has dimension 2/3 a.s., strengthening the last statement in Theorem 1.4.

   For $\ell > 1$, there is a further small complication when $\ell$ is odd: a bit can be pivotal for the $\ell$-arm event even without having the exact 4-arm event around its tile. (That is why we restricted Proposition 4.7 to the $\ell \in \{1\} \cup 2\mathbb{N}_+$ case.) Resolving this technicality and the non-monotonicity problem would imply the existence of exceptional times where there are polychromatic three arms from 0 to infinity (the dimension of this set of exceptional times would then be 1/9). We cannot prove the existence of such times with the results of the present paper.

9. Let us conclude with a computational problem: find any "efficient" algorithmic way to sample $\mathscr{S}$ in the case of percolation, say (in order, for instance, to make pictures of it), or prove that such an algorithm does not exist.

# 12 Appendix: an inequality for multi-arm probabilities

We prove here an estimate regarding the multi-arm crossing probabilities for annuli in critical bond percolation on $\mathbb{Z}^2$, which is due to Vincent Beffara (private communication) and included here with his permission.

**Proposition 12.1.** *Fix $k \in \mathbb{N}_+$ and consider bond percolation on $\mathbb{Z}^2$ with parameter $p = 1/2$. There are constants, $C, \epsilon > 0$, which may depend on $k$, such that for all $1 < r < R$,*

$$\alpha_{2k+1}(r,R) \le C\,\alpha_1(r,R)\,\alpha_{2k}(r,R)\,(r/R)^{\epsilon}. \qquad (12.1)$$

The method of proof can be generalized to give a few similar results. However, new ideas seem to be necessary for the corresponding statement with $k = 1/2$. The case $k = 1/2$ is of particular significance: it was proved in [SS05] that it implies the existence of exceptional times for Bernoulli(1/2) bond percolation on $\mathbb{Z}^2$. In the present paper we prove their existence using (12.1) instead.

**Proof.** For simplicity of notation, we will restrict the proof to the case $k = 2$, which is the case we need, but the proof very easily carries over to the general case. Let $A(r, R)$ denote the annulus which is the closure of $B(0, R) \setminus B(0, r)$. If we have the 4-arm event in $A(r, R)$, i.e., four crossings of alternating colors between the two boundary components of $A(r, R)$, with white (primal) and black (dual) colors on the tiles given in Subsection 2.1, then there are at least 4 interfaces $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ separating these clusters. These interfaces are simple paths on the grid $\mathbb{Z}^2 + (1/4, 1/4)$ in $A(r, R)$, and each of them has one point on each boundary component $A(r, R)$.

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ be a triple of 3 simple paths that can arise as 3 consecutive interfaces in cyclic order. Let $\mathcal{A}_\gamma$ denote the event that these are actual interfaces between crossing clusters. Let $S := S_\gamma$ denote the connected component of $A(r, R) \setminus (\gamma_1 \cup \gamma_3)$ that does not contain $\gamma_2$, and let $\mathcal{B}_\gamma$ denote the event that $\mathcal{A}_\gamma$ occurs and there are at least two disjoint primal crossings in $S$. Our first goal is to prove that

$$\mathbb{P}\big[\mathcal{B}_\gamma \bigm| \mathcal{A}_\gamma\big] \le O(1)\, \alpha_1(r, R)\, (r/R)^\epsilon, \tag{12.2}$$

with some constant $\epsilon > 0$.

Note that on $\mathcal{A}_\gamma$, we have in $S$ at least one primal crossing and at least one dual crossing, which are adjacent to $\gamma_1$ and $\gamma_3$. For the sake of definiteness, we will assume that the primal crossing is adjacent to $\gamma_1$ and the dual crossing is adjacent to $\gamma_3$. (This can be determined from $\gamma$.) Let $S' = S'_\gamma$ denote the set of edges in $S$ that are not adjacent to $\gamma_1$. Then given $\mathcal{A}_\gamma$, we have $\mathcal{B}_\gamma$ if and only if there is a primal crossing also in $S'$. Therefore, (12.2) follows once we show that for every such $\gamma$ the probability that there is a crossing in $S'$ is bounded by the right hand side of (12.2).

We may consider a percolation configuration $\omega$ in the whole plane and also restrict it to $S'$. Let $\omega_\rho$ denote the restriction of $\omega$ to $B(0, \rho)$, for any $\rho \in [r, R]$. Write $\rho \leftrightarrow \rho'$ for the event that there is a crossing of $\omega$ between the two boundary component of the annulus $A(\rho, \rho')$, and write $\rho \overset{D}{\leftrightarrow} \rho'$ for the existence of such a crossing within some specified set $D$. We will prove that for some constant $a \in (0, 1)$ and every $\rho$ and $\rho'$ satisfying $r \vee 100 \le \rho \le \rho'/8 \le R/8$ we have

$$\mathbb{P}\big[r \overset{S'}{\leftrightarrow} \rho' \bigm| \omega_\rho,\, r \leftrightarrow R\big] \le a\,. \tag{12.3}$$

Using induction, this implies (12.2) with $\epsilon = \log_8(1/a)$.

The interface $\gamma_1$ crosses the annulus $A(3\,\rho, 4\,\rho)$ one or more times. Let $w$ denote the winding number around 0 of one of these crossings, that is, the signed change of the argument along the crossing divided by $2\,\pi$. Suppose that $\beta$ is a simple path in $A(3\,\rho, 4\,\rho)$ with one endpoint on each boundary component of the annulus and let $w_\beta$ denote the winding number of $\beta$. If $\beta \cap \gamma_1 = \emptyset$, then we may adjoin to $\beta \cup \gamma_1$ two arcs on the boundary components of the annulus to form a simple closed curve which has winding number in $\{0, \pm 1\}$. Therefore, we see that $|w - w_\beta| > 3$ implies $\beta \cap \gamma_1 \neq \emptyset$.

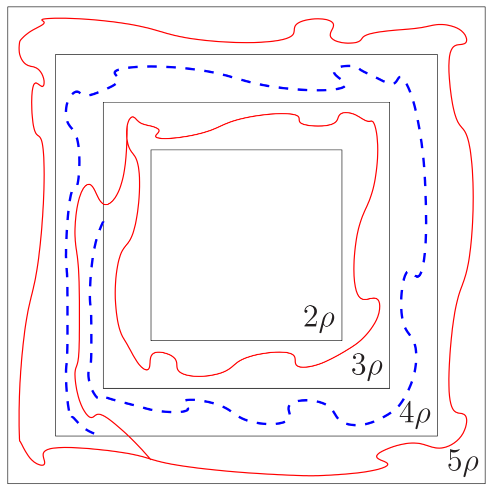



Figure 12.1: The event $\mathcal{D}_j$ with $j = 1$.

Let $\mathcal{D}_j$ denote the event that there is a dual crossing in $\omega$ of $A(3\,\rho, 4\,\rho)$ with winding number in the range $[j - 1/2, j + 1/2]$, and there are primal circuits in $A(2\,\rho, 3\,\rho)$ and in $A(4\,\rho, 5\,\rho)$, each of them separating the two boundary component of its annulus, and these primal circuits are connected to each other in $\omega$; see Figure 12.1. By the RSW theorem, there is constant $\delta > 0$ such that $\mathbb{P}\big[\mathcal{D}_{\pm 10}\big] \geq \delta$. We claim that

$$\mathbb{P}\big[\mathcal{D}_{\pm 10} \bigm| r \leftrightarrow R,\, \omega_\rho\big] \geq \delta\,. \tag{12.4}$$

If we condition of $\mathcal{D}_{10}$, and we further condition on the outermost primal circuit $\alpha_0$ in $A(2\,\rho, 3\,\rho)$ and on the innermost primal circuit $\alpha_1$ in $A(4\,\rho, 5\,\rho)$, then the configuration inside $\alpha_0$ and the configuration outside $\alpha_1$ remains unbiased, and if additionally $\alpha_0$ is connected to the inner boundary component of $A(r,R)$ and $\alpha_1$ is connected to the outer boundary component of $A(r,R)$, then we also have $r \leftrightarrow R$. This implies $\mathbb{P}\big[r \leftrightarrow R \bigm| \omega_\rho,\, \mathcal{D}_{10}\big] \geq \mathbb{P}\big[r \leftrightarrow R \bigm| \omega_\rho\big]$. The same holds for $\mathcal{D}_{-10}$, and since $\mathcal{D}_{\pm 10}$ is independent of $\omega_\rho$, the inequality (12.4) easily follows.

Now note that if $\mathcal{D}_j$ holds, then every primal crossing of $A(3\,\rho, 4\,\rho)$ in $\omega$ is with winding number in the range $[j-4, j+4]$. Hence, if $|j-w| > 7$, then $\mathcal{D}_j \cap \{3\,\rho \overset{S'}{\leftrightarrow} 4\,\rho\} = \emptyset$. Consequently, at least one of the two events $\mathcal{D}_{\pm 10}$ is disjoint from $\{\rho \overset{S'}{\leftrightarrow} 4\,\rho\}$. This gives (12.3) with $a := 1-\delta$. As we have argued before, (12.2) follows.

The 5-arm crossing event is certainly contained in $\bigcup_\gamma \mathbb{P}\big[\mathcal{B}_\gamma\big]$, where the union ranges over all $\gamma$ as above. Hence, (12.2) gives

$$\alpha_5(r,R) \leq O(1)\,(r/R)^\epsilon\,\alpha_1(r,R)\,\mathbb{E}\Big[\sum_\gamma 1_{\mathcal{A}_\gamma}\Big]. \tag{12.5}$$

If $X$ is the number of interfaces crossing the annulus $A(r,R)$ (which is necessarily even), then $\sum_\gamma 1_{\mathcal{A}_\gamma}$ is bounded by $X^3\,1_{X\geq 4}$. Since for all $j \in \mathbb{N}$ we have $\mathbb{P}\big[X \geq j\big] \leq O(1)\,(r/R)^{j\,\epsilon_0}$ with some constant $\epsilon_0 > 0$ (by RSW and BK) and $\mathbb{P}\big[X \geq 4\big] \geq (r/R)^{\epsilon_1}/O(1)$ for some $\epsilon_1 \in (0,\infty)$, we have $\mathbb{E}\big[X^3\,1_{X\geq 4}\big] \leq O(1)\,\mathbb{P}\big[X \geq 4\big]$ when $R > 2\,r$, say. Therefore,

$$\mathbb{E}\Big[\sum_\gamma 1_{\mathcal{A}_\gamma}\Big] \leq \mathbb{E}\big[X^3\,1_{X\geq 4}\big] \leq O(1)\,\mathbb{P}\big[X \geq 4\big] = O(1)\,\alpha_4(r,R)\,.$$

When combined with (12.5), this proves the proposition in the case $k=2$. The general case is similarly obtained. ☐

# Chapter VI

# Scaling limit of near-critical and dynamical percolation

Ongoing project with *Gábor Pete* and *Oded Schramm*.

This chapter is about an ongoing project where we plan to prove that near-critical percolation and dynamical percolation, properly renormalized, have a scaling limit. We will provide here two theorems (of independent interest) which as we will explain, will constitute key steps in the larger project.

# 1 Introduction

As we motivated in the global introduction, near-critical percolation is a way to extend our understanding of the critical point to the properties of percolation in the "neighborhood" of the phase transition.

For simplicity, in this chapter we will mostly restrict our study to the case of the triangular grid. Some partial results remain valid for $\mathbb{Z}^2$ (for instance concerning subsequential scaling limits of $\mathbb{Z}^2$ percolation but we will not discuss these in detail here). The mesh of the grid will be denoted by $\eta$ (rather than $1/n$ in the global introduction), so our configurations will be on the grids $\eta\mathbb{T}$.

In order to make our models of dynamical percolation and near-critical percolation look more similar, we will define our coupling of near-critical configurations in a slightly different way as we did in the (global) introduction (where we defined a coupling $(\hat{\omega}^n_\lambda)_{\lambda\in\mathbb{R}}$, by "slowing down" the standard coupling on percolation configurations).

More precisely, for any $\eta > 0$, let $(\omega_\eta(\lambda))_{\lambda\in\mathbb{R}}$ be the coupling of configurations defined as follows: start with a critical site percolation $\omega_\eta(0)$ on the rescaled grid $\eta\mathbb{T}$, and along the "time dynamic" $\lambda > 0$, independently for each site $x \in \eta\mathbb{T}$, let $x$ switch from closed to open at rate $q_\eta := \eta^2\alpha_4(\eta, 1)^{-1}$; as well, along the negative axis, let each site independently switch from open to closed at the same rate $q_\eta$. This means that for any fixed parameter $\lambda \in \mathbb{R}$, the configuration $\omega_\eta(\lambda)$ follows the law of i.i.d site percolation on the rescaled grid $\eta\mathbb{T}$, with parameter

$$p_\eta(\lambda) := \frac{1}{2} + \frac{1}{2}(1 - \exp(-\lambda\eta^2\alpha_4(\eta, 1)^{-1}) \sim \frac{1}{2} + \frac{1}{2}\lambda\,\eta^2\alpha_4(\eta, 1)^{-1}.$$

So we are indeed in the near-critical regime we introduced in the global introduction. It is straightforward to check that both descriptions are equivalent at the limit $\eta \to 0$ (up to a factor of 2 in our definitions).

We pan to prove that these near-critical percolation configurations have a scaling limit when the mesh $\eta$ goes to zero. Since we do not focus on a single interface but rather on the whole percolation process, we need to have a good way to "look at" our configurations; a way that would remain significant in the limit. Indeed, at the scaling limit, we will not have "bits" any more, so what are we willing to keep ? There are several possibilities here, each having there own advantages.

- In [Aiz95], Aizenman suggested to use the random set of all open paths of the percolation configuration (and to use tightness arguments to control this rather wild set).

- In [CN06], Camia and Newman suggested to keep all the interface loops around the clusters, and proved that "seen as a set of nested loops", discrete percolation configurations have a scaling limit when the mesh $\eta$ goes to zero, to a continuum percolation which consists of nested $\mathrm{SLE}_6$ loops. This is a rather natural generalization of the description of one interface via $\mathrm{SLE}_6$.

- In [SS], Schramm and Smirnov, inspired by Tsirelson's theory of noises, suggested to keep the information about connectivity by considering the random set consisting of all the quads (or tubes) that are crossed by the configuration. They prove that percolation has a scaling limit in that sense (with the help of elegant topological arguments), and that it leads to a noise in the sense of Tsirelson.

Of course all these "descriptions" are related to each other. A common point in all these choices is that one keeps only "macroscopic" information. Depending on the problem one is facing, a setup might be more adapted than some other.

For the particular purpose of our work, the third one seems more appropriate to us. Indeed, since it is intimately related with "noising" a continuum percolation, is is natural to follow Tsirelson's approach: knowing that continuum percolation can be seen as a noise means that the filtration of the continuum percolation process factorizes well (which is obvious on the discrete level but very non trivial once at the scaling limit). In particular, one can resample the continuum configuration "box by box" (say for small boxes of sidelength $\epsilon$) at some rate as one does with discrete percolation. For instance, using this procedure, Tsirelson defines a generalized Ornstein-Uhlenbeck operator (note though that in our case of a black noise, that operator is trivial). An other reason being that if we want to prove, say, ergodicity of dynamical percolation at the scaling limit, with the third setup, it boils down to proving asymptotic noise sensitivity of crossing events, and it is easier to read noise sensitivity using the Spectrum measure of some crossing event rather than some property about interfaces for instance.

Therefore we will define in section 3 a topology $\mathcal{T}$ on the space $\mathscr{H}$ of all percolation configurations (which includes both discrete and continuum type of percolation configurations). It is proved in [SS] that the topological space $(\mathscr{H}, \mathcal{T})$ is compact.

In the larger project, we plan to prove that near-critical models have a scaling limit. More exactly

**Theorem 1.1.** *For any fixed level $\lambda \neq 0$, near-critical percolation configurations $\omega_\eta(\lambda)$ on $\eta\mathbb{T}$ have a scaling limit when $\eta \to 0$ (under the topology $\mathcal{T}$ on the space of configurations $\mathscr{H}$).*

From [NW08], we know that in our limiting model, the interfaces between the clusters are singular with respect to $\mathrm{SLE}_6$ (this was proved to hold for any subsequential limit in [NW08] and therefore holds for the (unique) scaling limit above). In particular, the scaling limit in the above theorem 1.1 is different from the scaling limit at criticality (scaling limit of $\omega_\eta(0)$).

We plan to prove more than the "fixed-level" near-critical percolation scaling limit. Indeed one can consider the process $(\omega_\eta(\lambda))_{\lambda\in\mathbb{R}}$ as a cádlág process $\mathbb{R} \to \mathbb{H}$. We equip the space of such processes with the topology of locally uniform convergence that we denote $\widehat{\mathcal{T}}$. We plan to prove

**Theorem 1.2.** *The rescaled family of (monotone) configurations $(\omega_\eta(\lambda))_{\lambda\in\mathbb{R}}$ seen as random cádlág processes on $\mathscr{H}$ have a scaling limit when $\eta \to 0$. They converge in law under the topology of locally uniform convergence ($\widehat{\mathcal{T}}$) to a (monotone) family of continuum percolations $(\omega(\lambda))_{\lambda\in\mathbb{R}}$.*

Using the same approach, one can prove that dynamical percolation has a scaling limit (dynamical percolation scaling limit was also among the conjectures in [CFN06]). As was explained in the global introduction, when we rescale the lattice ($\eta$ goes to zero), because of the noise sensitivity of percolation, on has to slow down the time. The right scaling (up to constant) is as follows: Let $(\omega_\eta(t))_{t\geq 0}$ be a dynamical percolation on $\eta\mathbb{T}$, where each site $x \in \eta\mathbb{T}$ is updated according to a Poisson clock of rate $q_\eta := \eta^2 \alpha_4(\eta, 1)^{-1} = \eta^{3/4+o(1)}$. We plan to prove

**Theorem 1.3.** *The rescaled dynamical percolation processes $(\omega_\eta(t))_{t\geq 0}$ seen as random cádlág processes on $\mathscr{H}$ have a scaling limit when $\eta \to 0$. They converge in law under the topology of locally uniform convergence ($\widehat{\mathcal{T}}$) to a continuum dynamical percolation $(\omega(t))_{t\geq 0}$ (or $(\omega_t)_{t\geq 0}$).*

Let us now explain how we plan to prove these results and what are the steps we include in this chapter. Our project partly follows (and prove) a program developed by Camia, Fontes and Newman in [CFN06]. They had the idea to build the limiting coupling of near-critical configurations $(\omega(\lambda))_{\lambda\in\mathbb{R}}$ out of the critical "slice" $\omega(0)$. To sample $\omega(\lambda)$ (for some $\lambda > 0$, say) using $\omega(0)$, many "sites" should switch (in a random way) from closed to open; the trouble is: there are no sites anymore in the limit, except somehow the pivotal points (still "visible" at the scaling limit). In [CFN06], the authors explain that it should be enough in principle to follow the status of these "important" points. Assuming it is enough, we still have to sample which points among the set of all important points will switch from the $\omega(0)$ to the $\omega(\lambda)$ configuration. If we look at the set of all pivotal points (at the scaling limit), it is easy to see that it is a.s. a dense set of dimension 3/4, and one can check that infinitely (though countably many) of these sites will switch from closed to open. If one starts with some configuration $\omega(0)$ and that infinitely many of its pivotal points switch, it is hard to "read" what will be the configuration $\omega(\lambda)$.

This is why we need to introduce some "cut-off"; a natural one is to consider for any small $\epsilon > 0$, the set of pivotal points which are important at least up to distance $\epsilon$ (i.e. the set of points $x$ which satisfy a four-arms event in $B(x,\epsilon)$); let $\mathcal{P}_\epsilon$ denote this random set of $\epsilon$-important points in $\mathbb{C}$. This is still some set of dimension 3/4, but at least is not dense and has nice random "geometries". More importantly it turns out that on any compact set, only finitely many points in $\mathcal{P}_\epsilon$ will switch from closed to open.

The counterpart of working with such a cut-off is that not only we need to prove that it is enough to follow the status of pivotal points (as explained above), but we need more, i.e. to prove that if we only follow the status of the $\epsilon$-important points then we can predict with good accuracy (when $\epsilon \to 0$) what is the configuration $\omega(\lambda)$. Of course following only the $\epsilon$-important points will not allow us to predict the small scales connectivities of $\omega(\lambda)$, but if for any $\delta > 0$, we can predict with high accuracy when $\epsilon \to 0$ all the connectivity properties of $\omega(\lambda)$ up to a precision of $\delta$, then we are fine.

Let us return to the discrete picture to gain some intuition. If $\eta$ is some small mesh, let $X_\eta^\epsilon$ be the number of $\epsilon$-important closed points for $\omega_\eta(0)$ in the square $[0,1]^2$. $X_\eta^\epsilon$ is of order $O(1)\eta^{-2}\alpha_4(\eta,\epsilon)$. Each point among these $X_\eta^\epsilon$ points independently switches from closed to open with probability $1-\exp(-\lambda\eta^2\alpha_4(\eta,1)^{-1}) \sim \lambda\eta^2\alpha_4(\epsilon,1)^{-1}$. Therefore the total number of switches in the unit square is well approximated (when $\eta \to 0$) by a Poisson variable of parameter $\lambda \frac{X_\eta^\epsilon}{\eta^{-2}\alpha_4(\eta,1)}$. Hence as was argued in [CFN06], the pivotal points which switch from $\omega(0)$ to $\omega(\lambda)$ correspond to a certain "Poissonian" cloud over the set of pivotal points, according to some limiting counting measure on this set of pivotal points.

We summarize the program/strategy as follows:

- Prove that in order to predict with good accuracy what is the configuration $\omega(\lambda)$, it is enough to follow the status of pivotal points (at the scaling limit) which are at least $\epsilon$-important for the configuration $\omega(0)$ ($\epsilon$ being chosen sufficiently small depending on the accuracy we want). Furthermore we need a discrete version of this statement in order to prove the scaling limit of our discrete configurations to the continuous one.

- In order to sample which points among the $\epsilon$-important points (in $\omega(0)$) will switch, define a natural "counting" measure on the set $\mathcal{P}_\epsilon$ of all $\epsilon$-important points of $\omega(0)$. This measure should be the scaling limit of the normalized counting measures on the set of $\epsilon$-important points for the $\omega_\eta(0)$-configurations.

The same program applies for the scaling limit of dynamical percolation $(\omega_t)_{t\geq 0}$.

Let us first explain the issues behind the first item and the result we prove. What we need to rule out if one follows only points which initially were at least $\epsilon$-importants is what we might call "cascade of importance". For instance, it could be that some point in $\omega(0)$ is of very small importance (for $\omega(0)$), but that under the $\lambda$-"evolution" it gets "promoted" to a much larger importance. Not only it could, but it actually happens everywhere! Indeed each time a "big" pivotal switches, there are many points around which are promoted while many others are retrograded. So we cannot avoid such sharp changes of importance. Fortunately for us, even though some points that we are not looking at (were less than $\epsilon$-important in $\omega(0)$) get promoted, in order to affect our prediction of $\omega(\lambda)$, they would also need to switch their status. And we will prove that it is very unlikely that between level 0 and level $\lambda$, there is some point of initial ($\omega(0)$) very small importance which gets promoted and also switches its status between $\omega(0)$ and $\omega(\lambda)$. If we prove that such points do not exist with high probability, then it will be possible with good accuracy to predict things following only the $\epsilon$-important points.

This is essentially the statement we will prove. Since its statement requires some additional notations, we refer to section 2 and in particular to Lemma 2.2 and Proposition 2.3 for more precise statements.

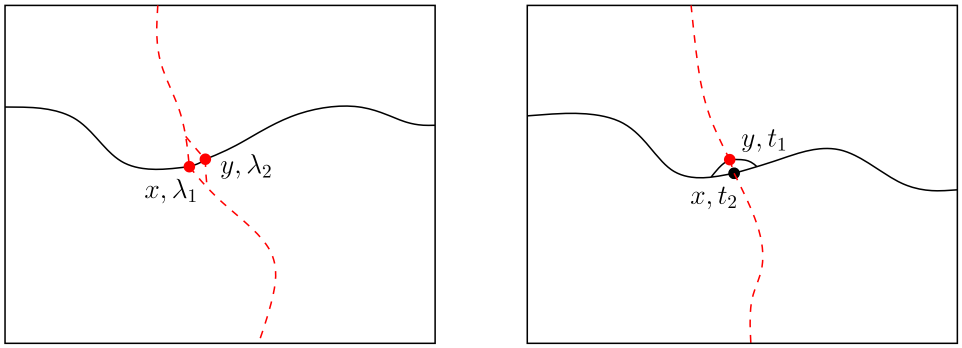



Figure 1.1: Two "cascade" configurations: on the left at $\lambda = 0$, there is no Left-Right crossing and both points $x$ and $y$ have low importance but at the level $\lambda_2 > \lambda_1$, there is a left-right crossing that we could not predict if we are not looking at low important points. Similarly, on the right, with $t_1 < t_2$, the low important point $y$ switches first then followed by the important one. If one does not look at low important points one would wrongly predict that the left-right crossing stops to occur, while it still does thanks to $y$. Note that the second configuration could occur only for dynamical percolation which does not have monotonicity in its dynamic.

The pictures 1.1 represent situations both for near-critical and dynamical percolation where cascade of importance would prevent us from predicting with good accuracy the outcome $\omega(\lambda)$ or $\omega_t$.

These pictures represent the extreme case where one has some point $x$ of macroscopic ($O(1)$) importance, as well as a closeby point $y$, initially of importance less than $\epsilon$ but promoted in between, both of them switching their status between levels 0 and $\lambda$ (or between times 0 and $t$). Let us estimate what is the probability of this extreme "bad" situation. In the square $[0, 1]^2$, there are $O(1)\eta^{-2} \times \epsilon^2\eta^{-2}$ ways to choose the pair $(x, y)$. Now $x$ and $y$ are respectively 1-important and $\epsilon$-important (to be rigorous, one should sum over the distances and thus importances $2^k\eta, 1 \leq k \leq \log_2(\epsilon/\eta)$ between $x$ and $y$) with probability $O(1)\alpha_4(\eta, 1) \times \alpha_4(\eta, \epsilon)$. Finally $x$ and $y$ both switch their status with probability $O(1)\eta^4\alpha_4(\eta, 1)^{-2}$. All together, the probability of the above situation is bounded by $O(1)\epsilon^2\alpha_4(\epsilon, 1)^{-1}$ (here we used quasimultiplicativity property) and this goes to zero when $\epsilon$ goes to zero, uniformly in $\eta < \epsilon$.

Of course this example is just an extreme case, and we need to control all types of multi-cascades which could occur at different scales.

Now for the second statement, for any $\epsilon > 0$, we consider the (random) counting measure $\mu_\eta^\epsilon$ on the $\epsilon$-important points normalized by $\eta^2 \alpha_4(\eta, 1)^{-1}$. Hence $\mu_\eta^\epsilon$ is defined as follows

$$\mu_\eta^\epsilon = \mu_\eta^\epsilon(\omega_\eta) = \sum_{x \in \eta\mathbb{T} \text{ is } \epsilon\text{-important}} \delta_x \eta^2 \alpha_4(\eta, 1)^{-1}.$$

We will prove the following Theorem

**Theorem 1.4.** *When $\eta \to 0$, the random variable $(\omega_\eta, \mu_\eta^\epsilon)$ converges in law to some $(\omega, \mu^\epsilon)$, where $\omega$ is the scaling limit of critical percolation, and the Borel measure $\mu^\epsilon = \mu^\epsilon(\omega)$ is a measurable function of $\omega$.*

We will provide a proof for both theorems. For the complete proof of the scaling limit, one would still need to "bring pieces" together, which still requires some further work. Nevertheless we hope to convince the reader with these two results, that the program is now close from completion.

The remaining of the chapter is structured as follows: in section 2 we prove the results which show that there are no "cascade of importance". In section 3 we describe the setup of the scaling limit (i.e. the topological space $(\mathscr{H}, \mathcal{T})$). Then in section 4 we provide a coupling result which will be needed in the proof of convergence of the discrete counting measures. Section 5 provides the proof of this convergence. Finally section 6 derives the conformal covariance properties of this limiting measure.

# 2 Stability

Given a percolation configuration $\omega$ and a site $z$, let $Z(z) = Z_\omega(z)$ denote the maximal radius $r$ such that the four arms event holds from the hexagon of $z$ to distance $r$ away. This is also the maximum $r$ for which changing the value of $\omega(z)$ will change the white connectivity in $\omega$ between two white points at distance $r$ away from $z$, or will change the black connectivity between two black points at distance $r$ away from $z$. The quantity $Z(z)$ will also be called the **importance** of $z$ in $\omega$.

Fix $\lambda \geq 0$. Let $X = X_\lambda$ be a random collection of sites independent from the percolation configuration $\omega$, where each site is in $X$ with probability $q_\lambda$ independently, and $q_\lambda$ is chosen so that the expected number of 1-important points in $X \cap [0,1]^2$ is $\lambda$. (This only works for $\eta$ smaller than some positive constant depending on $\lambda$.) Note that $q_\lambda$ is of order $\lambda\,\eta^2\,\alpha_4(1)^{-1}$. Let $\Omega(\omega, X)$ denote the set of percolation configurations $\omega'$ such that $\omega'(x) = \omega(x)$ for all $x \notin X$. Let $\mathcal{A}_4(z, r, r')$ denote the 4-arm event in the annulus $A(z, r, r')$.

**Lemma 2.1.** *Set $r_i := 2^i\,\eta$, $N := \lfloor \log_2(1/\eta) \rfloor$. Let $\mathcal{W}_z(i,j)$ denote the event that there is some $\omega' \in \Omega(\omega, X)$ satisfying $\mathcal{A}_4(z, r_i, r_j)$. Then for every integers $i, j$ satisfying $0 \leq i < j < N$ and every $z \in \mathbb{R}^2$*

$$\mathbb{P}\big[\mathcal{W}_z(i,j)\big] \leq C_1\,\alpha_4(r_i, r_j)\,, \tag{2.1}$$

*where $C_1 = C_1(\lambda)$ is a constant that may depend only on $\lambda$.*

**Proof.** Let $\mathcal{D}$ denote the event that $\omega$ does not satisfy $\mathcal{A}_4(z, r_{i+1}, r_{j-1})$. Suppose that $\mathcal{W}_z(i,j) \cap \mathcal{D}$ holds, and let $\omega' \in \Omega(\omega, X)$ satisfy $\mathcal{A}_4(z, r_i, r_j)$. Let $Y_0 := X \setminus A(z, r_{i+1}, r_{j-1})$, and let $\{x_1, x_2, \ldots, x_m\}$ be some ordering of $X \cap A(z, r_{i+1}, r_{j-1})$. Let $Y_k = Y_0 \cup \{x_1, x_2, \ldots, x_k\}$, $k = 1, 2, \ldots, m$, and let $\omega_k$ be the configuration that agrees with $\omega'$ on $Y_k$ and is equal to $\omega$ elsewhere. Then $\omega_0$ does not satisfy $\mathcal{A}_4(z, r_{i+1}, r_{j-1})$, and therefore also does not satisfy $\mathcal{A}_4(z, r_i, r_j)$. On the other hand, $\omega_m = \omega'$ satisfies $\mathcal{A}_4(z, r_i, r_j)$. Let $q \in \{1, 2, \ldots, m\}$ be minimal with the property that $\mathcal{A}_4(z, r_i, r_j)$ holds in $\omega_q$, and let $n \in \mathbb{N} \cap [i+1, j-2]$ be chosen so that $x_q \in A(z, r_n, r_{n+1})$. Then $x_q$ is pivotal in $\omega_q$ for $\mathcal{A}_4(z, r_i, r_j)$. Since $B(x_q, r_{n-1}) \subset A(z, r_i, r_j)$, this implies that $\omega_q$ satisfies $\mathcal{A}_4(x_q, 2\,\eta, r_{n-1})$. Hence, we get the bound

$$\mathbb{P}\big[\mathcal{W}_z(i,j),\, \mathcal{D}\big] \leq \sum_{n=i+1}^{j-2} \sum_{x \in A(z, r_n, r_{n+1})} \mathbb{P}\big[x \in X,\, \mathcal{W}_x(1, n-1),\, \mathcal{W}_z(i,j)\big].$$

Since $\mathcal{W}_z(i,j) \subset \mathcal{W}_z(i, n-1) \cap \mathcal{W}_z(n+2, j)$ and since $B(x, r_{n-1}) \subset A(z, r_{n-1}, r_{n+2})$, independence on disjoint sets gives

$$\begin{aligned}
&\mathbb{P}\big[x \in X,\, \mathcal{W}_x(1, n-1),\, \mathcal{W}_z(i,j)\big] \\
&\qquad \leq \mathbb{P}\big[x \in X,\, \mathcal{W}_x(1, n-1),\, \mathcal{W}_z(i, n-1),\, \mathcal{W}_z(n+2, j)\big] \\
&\qquad\quad = \mathbb{P}[x \in X]\,\mathbb{P}\big[\mathcal{W}_x(1, n-1)\big]\mathbb{P}\big[\mathcal{W}_z(i, n-1)\big]\,\mathbb{P}\big[\mathcal{W}_z(n+2, j)\big].
\end{aligned}$$

Now set $b_i^j := \sup_z \mathbb{P}\big[\mathcal{W}_z(i,j)\big]$. The above gives

$$\mathbb{P}\big[\mathcal{W}_z(i,j),\,\mathcal{D}\big] \le O(\lambda) \sum_{n=i+1}^{j-2} (r_n/\eta)^2\,\eta^2\,\alpha_4(1)^{-1}\,b_1^{n-1}\,b_i^{n-1}\,b_{n+2}^j\,.$$

Since $\mathbb{P}\big[\mathcal{W}_z(i,j)\big] \le \mathbb{P}\big[\neg\mathcal{D}\big] + \mathbb{P}\big[\mathcal{W}_z(i,j),\,\mathcal{D})\big]$, The above shows that for some absolute constant $C_0 > 0$, we have

$$\begin{aligned} b_i^j/C_0 &\le \alpha_4(r_i,r_j) + \lambda \sum_{n=i+1}^{j-2} r_n^2\,\alpha_4(1)^{-1}\,b_1^{n-1}\,b_i^{n-1}\,b_{n+2}^j \\ &\le \alpha_4(r_i,r_j) + \lambda\,\alpha_4(1)^{-1} \sum_{n=i+1}^{j-1} r_n^2\,b_1^{n-1}\,b_i^{n-1}\,b_{n+2}^j. \end{aligned} \tag{2.2}$$

We now claim that (2.1) holds with some fixed constant $C_1 = C_1(\lambda)$, to be later determined. This will be proved by induction on $j$, and for a fixed $j$ by induction on $j-i$. In the case where $j-i \le 5$, say, this can be guaranteed by an appropriate choice of $C_1$. Therefore, assume that the claim holds for all smaller $j$ and for the same $j$ with all larger $i$. The inductive hypothesis can be applied to estimate the right hand side of (2.2), to yield

$$\begin{aligned} b_i^j \le{}& C_0\,\alpha_4(r_i,r_j) + \\ &+ \lambda\,C_0\,C_1^3\,\alpha_4(1)^{-1} \sum_{n=i+1}^{j-1} r_n^2\,\alpha_4(r_1,r_{n-1})\,\alpha_4(r_i,r_{n-1})\,\alpha_4(r_{n+2},r_j)\,. \end{aligned}$$

By the familiar multiplicative properties of $\alpha_4$, we obtain

$$b_i^j \le C_2\,\alpha_4(r_i,r_j)\Big(1 + \lambda\,C_1^3 \sum_{n=i+1}^{j-1} \frac{r_n^2}{\alpha_4(r_n,1)}\Big), \tag{2.3}$$

for some constant $C_2$. Since $O(1)\,\alpha_4(r_n,1) > r_n^{2-\epsilon}$ for some constant $\epsilon > 0$, it is clear that when $N-j$ is larger than some fixed constant $M = M(\lambda) \in \mathbb{N}$, we have

$$\lambda\,(2\,C_2)^3 \sum_{n=i+1}^{j-1} \frac{r_n^2}{\alpha_4(r_n,1)} \le 1\,.$$

This shows that (2.3) completes the inductive step if we choose $C_1 = 2\,C_2$ and if $N-j > M$. (Note that in the proof of the induction step when

$N - j > M$, we have not relied on the inductive assumption in which this condition does not hold.) To handle the case $N - j \le M$, we just note that $b_i^j \le b_i^{N-M-1}$, and the estimate that we have for $b_i^{N-M-1}$ is within a constant factor (depending on $\lambda$) of our claimed estimate for $b_i^j$, since $M$ depends only on $\lambda$. □

Set

$$Z^X(z) := \sup_{\omega' \in \Omega(\omega, X)} Z_{\omega'}(z)\,.$$

**Lemma 2.2.** *For every site $z$ and every $\epsilon$ and $r$ satisfying $2\,\eta < \epsilon < 2^4\,\epsilon < r \le 1$, we have*

$$\mathbb{P}\big[Z^X(z) \ge r,\, Z_\omega(z) \le \epsilon\big] \le O_\lambda(1)\,\epsilon^2\,\alpha_4(\epsilon)\,\alpha_4(r,1)^{-1}\,.$$

The proof uses some of the ideas going into the proof of Lemma 2.1 as well as the estimate provided by that lemma.

**Proof.** Fix $z$, $\epsilon$ and $r$ as above. Suppose that $Z^X(z) \ge r$ and $Z_\omega(z) \le \epsilon$ both hold. Let $\omega' \in \Omega(\omega, X)$ be such that $Z_{\omega'}(z) \ge r$. Let $x_1, x_2, \ldots, x_m$ be the sites in $B_\eta(z, \epsilon)$ where $\omega' \ne \omega$. (We use some arbitrary but fixed rule to choose $\omega'$ and the sequence $x_j$ among the allowable possibilities.) For each $j = 0, 1, \ldots, m$, let $\omega_j$ denote the configuration that agrees with $\omega'$ on every site different from $x_{j+1}, x_{j+2}, \ldots, x_m$, and agrees with $\omega$ on $x_{j+1}, \ldots, x_m$. Then $\omega_m = \omega'$ and $Z_{\omega_0}(z) < \epsilon$. Let $k$ be the first $j$ such that $Z_{\omega_j}(z) > r$.

Fix some site $x$ satisfying $r^x := |z - x| \le \epsilon$. In order for $Z^X(z) \ge r$, $Z_\omega(z) \le \epsilon$ and $x_k = x$ to hold, the following four events must occur: $x \in X$, $Z^X(z) \ge r^x/2$, $Z^X(x) \ge r^x/2$, and $\mathcal{W}_z(2 + \lceil \log_2 r^x \rceil, \lfloor \log_2 r \rfloor)$ (using the notation of Lemma 2.1). We have $\mathbb{P}\big[x \in X\big] = q_\lambda = O(\lambda)\,\eta^2\,\alpha_4(1)^{-1}$, while the probabilities of the latter three events are bounded by Lemma 2.1. Combining these bounds, we get

$$\begin{aligned}\mathbb{P}\big[Z^X(z) \ge 1,\, Z(z) \le \epsilon,\, x_k = x\big] &\le O_\lambda(1)\,\alpha_4(r^x)^2\,\eta^2\,\alpha_4(1)^{-1}\,\alpha_4(r^x, r)\\ &= O_\lambda(1)\,\alpha_4(r^x)\,\eta^2\,\alpha_4(r,1)^{-1}\,.\end{aligned}$$

Summing this bound over all sites $x$ satisfying $|z - x| \le \epsilon$ yields the lemma. □

A **quad** is defined as a simple close path, or, more specifically, as an injective continuous map from the unit circle $\partial\mathbb{U}$ to $\mathbb{R}^2$. The closure of the domain surrounded by a quad $Q$ will be denoted by $\hat{Q}$. The boundary $\partial\hat{Q}$ of $\hat{Q}$ is partitioned into four distinguished arcs denote by

$$\partial_j Q := \big\{Q(e^{i\,\theta+j\pi/2}) : \theta\in[0,\pi/2]\big\}, \qquad j\in\{0,1,2,3\}\,.$$

We say that $Q$ is **crossed** by a percolation configuration $\omega$, if there is an $\omega$-white path inside $\hat{Q}$ that connects $\partial_0 Q$ and $\partial_2 Q$. If $r>0$ is smaller than the minimal distance from $\partial_0 Q$ to $\partial_2 Q$, then we say that $\omega$ is $r$**-almost crossed** by $\omega$, if there is an $\omega$-white path in the $r$-neighborhood of $\hat{Q}$ that comes within distance $r$ of each of the two arcs $\partial_0 Q$ and $\partial_2 Q$.

**Proposition 2.3.** *Let $\lambda$ and $X$ be as above, and fix some quad $Q$. Let $r>0$ be smaller than the minimal distance between $\partial_0 Q$ and $\partial_2 Q$, and suppose that $0<\eta<2\,\eta<\epsilon<2^5\,\epsilon<r\le 1$. Then the probability that there are some $\omega',\omega''\in\Omega(\omega,X)$ such that (a) $Q$ is crossed by $\omega'$, (b) $Q$ is not $r$-almost crossed by $\omega''$, and (c) $\omega'(z)=\omega''(z)$ for every site $z$ satisfying $Z_\omega(z)\ge\epsilon$ is at most*

$$O_{\lambda,Q}(\epsilon^2)\,\alpha_4(\epsilon,1)^{-1}\,\alpha_4(r,1)^{-1}.$$

**Proof.** Suppose that there are such $\omega'$ and $\omega''$. Let $Y$ denote the set of sites whose hexagons are contained in the $r$-neighborhood of $\partial_0 Q\cup\partial_2 Q$, and let $\{x_1,x_2,\dots,x_m\}$ denote the sites not in $Y$ whose hexagons intersect $\hat{Q}$. For $j=0,1,\dots,m$, let $\omega_j$ denote the configuration that agrees with $\omega'$ on $Y\cup\{x_{j+1},x_{j+2},\dots,x_m\}$, and agrees with $\omega''$ elsewhere. Then $Q$ is crossed by $\omega_0$ (since $\omega_0$ agrees with $\omega'$ on all hexagons intersecting $\hat{Q}$), but is not $r$-almost crossed by $\omega_m$ (since $\omega_m$ agrees with $\omega''$ on all hexagons except those contained in the $r$-neighborhood of $\partial_0 Q\cup\partial_2 Q$). Let $k$ be the least index $j$ such that there is no $\omega_j$-white path connecting $\partial_0 Q$ and $\partial_2 Q$ within the $r$-neighborhood of $\hat{Q}$. Then $\omega_{k-1}$ and $\omega_k$ differ only in the color of $x_k$. Since a flip of $x_k$ modifies the connectivity between $\partial_0 Q$ and $\partial_2 Q$ within the $r$-neighborhood of $\hat{Q}$, and since the hexagon of $x_k$ intersects $\hat{Q}$ and is not contained in the $r$-neighborhood of $\partial_0 Q\cup\partial_2 Q$, it follows that $Z_{\omega_{k-1}}(x_k)>r/2$. Consequently, $Z^X(x_k)>r/2$. If $x$ is any site, then in order to have $x=x_k$, we must have (i) $Z^X(x)>r/2$, (ii) $Z_\omega(x)<\epsilon$, (iii) $x\in X$, and (iv) the hexagon of $x$ intersects $\hat{Q}$. There are $O_Q(\eta^{-2})$ sites satisfying (iv). The event (iii) has probability $q_\lambda$ and is independent from the intersection of (i) and (ii), while Lemma 2.2 bounds the probability of this intersection. The proposition now follows easily by summing the bound we get for $\mathbb{P}[x_k=x]$ over all possible $x$. ☐

# 3 Setup of the scaling limit

## 3.1 Setup for static percolation

First of all, we define what we mean by the scaling limit of critical percolation. We will work with the setup introduced in [SS], which describes the scaling limit using "left-right" crossing events in generalized quadrilaterals.

Let $D \subset \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be open. A **quad** in $D$ is a homeomorphism $\mathcal{Q}$ from $[0,1]^2$ into $D$. The space of all quads in $D$, denoted by $\mathscr{Q}_D$, can be equipped with the uniform metric $d(\mathcal{Q}_1, \mathcal{Q}_2) := \sup_{z \in [0,1]^2} |\mathcal{Q}_1(z) - \mathcal{Q}_2(z)|$. A **crossing** of a quad $\mathcal{Q}$ is a connected closed subset of $[\mathcal{Q}] := \mathcal{Q}([0,1]^2)$ that intersects both $\mathcal{Q}(\{0\} \times [0,1])$ and $\mathcal{Q}(\{1\} \times [0,1])$.

From the point of view of crossings, there is a natural partial order on $\mathscr{Q}_D$: we write $\mathcal{Q}_1 \le \mathcal{Q}_2$ if any crossing of $\mathcal{Q}_2$ contains a crossing of $\mathcal{Q}_1$. See Figure 3.1. A subset $S \subset \mathscr{Q}_D$ is called **hereditary** if whenever $\mathcal{Q} \in S$ and $\mathcal{Q}' \in \mathscr{Q}_D$ satisfies $\mathcal{Q}' \le \mathcal{Q}$ , we also have $\mathcal{Q}' \in S$. The collection of all closed hereditary subsets of $\mathscr{Q}_D$ will be denoted by $\mathscr{H}_D$. Any discrete percolation configuration $\omega_\eta$ of mesh $\eta > 0$, considered as a union of the topologically closed percolation-wise open hexagons in the plane, naturally defines an element $S(\omega_\eta)$ of $\mathscr{H}_D$: the set of all quads for which $\omega_\eta$ contains a crossing. Thus, in particular, critical percolation induces a probability measure on $\mathscr{H}_D$, which will be denoted by $\mathbb{P}_\eta$.

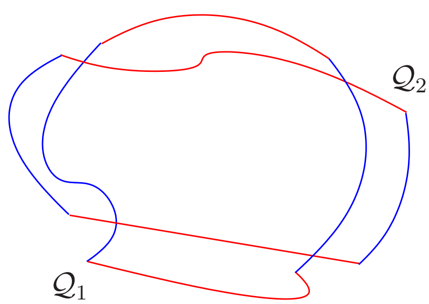



Figure 3.1: Two quads, $\mathcal{Q}_1 \le \mathcal{Q}_2$.

Hereditary subsets can be thought of as Dedekind cuts in the setting of partially ordered sets (instead of totally ordered sets, as usual). It can be therefore hoped that by introducing a natural topology, $\mathscr{H}_D$ can be made into a compact metric space. Indeed, let us consider the following subsets of $\mathscr{H}_D$. For any $\mathcal{Q} \in \mathscr{Q}_D$, let $\boxminus_\mathcal{Q} := \{S \in \mathscr{H}_D : \mathcal{Q} \in S\}$, and for any open

$U \subset \mathscr{Q}_D$, let $\boxminus_U := \{S \in \mathscr{H}_D : S \cap U = \emptyset\}$. It is easy to see that these sets have to be considered closed if we want $\mathscr{H}_D$ to be compact, therefore we define $\mathcal{T}_D$ to be the minimal topology that contains every $\boxminus_\mathcal{Q}^c$ and $\boxminus_U^c$ as open sets. It is proved in [SS] that for any nonempty open $D$, the topological space $(\mathscr{H}_D, \mathcal{T}_D)$ is compact, Hausdorff, and metrizable. Furthermore, for any dense $\mathscr{Q}_0 \subset \mathscr{Q}_D$, the events $\{\boxminus_\mathcal{Q} : \mathcal{Q} \in \mathscr{Q}_0\}$ generate the Borel $\sigma$-field of $\mathscr{H}_D$. It is then proved that the scaling limit of $\mathbb{P}_\eta$ exists as a Borel probability measure on $\mathscr{H}_\mathbb{C}$, i.e., for any finite collection of events $\boxminus_\mathcal{Q}$ and $\boxminus_\mathcal{Q}^c$, the joint probabilities converge. In this section, we will denote the scaling limit measure by $\mathbb{P}_0$.

Of course, the choice of the space $\mathscr{H}_D$ already poses restrictions on what events one can work with. Note, for instance, that $\mathcal{A} := \{\exists$ neighborhood $U$ of the origin $0 \in \mathbb{C}$ s.t. all quads $\mathcal{Q} \subset U$ are crossed$\}$ is clearly in the Borel $\sigma$-field of $(\mathscr{H}_D, \mathcal{T}_D)$, and it is easy to see that $\mathbb{P}_0[\mathcal{A}] = 0$, but if the sequence of $\eta$-lattices is such that 0 is always the center of a hexagonal tile, then $\mathbb{P}_\eta[\mathcal{A}] = 1/2$. Similarly, using events like $\mathcal{A}$, one can easily write down a Borel-measurable event in $(\mathscr{H}_D, \mathcal{T}_D)$ that will mean, for any $\eta > 0$, the event that there are more open than closed $\eta$-hexagons in the percolation configuration $\omega_\eta$ in $D$, but in the scaling limit it will be a trivial event, with no meaning similar to "majority of the bits is open", and having $\mathbb{P}_0$-measure 0 or 1 (depending on the exact definition). Therefore, it is important to know that for a lot of natural events this problem does not occur. For any topological annulus $A \subset D$ with piecewise smooth boundaries, we define the **(alternating) 4-arm event** in $A$ as the existence of disjoint quads $\mathcal{Q}_i \subset A$, $i = 1, 2, 3, 4$, with $\mathcal{Q}_i(\{0\} \times [0, 1]) \subset \partial_2 A$ and $\mathcal{Q}_i(\{1\} \times [0, 1]) \subset \partial_1 A$ for $i = 1, 3$, while $\mathcal{Q}_i([0, 1] \times \{0\}) \subset \partial_2 A$ and $\mathcal{Q}_i([0, 1] \times \{1\}) \subset \partial_1 A$ for $i = 2, 4$, ordered cyclically around $A$ according to their indices, such that the odd ones are crossed, while the even ones are not crossed (which means they contain dual crossings between the boundary pieces of $A$). The definitions of alternating (or polychromatic, in general) and monochromatic $k$-arm events in $A$ are of course analogous.

**Lemma 3.1.** *Let $A \subset D$ be a piecewise smooth topological annulus. Then the 1-arm and alternating 4-arm events in $A$, denoted by $\mathcal{A}_1$ and $\mathcal{A}_4$, are measurable w.r.t. the scaling limit of critical percolation in $D$, moreover,* $\lim_{\eta \to 0} \mathbb{P}_\eta[\mathcal{A}_i] = \mathbb{P}[\mathcal{A}_i]$.

**Proof.** Using a countable dense subset of $\mathscr{Q}_D$, it is clear that $\mathcal{A}_i$ is in the Borel $\sigma$-field of $(\mathscr{H}_D, \mathcal{T}_D)$. So we prove $\lim_{\eta \to 0} \mathbb{P}_\eta[\mathcal{A}_i] = \mathbb{P}[\mathcal{A}_i]$, starting with the $i = 4$ case. It is enough to show that if $\mathcal{A}_4$ holds in $\omega_\eta$, then for any $\epsilon > 0$ there is a $\delta > 0$ such that with conditional probability at least $1-\epsilon$, uniformly in $\eta$, there are four quads with disjoint $\delta$-neighborhoods that contain the four arms — then we can detect the four arms with disjoint quads (as required) even in the scaling limit.

Indeed, if we have $\mathcal{A}_4$ in $\omega_\eta$, then let us choose 4 alternating arms arbitrarily, then consider the counterclockwise boundaries of their connected components (in their own color). It is easy to see that if two of these boundaries come $\delta$-close to each other, then we have a 6-arm event from this $\delta$-ball to the boundaries of $A$. However, using the 6-arm probabilities in the plane and 3-arm probabilities in the half-plane, we see that this happens in $A$ only with a probability that goes to 0 as $\delta$ goes to 0, uniformly in $\eta$. Therefore, the four quads that the annulus is divided into by the four counterclockwise boundaries have the properties we required, and we are done.

For $\mathcal{A}_1$, the above proof does not work, because if there is no closed crossing in $A$, i.e., there is an open circuit, then the "counterclockwise boundary of the component of the open crossing" is not well-defined. So, let us take a radial exploration interface: start from the inner boundary $\partial_1 A$, with open hexagons on the right, closed hexagons on the left of the interface. If the interface makes a clockwise loop around $\partial_1 A$, then we have discovered a closed circuit, so there is no open crossing. Whenever the interface makes a counterclockwise all-around-loop, let us pretend that the last closed hexagon discovered was open, so turn to the right before it, and continue the interface. This interfaces reaches $\partial_2 A$ if and only if there is an open crossing. Given this event, erase chronologically the counterclockwise all-around-loops it has completed. The resulting interface has an open (non-simple) crossing path on its right without all-around-loops. Taking the right (clockwise) boundary of it, we get a simple open path. Now if this path comes $\delta$-close to itself, that is already a 6-arm whole-plane event, which happens in the annulus only with probability $o_{\delta\to 1}(1)$, uniformly in $\eta$. In particular, this open path does not make an all-around-loop in the scaling limit, and hence we can detect it with the event of crossing a quad, and we are done. □

*Remark* 3.2. Recall that [SS] proved the $\eta \to 0$ convergence of the $\mathbb{P}_\eta$-probability of the intersection of any finite set of quad crossing and non-crossing events. Therefore, the proof of the above lemma can be extended to the convergence of the joint $\mathbb{P}_\eta$-distribution of any finite set of macroscopic 4-arm events: the main point is that, in the scaling limit, the planar 6-arm and half-plane 3-arm events do not occur anywhere in the finite set of annuli, hence the convergence of quad crossings and non-crossings really measures what we want.

With similar arguments, one can show that a half-plane exploration interface in the scaling limit has the properties that characterize chordal $SLE_6$, hence it has $SLE_6$ as a scaling limit. See [SS] for more details.

### 3.2 Setup for dynamical percolation

Now, dynamical percolation is a probability measure on càdlàg paths $\mathbb{R} \longrightarrow \mathscr{H}_D$; i.e., at a switch time $t \in \mathbb{R}$ we define $S(\omega_t)$ to be the configuration after the switch. We equip this set of paths with the topology of locally uniform convergence (uniform convergence on compact subsets of $\mathbb{R}$), and will prove that the properly scaled process has a weak limit w.r.t. this topology.

## 4 Coupling argument

Let us consider some annulus $A = A(u, v)$ of inner radius $u$ and outer radius $v > u$. Let $\omega_\eta$ be a percolation configuration on the triangular grid of mesh $\eta$ inside $A$. Call $\Gamma$ the set of percolation interfaces which cross $A$ (it might be that there are no such interfaces, in which case $\Gamma = \emptyset$). We will need to measure how well separated the interfaces are on the inside boundary $\partial_1 A$. For that purpose we define a measure of the **(interior) quality** $\mathbf{Q}(\omega_\eta) = \mathbf{Q}(\Gamma)$ to be the least distance between the endpoints of $\Gamma$ on $\partial_1 A$ normalized by $u$. More precisely, if there are $p \geq 2$ interfaces crossing $A$ and if $x_1, \ldots, x_p$ denote the endpoints of these interfaces on $\partial_1 A$, then we define

$$\mathbf{Q}(\omega_\eta) = \mathbf{Q}(\Gamma) = \frac{1}{u} \inf_{k \neq l} |x_k - x_l|,$$

and if $\Gamma = \emptyset$, we define $\mathbf{Q}(\Gamma)$ to be $+\infty$.

We define similarly the **exterior quality** $\mathbf{Q}^+(\omega_\eta) = \mathbf{Q}^+(\Gamma)$ to be the least distance between the endpoints of $\Gamma$ on $\partial_2 A$ normalized by $v$ (and set to be $\infty$ if there are no such interfaces). For any $\alpha > 0$, let $\mathcal{T}^\alpha(u, v)$ be the event that the quality $\mathbf{Q}(\Gamma)$ is bigger than $\alpha$; also, $\mathcal{T}^\alpha_+(u, v)$ will denote the event that $\mathbf{Q}^+(\Gamma) > \alpha$.

For a square of radius $R$, we define a notion of **faces** around that square. Let $x_1, \ldots, x_4$ be four distinct points on the square of radius $R$ chosen in a counterclockwise order. We will adopt here cyclic notation, i.e., for any $j \in \mathbb{Z}$, we have $x_j = x_i$ if $j \equiv i[4]$. For any $i \in \mathbb{Z}$, let $\theta_i$ be a simple path of hexagons joining $x_i$ to $x_{i+1}$, i.e., a sequence of hexagons $h_1, \ldots, h_n$ such that $h_i$ and $h_j$ are neighbors if and only if $|i - j| = 1$. We assume furthermore that there are no hexagons in $\theta_i$ which are entirely contained in the square of radius $R$ (they might still intersect $\partial_1 A$) and that all the hexagons in $\theta_i$ are white (=open) if $i$ is odd and black (=closed) otherwise. If a set of paths $\Theta = \{\theta_1, \ldots, \theta_4\}$ satisfies the above conditions, $\Theta$ will be called a configuration of **faces** with endpoints $x_1, \ldots, x_4$. We define similarly the quality of a configuration of faces $\mathbf{Q}(\Theta)$ to be the least distance between the endpoints of the faces, normalized by $R$.

Let $\mathcal{G}(u, v)$ be the event that there are exactly 4 alternating arms from radius $u$ to $v$ and no extra arm crossing $A$. In particular, on the event $\mathcal{G}(u, v)$, the set of interfaces $\Gamma$ consists of exactly 4 interfaces, and furthermore any two consecutive interfaces have to share at least one hexagon in common (if not there would be at least 5 arms from $u$ to $v$). It is easy to see that on the event $\mathcal{G}(u, v)$, the 4 interfaces of $\Gamma$ induce a natural configuration of faces $\Theta = \Theta(\Gamma)$ at radius $u$ with same endpoints $x_1, \ldots, x_4$ as $\Gamma$. More precisely, if $\mathcal{H}$ is the set of all the hexagons neighboring the 4 interfaces in $\Gamma$, then by the definition of $\mathcal{G}(u, v)$, the connected component of $\mathbb{C} \setminus \mathcal{H}$ which contains the center of the annulus $A$ is a bounded domain; the set of hexagons which lie on the boundary of this domain form the 4 faces of $\Theta$. As we will do later, if we condition on the event $\mathcal{G}(u, v)$, we might as well condition on the configuration of faces $\Theta$. (Notice that by definition, on the event $\mathcal{G}(u, v)$, we have $\mathbf{Q}(\Gamma) = \mathbf{Q}(\Theta)$).

If we are given a square of radius $R$ and a configuration of faces $\Theta = \{\theta_1, \ldots, \theta_4\}$ around that square; $\mathcal{D} = \mathcal{D}_\Theta$ will denote the bounded component of $\mathbb{C} \setminus \Theta$ (this is a finite set of $\eta$-hexagons). Call $U = U_\Theta$ the random variable which is set to be one if there is an open crossing from $\theta_1$ to $\theta_3$ inside $\mathcal{D}_\Theta$ and 0 otherwise. For any radius $r < R$, let $\mathcal{A}_\Theta(r, R)$ be the event that there are open arms from the box of radius $r$ to the open faces $\theta_1$ and $\theta_3$ and there are closed arms from $r$ to the closed faces $\theta_2, \theta_4$ (in other words, if one starts 4 interfaces in $\mathcal{D}_\Theta$ at the endpoints of $\Theta$ at radius $R$, then these interfaces go all the way to radius $r$).

Let $Q$ be the square of radius 1 around the origin. As a particular case of the above, for any $\eta < r < 1$, we define $\mathcal{A}_0(r, 1)$ to be the event that there are open arms from the square of radius $r$ to the left and right edges of $Q$ and there are closed arms from $r$ to the top and bottom edges of $Q$. Similarly,

$U_0$ will be the indicator function of a left-right crossing in $Q$.

**Proposition 4.1.** *Let $\alpha > 0$, $\tau \in \{0, 1\}$, and let $A = A(r, R)$ be the annulus centered around 0 with radii $100\eta < 10r < R \leq 1$. Assume we are given any configuration of faces $\Theta = \{\theta_1, \ldots, \theta_4\}$ around the square of radius $R$ satisfying $\mathbf{Q}(\Theta) > \alpha$. Let $\nu_\Theta$ be the law of the percolation configuration inside $\mathcal{D}_\Theta$ conditioned on the events $\mathcal{A}_\Theta(r, R)$ and $\{U_\Theta = \tau\}$.*

*Let $\nu_0$ be the law of the percolation configuration in the square $Q$ under the conditioning that $\mathcal{A}_0(r, 1)$ and $\{U_0 = 1\}$ hold.*

*Then if $\tau = 1$ there is a coupling of the conditional laws $\nu_\Theta, \nu_0$ so that with (conditional) probability at least $1 - (r/R)^{k(\alpha)}$, the event $\mathcal{G}(r, R)$ is satisfied for both configurations and the induced faces at radius $r$: $\Theta(r)$ and $\Theta_0(r)$ are identical. If $\tau = 0$, there is also such a coupling except that the faces $\Theta(r)$ and $\Theta_0(r)$ are (with probability at least $1 - (r/R)^{k(\alpha)}$) identical but with reversed color. Here, the exponent $k = k(\alpha)$ only depends on the quality of the initial configuration of faces $\Theta$.*

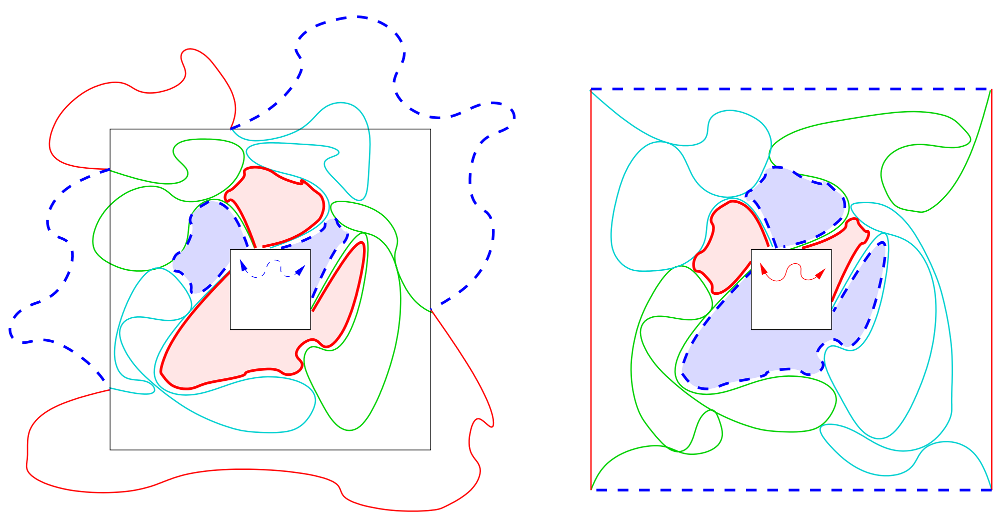


Figure 4.1: Successful coupling of $\nu_\Theta$ and $\nu_0$ when $\tau = 0$: the four interfaces induce the same configuration of faces at radius $r$, only with reversed colors.

*Remark* 4.2. The proposition also holds if the annulus $A$ is not centered around the origin, but there are issues here coming from the discrete lattice: indeed the set of $\eta$-hexagons intersecting a square $S$ is not invariant under translations of the square $S$. We will deal with this issue in the next section.

**Proof of Proposition 4.1.** We first prove the proposition in the case where $\tau = 1$; the case $\tau = 0$ will need an additional color switching argument. Let $N = \lfloor \log_4(\frac{R}{r}) \rfloor$. For $0 \leq i \leq N$, let $r_i = 4^{N-i} r$.

Let $\omega_\Theta$ and $\omega_0$ be percolation configurations in $\mathcal{D}_\Theta$ and the square $Q$ sampled according to $\nu_\Theta$ and $\nu_0$. For all $1 \leq i \leq N$, let $\Gamma^i = \Gamma^i(\omega_\Theta)$ denote the set of interfaces from the 4 endpoints of $\Theta$ inside $\mathcal{D}_\Theta$ until they reach radius $r_i$; also let $\Upsilon^i = \Upsilon^i(\omega_\Theta)$ denote the set of all interfaces crossing from $2r_i$ to $r_i$ (therefore by definition of $\nu_\Theta$, there are at least 4 such interfaces). Define in the same way the set of interfaces $\Gamma_0^i = \Gamma_0^i(\omega_0)$ and $\Upsilon_0^i = \Upsilon_0^i(\omega_0)$.

Let $0 \leq i < N$; assume that we sampled under $\nu_\Theta$ the set of interfaces $\Gamma^i$ and that we sampled under $\nu_0$ the set of interfaces $\Gamma_0^i$. It is straightforward to check that the initial configurations of faces, $\Theta$ and $\partial Q$, plus the sets of interfaces $\Gamma^i$ and $\Gamma_0^i$ induce two configurations of faces, $\Theta^i$ and $\Theta_0^i$, around radius $r_i$. Now the law of $\nu_\Theta$ in $\mathcal{D}_{\Theta^i}$ conditioned on the explored interfaces $\Gamma^i$ is exactly the same as the law of the percolation in $\mathcal{D}_{\Theta^i}$ conditioned on $\mathcal{A}_{\Theta^i}(r, r_i)$ and $\{U_{\Theta^i} = \tau = 1\}$ (a law which depends only on the faces in $\Theta^i$). Let $\nu^i$ (and similarly $\nu_0^i$) denote this law. Let $\epsilon > 0$, whose value will be fixed later on. Let $\mathcal{W}^i$ be the event that the set of interfaces $\Gamma^i$ has quality at least $\epsilon > 0$ ($\mathcal{W}^i = \{\mathbf{Q}(\Gamma^i) > \epsilon\}$). Define similarly the event $\mathcal{W}_0^i$. There is a slight abuse of notation here, since we defined the quality only for a set of interfaces inside an annulus, but it of course generalizes to the case of interfaces starting at the endpoints of 4 faces until they reach some smaller radius. Note that $\neg\mathcal{W}^i \subset \{\mathbf{Q}(\Upsilon^i) \leq \epsilon\}$, which will be useful later on in the analysis.

Define $\mathcal{R}_i$ to be the event that for a percolation configuration in the annulus $A(r_{i+1}, 2r_{i+1})$, the set of all its interfaces which cross the annulus satisfies the events $\mathcal{G}(r_{i+1}, 2r_{i+1})$, $\mathcal{T}^{1/4}(r_{i+1}, 2r_{i+1})$ and $\mathcal{T}_+^{1/4}(r_{i+1}, 2r_{i+1})$ (in particular if $\mathcal{R}_i$ holds, then there are exactly 4 interfaces crossing from $2r_{i+1}$ to $r_{i+1}$). Now we sample the sets of interfaces $\Upsilon^{i+1}$ and $\Upsilon_0^{i+1}$ according to the laws $\nu^i$ and $\nu_0^i$. Note that conditioning on $\Upsilon^{i+1}$ (or $\Upsilon_0^{i+1}$) determines the color of the hexagons neighboring the interfaces in $\Upsilon^{i+1}$ ($\Upsilon_0^{i+1}$). Let then $S^i$ ($S_0^i$) be the union of all hexagons whose color is determined by $\Upsilon^{i+1}$ ($\Upsilon_0^{i+1}$). Let $S$ be a possible value for $S^i$ such that $\mathcal{R}_i$ holds (i.e. the interfaces determined by $S$ satisfy $\mathcal{R}_i$). Let $m_S$ be the number of hexagons that are in $S$. Clearly, without conditioning, $\mathbb{P}\big[S^i = S\big] = 2^{-m_S}$. We claim that if $\mathcal{W}^i$ holds then there is a universal constant $c = c(\epsilon) > 0$ such that:

$$2^{-m_S}/c \leq \mathbb{P}\big[S^i = S \;\big|\; \mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau,\, \Theta^i\big] = \nu^i[S^i = S] \leq c\, 2^{-m_S}. \quad (4.1)$$

Also, if $\mathcal{W}_0^i$ holds then for the same constant $c = c(\epsilon)$:

$$2^{-m_S}/c \le \nu_0^i[S_0^i = S] \le c\, 2^{-m_S}. \tag{4.2}$$

Indeed, on the event $\mathcal{W}_i$,

$$\begin{aligned}\nu^i[S^i = S] &= \frac{\mathbb{P}\big[\mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \bigm| S^i = S,\, \Theta^i\big]\mathbb{P}\big[S^i = S \bigm| \Theta^i\big]}{\mathbb{P}\big[\mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \mid \Theta^i\big]} \\ &= 2^{-m_S}\frac{\mathbb{P}\big[\mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \bigm| S^i = S,\, \Theta^i\big]}{\mathbb{P}\big[\mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \mid \Theta^i\big]}\,.\end{aligned}$$

Since $S$ satisfies the event $\mathcal{R}_i$, conditioning on it implies that the arms which cross the annulus $A(r_{i+1}, 2r_{i+1})$ are well-separated. The separation of arms phenomenon is a very useful tool introduced by Kesten [Kes87] in order to prove for example quasi-multiplicativity of multi-arms events. See, for instance, [SS05] or [Nol07] for more details on the concept of separation of arms and how to use it. Applied to our setting, it implies that we can glue interfaces (using RSW and FKG) on both ends of $A(r_{i+1}, 2r_{i+1})$ with a cost of only a constant factor, hence there is a positive constant $C = C(\epsilon) > 0$ such that

$$\begin{aligned}\alpha_4(r, r_i)/C &\le \mathbb{P}\big[\mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \bigm| S^i = S,\, \Theta^i\big] \le C\alpha_4(r, r_i) \\ \alpha_4(r, r_i)/C &\le \mathbb{P}\big[\mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \mid \Theta^i\big] \le C\alpha_4(r, r_i)\,.\end{aligned}$$

This implies our claim (4.1). Now by summing the claim (4.1) over the different $S$ for which $\mathcal{R}_i$ holds, we get that if $\mathcal{W}^i$ holds then:

$$\mathbb{P}\big[\mathcal{R}_i \bigm| \mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau \mid \Theta^i\big] \ge \frac{1}{c}\,\mathbb{P}\big[\mathcal{R}_i\big]\,.$$

We now use the following easy fact, which is part of the folklore, though we are not aware of an explicit proof in the literature:

**Lemma 4.3.** *In the annulus $A = A(r, 2r)$, there is a uniformly positive probability to have exactly 4 disjoint alternating arms crossing $A$, with uniformly positive distances between the endpoints of the resulting 4 interfaces on the two boundaries of $A$. That is, $\mathbb{P}\big[\mathcal{R}_i\big] > c$ for an absolute constant $c > 0$.*

**Proof.** Let us provide two proofs, the first using Reimer's inequality [Rei00], the second being longer, but completely elementary.

Consider $\mathcal{A}_4$, the alternating 4-arm event in $A$, and $\mathcal{B} = \{$there is an arm (open or closed) from $\partial_1 A$ to $\partial_2 A\}$. Then, by Reimer, $\mathbb{P}[\mathcal{A}_4 \Box \mathcal{B}] = \mathbb{P}[$polychromatic 5 arms in $A] \le \mathbb{P}[\mathcal{A}_4]\,\mathbb{P}[\mathcal{B}]$. But Russo-Seymour-Welsh implies $\mathbb{P}[\mathcal{B}^c] > c > 0$, so we get $\mathbb{P}[\mathcal{A}_4 \setminus \mathcal{A}_4 \Box \mathcal{B}] = \mathbb{P}[$exactly 4 arms in $A] > (1-c)\mathbb{P}[\mathcal{A}_4] > c' > 0$, and we are done.

For the elementary proof, we first show that in the square there is a uniformly positive probability of the event that there is a unique left-right crossing (which is the same as having at least one pivotal open bit), with a pivotal in the left two thirds sub-rectangle $R$ of the square, and from this pivotal there are closed arms inside $R$ to the upper and lower sides of $R$. See the left picture in Figure 4.2. By RSW, there is a positive probability for a left-right open crossing in $R$; the uppermost such crossing can be found by running an exploration process from the upper left corner to the upper right corner of $R$, with open bits on the right (below the interface), closed bits on the left. Any point on this interface has a closed arm to the upper side of $R$. Furthermore, this process did not explore anything below the open crossing it finds, hence RSW can be applied to get with positive probability a closed path inside the left half of $R$ from the bottom of $R$ to this interface. The bit where this path hits the interface is a pivotal for the left-right crossing in $R$. Now, again by RSW, with positive probability there is an open path inside the right two thirds of the square from the interface we have discovered in $R$ to the right side of the square. This open path can be glued to the open crossing found by that interface, to produce the desired open crossing of the square.

Now let us divide the annulus $A$ into four congruent sectors. Each sector is a nice quad, so we can translate the previous paragraph to get the following: with positive probability, each of the odd sectors contains a pivotal bit for an open crossing between the sides of the quad that are part of the annulus boundaries, and the closed arms emanating from these pivotals are contained in $A(4r/3, 2r)$. Similarly, with positive probability, each of the even sectors contain a pivotal bit for a closed crossing between the sides at radii $r$ and $2r$, and the open arms emanating from these pivotals are contained in $A(r, 5r/3)$. Moreover, with a small strengthening of the previous paragraph, with positive probability we can achieve that in the odd sectors the endpoints of the two open arms emanating from the pivotal we find are at a distance at most $r/10$ from the midpoints of their respective sides, while the endpoints of the two closed arms are in $A(5r/3, 2r)$. Similarly, in the even sectors the endpoints of the closed are arms are at most $r/10$ away from the midpoints of their

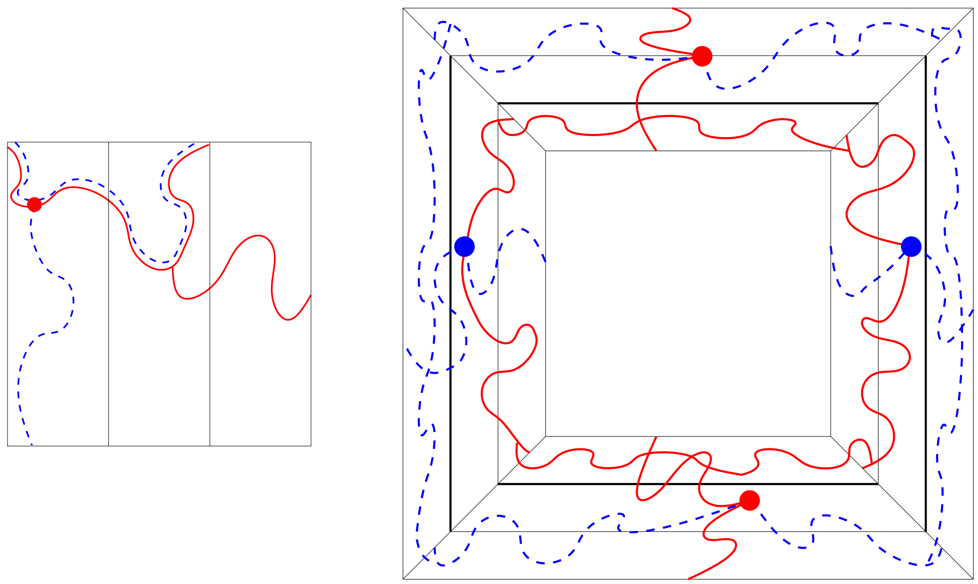


Figure 4.2: Constructing exactly four arms in an annulus.

sides, while the endpoints of the open arms are in $A(r, 4r/3)$. Then, using RSW and FKG, with positive probability we can glue the closed arms within $A(5r/3, 2r)$ with closed paths to get closed barriers that make sure that any open arm from $r$ to $2r$ must go through on of the pivotals in the odd sectors. Similarly, we can glue the open arms within $A(r, 4r/3)$ to obtain open barriers that force any close arm from $r$ to $2r$ to go through one of the pivotals in the even sectors. Altogether, with positive probability we achieved that there are only 4 disjoint arms, and their endpoints are well-separated. With further gluing we can also easily make sure that the resulting four interfaces have well-separated endpoints, and the proof is complete. See the right picture in Figure 4.2. □

Thus, there is a positive $c_2 = c_2(\epsilon) > 0$ so that on $\mathcal{W}^i$:

$$\mathbb{P}\big[\mathcal{R}_i \,\big|\, \mathcal{A}_{\Theta^i}(r, r_i),\, U_{\Theta^i} = \tau,\, \Theta^i\big] \geq c_2. \tag{4.3}$$

Similarly, on $\mathcal{W}_0^i$:

$$\mathbb{P}\big[\mathcal{R}_i \,\big|\, \mathcal{A}_{\Theta_0^i}(r, r_i),\, U_{\Theta_0^i} = 1,\, \Theta^i\big] \geq c_2. \tag{4.4}$$

The construction of the coupling between $\nu_\Theta$ and $\nu_0$ works as follows. We proceed by induction on $i \geq 1$. Assume we explored the sets of interfaces $\Gamma^i$ and $\Gamma_0^i$ (interfaces up to radius $r_i$) according to some coupling of $\nu_\Theta$ and $\nu_0$. We want to sample the continuations $\Gamma^{i+1}$ and $\Gamma_0^{i+1}$ of these interfaces up to radius $r_{i+1}$ according to what have been explored so far, i.e., according to the laws $\nu^i$ and $\nu_0^i$. If $\mathcal{W}^i$ or $\mathcal{W}_0^i$ does not hold, then we sample $\Gamma^{i+1} \setminus \Gamma^i$ and $\Gamma_0^{i+1} \setminus \Gamma_0^i$ independently, according to $\nu^i$ and $\nu_0^i$. Else, if both $\mathcal{W}^i$ and $\mathcal{W}_0^i$ hold, then by (4.3) and (4.4), with positive probability for both $\nu^i$ and $\nu_0^i$, the sets of all interfaces crossing from $r_{i+1}$ to $2r_{i+1}$ satisfy $\mathcal{R}_i$. Furthermore, by (4.1) and (4.2), any set of interfaces satisfying $\mathcal{R}_i$ has about the same probability under $\nu^i$ or $\nu_0^i$. This allows us to sample $\Gamma^{i+1}$ under $\nu^i$ and $\Gamma_0^{i+1}$ under $\nu_0^i$ and to couple these samples so that the conditional probability of the event $\mathcal{S}_i := \{\Upsilon^{i+1} = \Upsilon_0^{i+1} \in \mathcal{R}_i\}$ is greater than some absolute constant $c_3 = c_3(\epsilon)$. Notice here that some care is needed since it is not $\Upsilon^{i+1}$ that we are sampling according to $\nu^i$, but rather $\Gamma^{i+1} \setminus \Gamma^i$; indeed $\Upsilon^{i+1}$ could consist of more than 4 interfaces, but as we have seen, with positive probability there are only 4 of them, and in that case we have $\Upsilon^{i+1} \subset \Gamma^{i+1}$. It is easy to check that if $\mathcal{S}_i$ holds then the interfaces $\Gamma^{i+1}$ and $\Gamma_0^{i+1}$ induce exactly the same configuration of faces $\Theta^{i+1} = \Theta_0^{i+1}$ at radius $r_{i+1}$.

The induction stops when $\mathcal{S}_i$ has occurred or when $i$ reaches $N$, whichever happens first. Let $i^* \leq N$ denote the index where the induction stopped. Call $\mathcal{S}$ the event that the induction stopped at $i^* < N$, i.e., the event that the coupling succeeded. If $i^* < N$, then the configurations of faces at radius $r_{i^*+1}$ are identical for both partially discovered configurations under the coupled $\nu_\Theta$ and $\nu_0$. Let us call this configuration of faces $\Theta^*$. In both configurations, what remains to be sampled in $\mathcal{D}_{\Theta^*}$ depends only on the faces $\Theta^*$, and since we assumed $\tau = 1$, they follow exactly the same law. Therefore, we can sample identically for $\nu_\Theta$ and $\nu_0$ and conditioned on $\Theta^*$ the 4 interfaces which start at the 4 endpoints of $\Theta^*$ until they reach radius $r$. These interfaces, plus the faces $\Theta^*$ if needed, define the same faces around $r$ for the coupled $\nu_\Theta$ and $\nu_0$, as desired. It now remains to prove that the coupling succeeds (event $\mathcal{S}$) with high probability. Let $\hat{\nu}$ denote the law of the coupling $(\nu_\Theta, \nu_0)$ we have just constructed.

We want to bound from above the probability $\hat{\nu}(\neg\mathcal{S}) = \hat{\nu}(i^* = N)$. Let

$(\omega_\Theta, \omega_0)$ be sampled according to $\hat{\nu} = (\nu_\Theta, \nu_0)$; for any $1 \le i < N$, define

$$X^i = 1_{\{\mathbf{Q}(\Upsilon^i) > \epsilon\}}$$
$$X_0^i = 1_{\{\mathbf{Q}(\Upsilon_0^i) > \epsilon\}}.$$

Let $Z^i = X^i X_0^i$, and $Z = Z^1 + \ldots + Z^{i^*-1}$. If $Z^i = 1$, then in the construction of the coupling $\hat{\nu}$, both events $\mathcal{W}^i$ and $\mathcal{W}_0^i$ hold; therefore if the coupling did not succeed yet, it will succeed at scale $i$ with probability at least $c_3(\epsilon) > 0$, as discussed above. By the definition of $\hat{\nu}$ and $Z$, we deduce that for any $k \ge 0$,

$$\hat{\nu}(Z \ge k) \le (1 - c_3(\epsilon))^k \,. \tag{4.5}$$

For any $1 \le i < N$, call $M^i = 1 - Z^i$ and let $M := M^1 + \ldots M^{N-1}$. Note that on the event $\{i^* = N\}$, we have $M + Z = N - 1$. Hence,

$$\hat{\nu}(i^* = N) \le \hat{\nu}(M \ge N/3) + \hat{\nu}(Z \ge N/3). \tag{4.6}$$

In words: if the coupling fails ($i^* = N$), then either $Z$ is large, so a lot of times we had a good chance to couple but failed at all of them, or we were not that many times in a good position to couple, hence $M$ is large.

In order to bound $\hat{\nu}(M \ge N/3)$, let us introduce for any $1 \le i < N$,

$$\begin{aligned} Y^i &= 1 - X^i = 1_{\{\mathbf{Q}(\Upsilon^i) \le \epsilon\}} \\ Y_0^i &= 1 - X_0^i = 1_{\{\mathbf{Q}(\Upsilon_0^i) \le \epsilon\}} \\ Y &= Y^1 + \ldots + Y^{N-1} \\ Y_0 &= Y_0^1 + \ldots + Y_0^{N-1}. \end{aligned}$$

By definition, $M \le Y + Y_0$, so that

$$\begin{aligned} \hat{\nu}(M \ge N/3) &\le \hat{\nu}(Y \ge N/6) + \hat{\nu}(Y_0 \ge N/6) \\ &= \nu_\Theta(Y \ge N/6) + \nu_0(Y_0 \ge N/6). \end{aligned} \tag{4.7}$$

Now, using the definition of $\nu_\Theta$ as a conditional probability measure, we write

$$\begin{aligned} \nu_\Theta(Y \ge N/6) &= \mathbb{P}\big[Y \ge N/6 \,\big|\, \mathcal{A}_\Theta(r,R),\, U_\Theta = \tau,\, \Theta\big] \\ &\le \frac{\mathbb{P}\big[Y \ge N/6\big]}{\mathbb{P}\big[\mathcal{A}_\Theta(r,R),\, U_\Theta = \tau \mid \Theta\big]} \\ &\le \frac{C(\alpha)}{\alpha_4(r,R)}\, \mathbb{P}\big[Y \ge N/6\big], \end{aligned}$$

by quasi-multiplicativity and taking into account the fact that $\mathbf{Q}(\Theta) > \alpha$.

Using standard techniques (mainly the 3-arms exponent in $\mathbb{H}$ known for triangular lattice as well as for $\mathbb{Z}^2$), we obtain that if the mesh $\eta$ is small enough (we supposed $r > 10\eta$), then there is a function $h(\epsilon)$ such that for all $0 \leq i < N$, $\mathbb{P}\big[\mathbf{Q}(\Upsilon^i) \leq \epsilon\big] < h(\epsilon)$ uniformly in $\eta < r/10$, and furthermore $h(\epsilon)$ goes to zero when $\epsilon$ goes to zero (see Lemma A.2 in [SS05] for an actual proof of a stronger result). We want to bound $\mathbb{P}\big[Y^1 + \ldots Y^{N-1} \geq N/6\big]$ under the unconditional probability measure on percolation configurations in the annulus $A(r, R)$. By independence on disjoint sets, under $\mathbb{P}$ the variables $Y^i$ are independent, each satisfying $\mathbb{P}\big[Y^i = 1\big] < h(\epsilon)$. Hence $Y$ is dominated by a Binomial variable $\mathcal{B}(N-1, h(\epsilon))$. By a standard Large Deviations estimate, for all $\epsilon > 0$ small enough so that $h(\epsilon) < 1/10$, there is a function $g(\epsilon)$, going to zero when $\epsilon \to 0$, such that:

$$\begin{aligned}\mathbb{P}\big[Y \geq N/6\big] &\leq \mathbb{P}\big[\mathcal{B}(N-1, h(\epsilon)) \geq N/6\big] \\ &\leq g(\epsilon)^N.\end{aligned}$$

Therefore,

$$\begin{aligned}\nu_\Theta(Y \geq N/6) &\leq C(\alpha)\alpha_4(r, R)^{-1} g(\epsilon)^N \\ &\leq O(1) C(\alpha)(R/r)^2 e^{\log(g(\epsilon))\, \lfloor \log_4(\frac{R}{r}) \rfloor}.\end{aligned}$$

We can now fix the value of the parameter $\epsilon$ so that

$$\nu_\Theta(Y \geq N/6) \leq C(\alpha)\frac{r}{R},$$

for some new constant $C(\alpha) > 0$. Similarly (maybe by changing $\epsilon$ and $C(\alpha)$, which we can still do), we have that

$$\nu_0(Y_0 \geq N/6) \leq C(\alpha)\frac{r}{R}.$$

Therefore, by combining the above estimates on (4.7) with (4.5), the bound (4.6) becomes

$$\hat{\nu}(i^* = N) \leq (1 - c_3(\epsilon))^{N/3} + 2C(\alpha)\frac{r}{R}.$$

Since $\epsilon$ was chosen depending on the "quality" $\alpha$, this can be written

$$\hat{\nu}(i^* = N) \leq \Big(\frac{r}{R}\Big)^{k(\alpha)}$$

for some exponent $k(\alpha) > 0$, which proves Proposition 4.1 in the case $\tau = 1$.

It remains to prove the case where $\tau = 0$. The proof follows the exact same lines as in the case $\tau = 0$, plus the following color switching argument (we keep the same notations). In the construction of the coupling $\hat{\nu}$, suppose we are at scale $i < N$ and that we already sampled $\Gamma^i$ and $\Gamma_0^i$. If both $\mathcal{W}^i$ and $\mathcal{W}_0^i$ hold, this allows us to sample $\Gamma^{i+1} \setminus \Gamma^i$ under $\nu^i$ and $\Gamma_0^{i+1} \setminus \Gamma_0^{i+1}$ under $\nu_0^i$ and to couple these samples so that the conditional probability of the event $\tilde{\mathcal{S}}_i := \{\neg\Upsilon^{i+1} = \Upsilon_0^{i+1} \in \mathcal{R}_i\}$ is greater than some absolute constant $c_3(\epsilon)$, where $\neg\Upsilon$ is the color-switched of $\Upsilon$. With the same proof, the coupling succeeds with probability $1 - (r/R)^{k(\alpha)}$, and one ends up, as desired, with two identical faces $\Theta(r)$ and $\Theta_0(r)$ with reversed color. □

*Remark* 4.4. The proof adapts easily to the case of $\mathbb{Z}^2$, even for the color switching argument which sometimes can be troublesome with $\mathbb{Z}^2$.

We will also need the following proposition, which is very similar to Proposition 4.1.

**Proposition 4.5.** *Let $\Omega$ be some piecewise smooth simply connected with $0 \in \Omega$. Let $d > 0$ be the distance from 0 to the boundary $\partial\Omega$ and $d' = d \wedge 1$. For any $0 < r < d'$ call $\mathcal{A}(r, \partial\Omega)$ the event that there are four (alternating) arms from the square of radius $r$ to the boundary $\partial\Omega$. Call also $\mathcal{A}(r, 1)$ the event that there are four arms from radius $r$ to radius 1. For any $10\eta < r < \frac{d'}{10}$, call $\nu$ the law on the percolation configurations conditioned on the event $\mathcal{A}(r, \partial\Omega)$ and call $\nu_0$ the law conditioned on the event $\mathcal{A}(r, 1)$.*

*Then, there is a coupling of the conditional laws $\nu, \nu_0$, such that with (conditional) probability at least $1 - (r/d')^k$, the event $\mathcal{G}(r, d'/2)$ (as defined previously) is satisfied for both configurations and the induced faces at radius $r$ are identical. Here $k > 0$ is some absolute exponent.*

**Proof.** This is proved in the same way as Proposition 4.1. The only difference is that whenever in that proof we used RSW to connect arms to the faces of $\Theta$ with a positive probability that depended on the quality $\mathbf{Q}(\Theta) \geq \alpha$, we now do not have the restriction to use this fixed configuration of arms, hence we get a uniform positive probability, independently of everything. Consequently, the exponent $k > 0$ is an absolute constant now. □

# 5 A measure on pivotals which is measurable with respect to the scaling limit

In this section we define a natural scaling limit of the counting measure on pivotals, normalized so that the expected measure of the points in $[0,1]^2$ which are 1-important is one.

Of course, on the discrete level, for a given configuration, any point is pivotal (or important) on some scale (only $\eta$ if it is surrounded by 5 hexagons of the same color, for example); so in order to keep a meaningful measure at the limit, we need to keep track of points in the mesh which are pivotals on a macroscopic scale. Section 2 tells us roughly that there is no loss of information if we keep track of all points which are at least $\epsilon$-important. So we are interested in the scaling limit of the following (random) counting measures in the plane, parametrized by the scale "cut-off" $\rho > 0$:

$$\mu_\eta^\rho := \sum_{x:\,\rho-\text{important}} \delta_x\, \eta^2 \alpha_4(\eta, 1)^{-1}. \tag{5.1}$$

But for convenience, we will actually consider measures which are defined in a somewhat less symmetric way, but will turn out to be less sensitive to local effects in our proof. This needs some definitions:

*Definition* 5.1. Let $A$ be some closed topological annulus of the plane; the bounded component of $A^c$ will be called its **inside face**; $\partial_1 A$ and $\partial_2 A$ will be its inner and outer boundaries. We will say that $A$ is a **proper annulus** if its inner and outer boundaries are piecewise smooth curves.

*Definition* 5.2. Let $\mathcal{H}$ be a family of proper annuli in the plane. It will be called an **enhanced tiling** if the collection of the inner boundaries form a locally finite tiling of the plane. The **inner diameter** of $\mathcal{H}$ will be

$$\operatorname*{diam}_1 \mathcal{H} := \sup_{\Delta:\text{ inside face } \in \mathcal{H}} \{\operatorname{diam} \Delta\}.$$

The **outer diameter** of $\mathcal{H}$ will be

$$\operatorname*{diam}_2 \mathcal{H} := \sup_{A \in \mathcal{H}} \{\operatorname{diam} A\}.$$

Let $A$ be some proper annulus of the plane with inside face $\Delta$; a point $x \in \Delta$ in the triangular grid of mesh $\eta$ is called $A$**-important** if there are 4 arms in $\omega_\eta$ from $x$ to the outer boundary $\partial_2 A$ of $A$. We define the following measure:

$$\mu_\eta^A := \mu_\eta^A(\omega_\eta) = \sum_{x \in \Delta:\, A\text{-important}} \delta_x \, \eta^2 \alpha_4(\eta, 1)^{-1} \,. \tag{5.2}$$

Now, let $\mathcal{H}$ be some enhanced tiling, then define

$$\mu_\eta^{\mathcal{H}} := \mu_\eta^{\mathcal{H}}(\omega_\eta) = \sum_{A \in \mathcal{H}} \mu_\eta^A \,. \tag{5.3}$$

Note that in the so defined measure $\mu_\eta^{\mathcal{H}}$, we do not count the points which might lie on the boundary $\partial_1 A$ of some annulus $A \in \mathcal{H}$, but this has no effect: indeed for a fixed proper annulus $A$, it is straightforward to check that when $\eta$ goes to 0, the probability that there is some $A$-important hexagon $x$ intersecting $\partial_1 A$ is going to zero.

We will now prove that for any fixed proper annulus $A$, the random measure $\mu_\eta^A(\omega_\eta)$ has a scaling limit $\mu^A$ when $\eta$ goes to zero, and moreover $\mu^A = \mu^A(\omega)$ is a measurable function of the continuum percolation $\omega$, as defined in Section 3. More precisely, we have the following theorem:

**Theorem 5.3.** *Let $A$ be a fixed proper annulus of the plane. When $\eta \to 0$, the random variable $(\omega_\eta, \mu_\eta^A)$ converges in law to some $(\omega, \mu^A)$, where $\omega$ is the scaling limit of critical percolation, and $\mu^A = \mu^A(\omega)$ is a function of $\omega$.*

**Proof.** First we show that the family of variables $(\omega_\eta, \mu_\eta^A)_{\eta>0}$ is tight. In Section 3, we defined $\omega_\eta$ as a Borel measure on the compact separable metrizable space $(\mathscr{H}_D, \mathcal{T}_D)$, hence $(\omega_\eta)_{\eta>0}$ is obviously tight. It is a standard fact that if $(X_\eta)_{\eta>0}$, $(Y_\eta)_{\eta>0}$ are tight families of variables, then, in any coupling, the coupled family of variables $(X_\eta, Y_\eta)_{\eta>0}$ is tight, as well. So, it is enough to prove that the family of measures $(\mu_\eta^A)_{\eta>0}$ is tight. Since $\mu_\eta^A$ are finite measures supported on the inside face $\Delta$, proving tightness boils down to proving

$$\limsup_{\eta \to 0} \mathbb{E}\big[\mu_\eta^A(\Delta)\big] < \infty \,.$$

This is straightforward by the definition of $\mu_\eta^A$. Indeed, let $d > 0$ be the distance between $\partial_1 A$ and $\partial_2 A$; a point in $\Delta$ has to be $d$-important in order to be $A$-important. Therefore, if $d' = d \wedge 1/2$:

$$\begin{aligned}\mathbb{E}\big[\mu_\eta^A(\Delta)\big] &= \sum_{x\in\Delta} \mathbb{P}\big[x \,\text{is}\, A\text{--important}\big]\,\eta^2\,\alpha_4(\eta,1)^{-1} \\ &\le \sum_{x\in\Delta} \alpha_4(\eta,d')\,\eta^2\,\alpha_4(\eta,1)^{-1} \\ &\asymp \text{area}(\Delta)\,\alpha_4(d',1)^{-1} < \infty.\end{aligned}$$

The last inequality follows from quasi-multiplicativity and the fact that $\alpha_4(d',1) = \alpha_4^{(\eta)}(d',1)$ depends on $\eta$ but converges to the macroscopic probability $\alpha_4(d',1)$ (same notation) when $\eta$ goes to zero. This proves tightness of $(\omega_\eta, \mu_\eta^A)_{\eta>0}$.

Therefore, there exists some subsequential scaling limit $(\omega, \mu^A)$ along some subsequence $(\eta_k)_{k>0}$, where $\eta_k$ goes to 0. We will show that this $\mu^A$ can actually be recovered from $\omega$, which is the unique subsequential scaling limit of $(\omega_\eta)_{\eta>0}$, as we already know from Section 3. Consequently, the pair $(\omega, \mu^A)$ will also be unique.

Since the subsequential scaling limit $\mu^A$ is a measure, we will not need to check sigma-additivity etc., only need to characterize uniquely the law of this random measure. For this, it will be enough to determine, for any ball $B \subset \Delta$, the value of $\mu^A(B)$ as a function of $\omega$. The strategy of the proof is as follows. If $\eta$ is a small mesh, we are interested in $\mu_\eta^A(B)$, that is, in the number $X = X_\eta$ of points inside $B$ which are $A$-important (indeed, by definition, $\mu_\eta^A(B) = \frac{X}{\eta^{-2}\alpha_4(\eta,1)}$). We first need to show that $X$ can be "guessed" with arbitrary good precision from the macroscopic scale, since only the macroscopic information is preserved in the scaling limit $\omega$. In order to do so, fix any grid of squares of side-length $\epsilon$, and let $Y = Y_\eta^\epsilon$ be the number of squares $Q$ in this grid which are contained in $B$ and which satisfy a 4-arms event from $2Q$ to $\partial_2 A$. We will prove the following lemma.

**Lemma 5.4.** *For any proper annulus $A$, and for any ball $B \subset \Delta$, if we fix the $\epsilon$-grid $G$ to be the set of $\epsilon$-squares centered at the points of the lattice $2\epsilon\, e^{i\theta}\mathbb{Z}^2 + a$, with $\theta \in \mathbb{R}$ and $a \in \mathbb{C}$, then*

$$\mathbb{E}\left[\left(\frac{X_\eta}{\mathbb{E}\big[X_\eta\big]} - \frac{Y_\eta^\epsilon}{\mathbb{E}\big[Y_\eta^\epsilon\big]}\right)^2\right] = o(1)$$

*when $\epsilon$ and $\eta/\epsilon$ go to zero, where the constants involved do not depend on the choice of the parameters $(\theta,\, a)$ of the $\epsilon$-grid $G$.*

After this, we will still have to show that these macroscopic approximations of $\mu_\eta^A(B)$ actually converge as $\epsilon \to 0$ and $\eta/\epsilon \to 0$: that limit will be the unique limit $\mu^A(B)$. This will be done in Lemma 5.5.

**Proof of Lemma 5.4.** Let us start with a rough outline of the proof. The main idea is that, conditioned on an $\epsilon$-box $Q$ to have the 4-arm event to $\partial_2 A$, and conditioned on the entire percolation configuration outside an enlarged box $Q'$, with side-length $r \gg \epsilon$, the distribution of the set of points in $Q$ that are $A$-important should be almost independent of the configuration outside $Q'$ and of the location of the $\epsilon$-box within the inner face $\Delta$ of $A$. The justification for this idea is, of course, our coupling results in Section 4. Thus, if two $\epsilon$-boxes are farther from each other than $r \gg \epsilon$, and both have the 4-arm event to $\partial_2 A$, i.e., they are both counted in $Y$, then their contributions to $X$ should be the same on average, and should be almost independent from each other. On the other hand, if $r \ll 1$, then there are much more pairs of $\epsilon$-boxes that are $r$-far from each other than those that are close. This suggests that from $Y$ we should be able to guess $X$ with a small variance.

However, there are two issues about the above conditioning that need some care. Firstly, to apply the coupling of Section 4, we cannot really condition arbitrarily on the configuration outside $Q'$: we need to have only four faces around $Q'$. Secondly, even after we condition on the entire configuration outside some $Q'_i$, the set of $A$-important points inside some $Q_j \subset (Q'_i)^c$ might still depend on the connections inside $Q'_i$; so, we will have to condition not just on $(Q'_i)^c$ and on having the 4-arm event from $Q_i$ to $\partial_2 A$, but also on how the faces around $Q'_i$ are connected to each other from the inside. This extra conditioning would be hard to handle if there were many faces around $Q'_i$, so, just like with the first issue, we need to have only four faces around $Q'_i$. To solve this problem, we will take an intermediate box $Q''_i$ between $Q_i$ and $Q'_i$, with side-length $\gamma$ satisfying $\epsilon \ll \gamma \ll r$, and argue that having at least 5 arms in the thick annulus $Q'_i \setminus Q''_i$ is much more costly than having only 4, hence with good probability there are only 4 faces around $Q''_i$, and we can start the coupling from there. And then we can also condition on how the 4 arms around $Q''_i$ are connected to each other: this is the reason why we had the conditioning on $\{U_\Theta = \tau\}$ in Proposition 4.1.

In the following, a box $B(q, r)$ centered at $q$ (which will usually be the center of a tile) of radius $r$ will denote the set of tiles whose center is included inside $e^{i\theta}[-r, r)^2 + q$. Let us consider in particular the square $Q_0$ in $G$ centered around $a$ ($Q_0 = B(a, \epsilon)$ is not necessarily in $B$). Let $x_0$ be the number of points inside $Q_0$ which are $(B(a,1) \setminus Q_0)$-important. Following the notations of Section 4, call $\mathcal{A}_0(2\epsilon, 1)$ the event that there are 4 arms from $B(a, 2\epsilon)$ to $\partial B(a, 1)$; as well, call $U_0$ the indicator function that there is a left-right crossing in $B(a, 1)$ (here, the "left side" of $B(a, 1)$ is the image of the usual left side after multiplying by $e^{i\theta}$).

We define

$$\beta = \mathbb{E}\big[x_0 \bigm| \mathcal{A}_0(2\epsilon, 1),\, U_0 = 1\big]. \tag{5.4}$$

The lemma will follow from the following claim:

$$\mathbb{E}\big[(X - \beta Y)^2\big] = o(\mathbb{E}\big[X^2\big]), \tag{5.5}$$

when $\epsilon$ and $\eta/\epsilon$ go to 0. Indeed, assuming we have proved (5.5), and using the fact that there is a constant $C > 0$ (which might depend on the sets $A$ and $B$) such that $\mathbb{E}\big[X^2\big] < C\mathbb{E}\big[X\big]^2$ and $\mathbb{E}\big[Y^2\big] < C\mathbb{E}\big[Y\big]^2$ , we have

$$\begin{aligned}
\mathbb{E}\big[X - \beta Y\big] &\leq \sqrt{\mathbb{E}\big[(X - \beta Y)^2\big]} \quad \text{by Jensen's inequality} \\
&\leq \sqrt{o(\mathbb{E}\big[X^2\big])} \quad \text{by (5.5)} \\
&\leq o(\mathbb{E}\big[X\big]) \quad \text{by second moment for } X\,.
\end{aligned} \tag{5.6}$$

This allows us to conclude as follows:

$$\begin{aligned}
\left\| \frac{X}{\mathbb{E}[X]} - \frac{Y}{\mathbb{E}[Y]} \right\|_2 &\leq \left\| \frac{X - \beta Y}{\mathbb{E}[X]} \right\|_2 + \left\| \frac{\beta Y}{\mathbb{E}[X]} - \frac{Y}{\mathbb{E}[Y]} \right\|_2 \\
&\leq o(1) + \frac{|\beta \mathbb{E}[Y] - \mathbb{E}[X]|}{\mathbb{E}[X]\mathbb{E}[Y]} \sqrt{\mathbb{E}\big[Y^2\big]} \quad \text{by (5.5) and second moment for } X \\
&\leq o(1) \quad \text{by (5.6) and second moment for } Y.
\end{aligned}$$

It will be helpful to keep in mind the following easy estimates:

$$\begin{cases}
\beta & \asymp \quad \epsilon^2 \eta^{-2} \alpha_4(\eta, \epsilon) \\
\mathbb{E}\big[X\big] & \asymp \quad \eta^{-2} \alpha_4(\eta, 1) \\
\mathbb{E}\big[X^2\big] & \asymp \quad \eta^{-4} \alpha_4(\eta, 1)^2\,.
\end{cases}$$

We now start proving claim (5.5). Let $S_1, \ldots, S_p$ denote the list of $\epsilon$-squares in the grid $G$ which are contained in $B$. For each square $S_i$, let $Q_i$ be the set of $\eta$-tiles whose center is included in $S_i$ (to avoid multiple allocation, each square is considered to be $[-\epsilon, \epsilon)^2$, this convention being mapped by $z \mapsto e^{i\theta} z + a$). Therefore, the set of $\eta$-tiles inside $B$ is partitioned into the "tiling" $\{Q_i\}_i$ plus some boundary (or exterior) $\eta$-tiles which are at distance at most $2\epsilon$ from $\partial B$. Let $Q_{\text{ext}}$ be the set of these $\eta$-tiles. Because of issues coming from the discrete lattice (mesh $\eta$), we need to change slightly our definition of $Y$. For each $1 \leq i \leq p$, let $q_i$ be the closest $\eta$-tile, in any reasonable sense, to the center of $Q_i$; define $y_i$ to be the indicator function of the event that there are 4 arms from $B(q_i, 2\epsilon)$ to $\partial_2 A$. Notice that the tiles $B(q_i, 2\epsilon)$ for $1 \leq i \leq p$ are all translate of each other and are identical to $B(a, 2\epsilon)$ which is used in the definition of $\beta$. This will be relevant when we will apply the coupling argument (Proposition 4.1) to our situation. We now define $Y := y_1 + \ldots + y_p$. (Notice that, in the scaling limit, this definition matches with our original one, which is the only relevant thing to us.) Similarly, for any $1 \leq i \leq p$, let $x_i$ be the number of $A$-important points inside $Q_i$, and let $x_{\text{ext}}$ be the number of $A$-important points in $Q_{\text{ext}}$. So, the number $X$ of $A$-important points in $B$ is $X = x_1 + \ldots + x_p + x_{\text{ext}}$.

First, we briefly explain why the additional term $x_{\text{ext}}$ has a negligible contribution. Notice that the estimate of $\mathbb{E}\big[x_{\text{ext}}^2\big]$ is similar to the second moment of the number of points which are 1-important inside a band of length 1 and width $\epsilon$. Writing this second moment as a sum over couples of points, it is easy to check that in this case the main contribution comes from couples of points which are about $\epsilon$-close (unlike the second moment for the number of 1-important points in a square where the main contribution comes from "distant" points). Therefore,

$$\begin{aligned}\mathbb{E}\big[x_{\text{ext}}^2\big] &\leq O(1)\epsilon^{-1}\epsilon^4\eta^{-4}\alpha_4(\eta,1)^2/\alpha_4(\epsilon,1)\\ &\leq O(1)\epsilon^3\alpha_4(\epsilon,1)^{-1}\eta^{-4}\alpha_4(\eta,1)^2.\end{aligned}$$

Since $\epsilon^3\alpha_4(\epsilon,1)^{-1} = o(1)$, when $\epsilon$ goes to 0, this implies $\mathbb{E}\big[x_{\text{ext}}^2\big] = o(\mathbb{E}\big[X^2\big])$ (when $\epsilon$ goes to 0, uniformly in $\eta \in (0,\epsilon)$). Another way to check that $\mathbb{E}\big[x_{\text{ext}}^2\big]$ is negligible is to notice that with probability going to 1 when $\epsilon$ goes to zero, $x_{\text{ext}} = 0$ and when it is non zero it is obviously less (in average) than $\mathbb{E}\big[X^2\big]$.

Hence this allows us to restrict to $\bar{X} = x_1 + \ldots + x_p$, since

$$\|X - \beta Y\|_2 \leq \|\bar{X} - \beta Y\|_2 + \|x_{\text{ext}}\|_2\,,$$

and it is enough to prove that

$$\mathbb{E}\big[(\bar{X} - \beta Y)^2\big] = o(\mathbb{E}\big[X^2\big]).$$

Now let us fix some $\delta > 0$. We need to prove that for $\epsilon$ and $\eta/\epsilon$ small enough, we have

$$\mathbb{E}\big[(\bar{X} - \beta Y)^2\big] \le \delta\, \mathbb{E}\big[X^2\big]. \tag{5.7}$$

Let us write

$$\mathbb{E}\big[(\bar{X} - \beta Y)^2\big] = \sum_{i,j} \mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big]; \tag{5.8}$$

so, in order to prove (5.7), we need to control the correlations between the number of $A$-important points in squares $Q_i$ and $Q_j$. If the squares are close, then $x_i$ and $x_j$ are highly correlated, and some correlation is still there even if the squares are far away from each other, but at least in that case we will control their dependence well enough. So, if $r = r(\delta)$ is any distance that we will choose later depending on $\delta$, it will be convenient to split the above sum (5.8) into a "diagonal" term corresponding to nearby squares, plus a term corresponding to distant squares:

$$\begin{aligned} \mathbb{E}\big[(\bar{X} - \beta Y)^2\big] = &\sum_{d(Q_i,Q_j)\le r} \mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big] \\ &+ \sum_{d(Q_i,Q_j)> r} \mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big]\,. \end{aligned} \tag{5.9}$$

First, we estimate from above the first (diagonal) term. Take any $i, j$ such that $d(Q_i, Q_j) \le r$. We want to bound from above $\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big]$ (note that this might as well be negative, in which case it would "help" us). We have

$$\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big] \le \mathbb{E}\big[x_i x_j + \beta^2 y_i y_j\big] \tag{5.10}$$

We deal with the term $\sum_{d(Q_i,Q_j)\le r} \mathbb{E}\big[x_i x_j\big]$, the other one being treated in a similar way. There are $O(1)\epsilon^{-2}$ choices for the box $Q_i$ (where $O(1)$ depends on $B$). Choose one of the $Q_i$ boxes. For any $k \ge 0$, such that $2^k\epsilon < r$, there are $O(1)2^{2k}$ boxes $Q_j$ satisfying $2^k\epsilon \le d(Q_i, Q_j) < 2^{k+1}\epsilon$. For any of these boxes, we have

$$\begin{aligned} \mathbb{E}\big[x_i x_j\big] &= \sum_{x\in Q_i,\, y\in Q_j} \mathbb{P}\big[x,\, y \text{ are } A\text{-important}\big] \\ &\le O(1)\epsilon^4\eta^{-4}\frac{\alpha_4(\eta,1)^2}{\alpha_4(2^k\epsilon,1)}. \end{aligned}$$

So, this gives us

$$\begin{aligned}\sum_{d(Q_i,Q_j)\le r} \mathbb{E}\big[x_i x_j\big] &\le O(1)\epsilon^{-2} \sum_{k\le \log_2(r/\epsilon)} 2^{2k}\alpha_4(2^k\epsilon,1)^{-1}\epsilon^4\eta^{-4}\alpha_4(\eta,1)^2 \\ &\le O(1) r^2 \alpha_4(r,1)^{-1}\eta^{-4}\alpha_4(\eta,1)^2 \\ &\le O(1) r^2\alpha_4(r,1)^{-1}\mathbb{E}\big[X^2\big].\end{aligned}$$

Since $r^2\alpha_4(r,1)^{-1} = o(1)$ when $r$ goes to zero, by choosing $r = r(\delta)$ small enough and by applying the same argument to the other term of (5.10), we obtain that $\sum_{d(Q_i,Q_j)\le r}\mathbb{E}\big[(x_i-\beta y_i)(x_j-\beta y_j)\big] \le \delta/2\mathbb{E}\big[X^2\big]$.

We now turn to the second term $\sum_{d(Q_i,Q_j)>r}\mathbb{E}\big[(x_i-\beta y_i)(x_j-\beta y_j)\big]$ in (5.9). For the diagonal term, the strategy was to use the fact that there were few terms in the sum, and that each of the terms were of reasonable size; here we have many terms to deal with, so we need to proceed differently: we will prove that if $\epsilon$ and $\eta/\epsilon$ are small enough, then for any $i, j$ such that $l := d(Q_i, Q_j) > r$ we have:

$$\mathbb{E}\big[(x_i-\beta y_i)(x_j-\beta y_j)\big] \le \frac{\delta}{2}\,\mathbb{E}\big[x_i x_j\big].$$

Let $q_i$ and $q_j$ be the respective centers of these squares. Let $\gamma \in (2\epsilon, r/4)$ be some intermediate distance whose value will be fixed later. Following the notations of Section 4, let $\Upsilon$ be the set of all interfaces crossing the annulus $B(q_i, l/2)\setminus B(q_i,\gamma)$. As previously, $\mathbf{Q}(\Upsilon)$ will denote the least distance between the endpoints of the interfaces on $\partial B(q_i,\gamma)$ renormalized by $\gamma$. Let $\mathcal{A}_4 = \mathcal{A}_4(\gamma, l/2)$ be the event that there are at least 4 arms of alternating color in the annulus $B(q_i, l/2)\setminus B(q_i,\gamma)$. Furthermore, let $\mathcal{A}_5$ be the event that there are at least 5 arms in the same annulus, with four of them of alternating color. Recall that $\mathcal{G} = \mathcal{G}(\gamma, l/2)$ is the event that there are exactly 4 alternating arms (thus $\mathcal{G} = \mathcal{A}_4\setminus\mathcal{A}_5$), and $\mathcal{T}^\alpha = \mathcal{T}^\alpha(\gamma, l/2)$ is the event that $\{\mathbf{Q}(\Upsilon) > \alpha\}$. We now define the following disjoint events:

$$\begin{cases} \mathcal{W}_i &= \mathcal{G}\cap\mathcal{T}^\alpha \\ \mathcal{Z}_i &= \mathcal{A}_5 \cup (\mathcal{A}_4\cap\neg\mathcal{T}^\alpha)\,. \end{cases}$$

In order for $(x_i-\beta y_i)(x_j-\beta y_j)$ to be non-zero, either $\mathcal{W}_i$ or $\mathcal{Z}_i$ must hold. We can thus write

$$\begin{aligned}&\mathbb{E}\big[(x_i-\beta y_i)(x_j-\beta y_j)\big] \qquad (5.11)\\ &\quad = \mathbb{E}\big[(x_i-\beta y_i)(x_j-\beta y_j)1_{\mathcal{Z}_i}\big] + \mathbb{E}\big[(x_i-\beta y_i)(x_j-\beta y_j)1_{\mathcal{W}_i}\big].\end{aligned}$$

We first want to bound the first term. By definition, $\mathbb{P}\big[\mathcal{Z}_i\big] \le \mathbb{P}\big[\mathcal{A}_5\big] + \mathbb{P}\big[\mathcal{A}_4 \cap \neg \mathcal{T}^\alpha\big]$. For the second event, notice that

$$\begin{aligned}\mathcal{A}_4 \cap \neg \mathcal{T}^\alpha &= \mathcal{A}_4(\gamma, l/2) \cap \neg \mathcal{T}^\alpha(\gamma, l/2) \\ &\subset \mathcal{A}_4(2\gamma, l/2) \cap \neg \mathcal{T}^\alpha(\gamma, 2\gamma)\,,\end{aligned}$$

since there are more interfaces between radii $\gamma$ and $2\gamma$ than between $\gamma$ and $l/2$, therefore the quality of the set of interfaces is smaller for the annulus $B(q_i, 2\gamma) \setminus B(q_i, \gamma)$. But as we mentioned in Section 4 (see also Lemma A.2 in [SS05]), there is a function $h(\alpha)$ such that (uniformly in $\eta \in (0, \epsilon/10)$), $\mathbb{P}\big[\neg \mathcal{T}^\alpha(\gamma, 2\gamma)\big] < h(\alpha)$, and furthermore $h(\alpha)$ goes to zero when $\alpha$ goes to zero. We then deduce (by independence on disjoint sets)

$$\mathbb{P}\big[\mathcal{Z}_i\big] \le O(1)\alpha_5(\gamma, l) + O(1)h(\alpha)\alpha_4(\gamma, l). \tag{5.12}$$

We have

$$\begin{aligned}\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)1_{\mathcal{Z}_i}\big] &\le \mathbb{E}\big[(x_i x_j + \beta^2 y_i y_j)1_{\mathcal{Z}_i}\big] \\ &\le \mathbb{P}\big[\mathcal{Z}_i\big]\mathbb{E}\big[x_i x_j + \beta^2 y_i y_j \bigm| \mathcal{Z}_i\big].\end{aligned}$$

Let $\mathcal{A}_{i,j}$ be the event that there are four arms from $B(q_j, 2\epsilon)$ to $B(q_j, l/2)$, four arms from $B(q_i, 2\epsilon)$ to $B(q_i, \gamma)$ and four arms from $B(\frac{q_i+q_j}{2}, l)$ to $\partial_2 A$. By independence on disjoint sets, we can write

$$\begin{aligned}\mathbb{E}\big[x_i x_j + \beta^2 y_i y_j \bigm| \mathcal{Z}_i\big] &= \mathbb{P}\big[\mathcal{A}_{i,j} \bigm| \mathcal{Z}_i\big]\mathbb{E}\big[x_i x_j + \beta^2 y_i y_j \bigm| \mathcal{Z}_i, \mathcal{A}_{i,j}\big] \\ &= \mathbb{P}\big[\mathcal{A}_{i,j}\big]\mathbb{E}\big[x_i x_j + \beta^2 y_i y_j \bigm| \mathcal{Z}_i, \mathcal{A}_{i,j}\big]\,.\end{aligned}$$

In order to have more independence, let us introduce the number $\tilde{x}_i$ of points in $Q_i$ which have four arms to $\partial B(q_i, 2\epsilon)$; it is clear that $x_i \le \tilde{x}_i$. We define $\tilde{x_j}$ in the same way. Then,

$$\begin{aligned}\mathbb{E}\big[x_i x_j + \beta^2 y_i y_j \bigm| \mathcal{Z}_i\big] &\le \mathbb{P}\big[\mathcal{A}_{i,j}\big]\mathbb{E}\big[\tilde{x}_i \tilde{x_j} + \beta^2 y_i y_j \bigm| \mathcal{Z}_i, \mathcal{A}_{i,j}\big] \\ &\le O(1)\alpha_4(\epsilon, \gamma)\alpha_4(\epsilon, 1)(\mathbb{E}\big[\tilde{x}_i \tilde{x_j}\big] + \beta^2) \\ &\le O(1)\alpha_4(\epsilon, \gamma)\alpha_4(\epsilon, 1)(\epsilon/\eta)^4\alpha_4(\eta, \epsilon)^2. \qquad (5.13) \\ &\le O(1)\epsilon^4\eta^{-4}\frac{\alpha_4(\eta, 1)^2}{\alpha_4(\gamma, 1)} \quad \text{by quasi-multiplicativity.}\end{aligned}$$

Combining (5.12) and (5.13) gives the following bound on the first term in (5.11):

$$\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)1_{\mathcal{Z}_i}\big] \;\le\; O(1)\left(\frac{\alpha_5(\gamma, l)}{\alpha_4(\gamma, l)} + h(\alpha)\right)\epsilon^4\eta^{-4}\frac{\alpha_4(\eta, 1)^2}{\alpha_4(l, 1)}.$$

On the other hand, it is easy to check that $\mathbb{E}\big[x_i x_j\big] \asymp \epsilon^4 \eta^{-4} \frac{\alpha_4(\eta,1)^2}{\alpha_4(l,1)}$, hence

$$\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j) 1_{\mathcal{Z}_i}\big] \leq O(1) \left( \frac{\alpha_5(\gamma, l)}{\alpha_4(\gamma, l)} + h(\alpha) \right) \mathbb{E}\big[x_i x_j\big]. \tag{5.14}$$

We need to bound now the second term $\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j) 1_{\mathcal{W}_i}\big]$. We will use for that purpose the coupling argument (Proposition 4.1). Indeed, on the event $\mathcal{W}_i$, there are exactly 4 arms crossing the annulus $B(q_i, l/2) \setminus B(q_i, \gamma)$, therefore there are exactly 4 interfaces crossing this annulus and, as we have seen in Section 4, they induce a configuration of faces $\Theta = \{\theta_1, \ldots, \theta_4\}$ at radius $\gamma$ around $q_i$ (here $\theta_1, \theta_3$ are the open faces). As in Section 4, let $\mathcal{D}_\Theta$ be the bounded component of $\mathbb{C} \setminus \Theta$ (which is a finite set of $\eta$-tiles) and let $U = U_\Theta$ be the indicator function that there is an open crossing from $\theta_1$ to $\theta_3$ in $\mathcal{D}_\Theta$. Let $\mathcal{F}_\Theta$ be the $\sigma$-field generated by the tiles in $\mathcal{D}_\Theta^c$. On the event $\mathcal{W}_i$, we may condition on $\mathcal{F}_\Theta$ in order to "factorize" the information in the $Q_i$ and $Q_j$ boxes, but notice that even if we condition on $\mathcal{F}_\Theta$ (and thus, in particular, we know all the information inside $Q_j$), the number $x_j$ of $A$-important points in $Q_j$ might still depend on the connectivities inside $Q_i$. That is why we also condition on $U_\Theta$ which gives the only information that is significant outside $\mathcal{D}_\Theta$ about what the connectivities are inside $\mathcal{D}_\Theta$. We end up with

$$\begin{aligned}
\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j) 1_{\mathcal{W}_i}\big] &= \mathbb{P}\big[\mathcal{W}_i\big] \mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j) \bigm| \mathcal{W}_i\big] \\
&= \mathbb{P}\big[\mathcal{W}_i\big] \mathbb{E}\big[\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j) \bigm| \mathcal{F}_\Theta, U_\Theta\big] \bigm| \mathcal{W}_i\big] \\
&= \mathbb{P}\big[\mathcal{W}_i\big] \mathbb{E}\big[(x_j - \beta y_j) \mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta, U_\Theta\big] \bigm| \mathcal{W}_i\big],
\end{aligned}$$

since $x_j - \beta y_j$ is measurable with respect to the $\sigma$-field generated by $\mathcal{F}_\Theta$ and $U_\Theta$ (which allows us to "factorize" the $Q_i$ and $Q_j$ boxes). We have

$$\begin{aligned}
&\Big| \mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j) 1_{\mathcal{W}_i}\big] \Big| \leq \\
&\qquad \mathbb{P}\big[\mathcal{W}_i\big] \mathbb{E}\Big[(x_j + \beta y_j) \big| \mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta, U_\Theta\big] \big| \Bigm| \mathcal{W}_i\Big].
\end{aligned} \tag{5.15}$$

As in Section 4, let $\mathcal{A}_\Theta = \mathcal{A}_\Theta(2\epsilon, \gamma)$ be the event that there are open arms from $B(q_i, 2\epsilon)$ to the open faces $\theta_1, \theta_3$ and closed arms from $B(q_i, 2\epsilon)$ to the closed faces $\theta_2, \theta_4$. Let $\mathcal{X}_\Theta$ be the event that there are open arms in $\mathcal{D}_\Theta^c$ from $\theta_1, \theta_3$ to $\partial_2 A$ and closed arms from $\theta_2, \theta_4$ to $\partial_2 A$, so that on the event $\mathcal{W}_i$ we have

$$\begin{aligned}\mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta, U_\Theta\big] &= 1_{\mathcal{X}_\Theta}\,\mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta, U_\Theta\big] \\ &= 1_{\mathcal{X}_\Theta}\,\mathbb{E}\big[x_i - \beta 1_{\mathcal{A}_\Theta} \bigm| \mathcal{F}_\Theta, U_\Theta\big] \\ &= 1_{\mathcal{X}_\Theta}\,\mathbb{P}\big[\mathcal{A}_\Theta \bigm| U_\Theta\big]\mathbb{E}\big[x_i - \beta \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta\big],\end{aligned} \tag{5.16}$$

by independence on disjoint sets and since on $\mathcal{W}_i \cap \mathcal{X}_\Theta$, $y_i = 1_{\mathcal{A}_\Theta}$.

Recall that $\beta = \mathbb{E}\big[x_0 \bigm| \mathcal{A}_0(2\epsilon, 1),\, U_0 = 1\big]$. Recall also that, on the event $\mathcal{W}_i$, the faces are well-separated: $\mathbf{Q}(\Theta) > \alpha$. Therefore one wishes to use the coupling argument Proposition 4.1. Note that we will use here Proposition 4.1 in a slightly more general form since we have boxes which are rotated by an angle $\theta$. It is clear that the same proof as in Proposition 4.1 applies here where the constant $k(\alpha)$ can be chosen independently of $e^{i\theta}$.

Finally, we need to be careful with the issues coming from the discrete lattice: indeed, $x_i$ is the number of points in $Q_i$ which are $A$-important, but $\beta$ is defined as a (conditional) expected number of points in $Q_0 = B(a, \epsilon)$, or, by translation invariance, in $B(q_i, \epsilon)$. However, $Q_i$ and $B(q_i, \epsilon)$ do not exactly coincide (at the boundary points). Hence let us introduce $\hat{x}_i$ to be the number of $A$-important points in $B(q_i, \epsilon)$. We have

$$x_i = \hat{x}_i + \sum_{x \in Q_i \setminus B(q_i,\epsilon)} 1_{x \text{ is } A\text{-important}} - \sum_{y \in B(q_i,\epsilon) \setminus Q_i} 1_{y \text{ is } A\text{-important}}.$$

There are $O(1)\epsilon\eta^{-1}$ such boundary points, each of them on the event $\mathcal{W}_i \cap \mathcal{X}_\Theta$ and conditioned on $(\mathcal{A}_\Theta, U_\Theta)$ are $A$-important with probability of order $O(1)\alpha_4(\eta, \epsilon)$. Hence, on the event $\mathcal{W}_i \cap \mathcal{X}_\Theta$, we have

$$\mathbb{E}\big[x_i | \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta\big] = \mathbb{E}\big[\hat{x}_i | \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta\big] + O(1)\epsilon\eta^{-1}\alpha_4(\eta, \epsilon). \tag{5.17}$$

In order to apply Proposition 4.1, one needs to consider both cases $U_\Theta = 1$ and $U_\Theta = 0$. On the event $\mathcal{W}_i \cap \mathcal{X}_\Theta$, if $\{U_\Theta = 1\}$ holds, we have

$$\mathbb{E}\big[\hat{x}_i - \beta \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta\big] = \mathbb{E}\big[\hat{x}_i \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 1\big] - \mathbb{E}\big[x_0 \bigm| \mathcal{A}_0(2\epsilon, 1),\, U_0 = 1\big].$$

Proposition 4.1 says that one can couple the probability measure conditioned on $\mathcal{A}_\Theta, \{U_\Theta = 1\}$ with the probability measure conditioned on $\mathcal{A}_0(2\epsilon, 1)$ so that with probability at least $1 - (2\epsilon/\gamma)^{k(\alpha)}$, we have $\hat{x}_i = x_0$. Let, as in Section 4, $\mathcal{S}$ be the event that the coupling succeeds. Hence, on the event

$\mathcal{W}_i \cap \mathcal{X}_\Theta \cap \{U_\Theta = 1\}$,

$$|\mathbb{E}\big[\hat{x}_i - \beta \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 1\big]| \le \left(\frac{2\epsilon}{\gamma}\right)^{k(\alpha)} \big(\mathbb{E}\big[\hat{x}_i\, 1_{\neg\mathcal{S}} \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 1\big] + \mathbb{E}\big[x_0\, 1_{\neg\mathcal{S}} \bigm| \mathcal{A}_0(2\epsilon, \gamma), U_0 = 1\big]\big). \tag{5.18}$$

Now let $\tilde{x}_i$ be the number of points in $B(q_i, \epsilon)$ which have four arms to $\partial B(q_i, 2\epsilon)$. Then $\hat{x}_i \le \tilde{x}_i$ and $\tilde{x}_i$ is independent of $\mathcal{F}_\Theta, \mathcal{A}_\Theta$. Thus

$$\mathbb{E}\big[\hat{x}_i\, 1_{\neg\mathcal{S}} \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 1\big] \le \mathbb{E}\big[\tilde{x}_i \bigm| U_\Theta = 1\big].$$

But recall that $\mathbf{Q}(\Theta) > \alpha$, since we are on the event $\mathcal{W}_i$. Therefore, as it is straightforward to check, there is some $c(\alpha) > 0$ such that $\mathbb{P}\big[U_\Theta = 1\big] \wedge \mathbb{P}\big[U_\Theta = 0\big] > c(\alpha)$. This means that conditioning on either value of $U_\Theta$ cannot increase the expectation of a non-negative variable by a factor larger than $1/c(\alpha)$, hence

$$\mathbb{E}\big[\tilde{x}_i \bigm| U_\Theta = 1\big] \le O_\alpha(1)\epsilon^2 \alpha_4(\epsilon, 1),$$

where $O_\alpha(1)$ depends on the quality threshold $\alpha$. Also, by introducing $\tilde{x_0}$, the number of points in $Q_0 = B(a, \epsilon)$ which have four arms to $\partial B(a, 2\epsilon)$, we have that $\mathbb{E}\big[x_0\, 1_{\neg\mathcal{S}} \bigm| \mathcal{A}_0(2\epsilon, \gamma), U_0 = 1\big] \le O(1)\epsilon^2\eta^{-2}\alpha_4(\eta, \epsilon)$ (notice here that there is no dependence upon $\alpha$). Hence (5.18) becomes

$$|\mathbb{E}\big[\hat{x}_i - \beta \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 1\big]| \le O_\alpha(1) \left(\frac{2\epsilon}{\gamma}\right)^{k(\alpha)} \epsilon^2\eta^{-2}\alpha_4(\eta, \epsilon). \tag{5.19}$$

Now, on the event $\mathcal{W}_i \cap \mathcal{X}_\Theta$, if $\{U_\Theta = 0\}$ holds, we have

$$\mathbb{E}\big[\hat{x}_i - \beta \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta\big] = \mathbb{E}\big[\hat{x}_i \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 0\big] - \mathbb{E}\big[x_0 \bigm| \mathcal{A}_0(2\epsilon, 1),\, U_0 = 1\big].$$

Again using Proposition 4.1, one can couple the two conditional probability measures so that, with probability at least $1 - (2\epsilon/\gamma)^{k(\alpha)}$, we have $\hat{x}_i = x_0$ (here the colors are reversed but nevertheless the $A$-important points are the same). In the exact same fashion, one ends up with

$$|\mathbb{E}\big[\hat{x}_i - \beta \bigm| \mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 0\big]| \le O_\alpha(1) \left(\frac{2\epsilon}{\gamma}\right)^{k(\alpha)} \epsilon^2\eta^{-2}\alpha_4(\eta, \epsilon). \tag{5.20}$$

Summarizing: on the event $\mathcal{W}_i$, we have rewritten (5.16) as

$$\mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta, U_\Theta\big] \tag{5.21}$$
$$= \begin{cases} 1_{\mathcal{X}_\Theta, U_\Theta=1}\,\mathbb{P}\big[\mathcal{A}_\Theta \bigm| U_\Theta = 1\big](\mathbb{E}\big[x_i|\mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 1\big] - \beta) \\ +\, 1_{\mathcal{X}_\Theta, U_\Theta=0}\,\mathbb{P}\big[\mathcal{A}_\Theta \bigm| U_\Theta = 0\big](\mathbb{E}\big[x_i|\mathcal{F}_\Theta, \mathcal{A}_\Theta, U_\Theta = 0\big] - \beta)\,, \end{cases}$$

and have bounded its different factors. The last ingredient is that

$$\mathbb{P}\big[\mathcal{A}_\Theta \bigm| U_\Theta\big] < O_\alpha(1)\alpha_4(\epsilon,\gamma)\,,$$

which holds because, as we argued above, $\mathbb{P}\big[U_\Theta = 1\big] \wedge \mathbb{P}\big[U_\Theta = 0\big] > c(\alpha) > 0$. Therefore, combining (5.17),(5.19) and (5.20) in (5.21) gives (still on the event $\mathcal{W}_i$):

$$\begin{aligned} |\mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta, U_\Theta\big]| &\le 1_{\mathcal{X}_\Theta} O_\alpha(1)\alpha_4(\epsilon,\gamma)\left(\left(\frac{2\epsilon}{\gamma}\right)^{k(\alpha)} \epsilon^2\eta^{-2} + \epsilon\eta^{-1}\right)\alpha_4(\eta,\epsilon) \\ &\le O_\alpha(1)\epsilon^2\eta^{-2}\alpha_4(\eta,\gamma)\left((2\epsilon/\gamma)^{k(\alpha)} + \frac{\eta}{\epsilon}\right). \end{aligned}$$

It is straightforward to check that $\mathbb{P}\big[\mathcal{W}_i\big] \le O(1)\alpha_4(\gamma, l)$, hence (5.15) becomes

$$\begin{aligned} &|\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)1_{\mathcal{W}_i}\big]| \\ &\qquad \le \mathbb{P}\big[\mathcal{W}_i\big]\mathbb{E}\big[(x_j + \beta y_j)\,\big|\mathbb{E}\big[x_i - \beta y_i \bigm| \mathcal{F}_\Theta,\, U_\Theta\big]\big| \bigm| \mathcal{W}_i\big] \\ &\qquad \le O_\alpha(1)\epsilon^2\eta^{-2}\alpha_4(\eta, l)\mathbb{E}\big[x_j + \beta y_j \bigm| \mathcal{W}_i\big]\left((2\epsilon/\gamma)^{k(\alpha)} + \frac{\eta}{\epsilon}\right). \end{aligned}$$

In order to bound $\mathbb{E}\big[x_j|\mathcal{W}_i\big]$, we introduce $x_j^*$, the number of points in $Q_j$ which have four arms to $\partial B(q_j, l/2)$, and we let $\mathcal{G}$ be the event that there are four arms from $B(\frac{q_i+q_j}{2})$ to $\partial_2 A$. By definition, $x_j \le x_j^* 1_{\mathcal{G}}$, therefore, by independence on disjoints sets ($x_j^*$ and $\mathcal{G}$ do not depend on $\mathcal{W}_i$), we obtain $\mathbb{E}\Big[x_j \bigm| \mathcal{W}_i\Big] \le O(1)\epsilon^2\eta^{-2}\alpha_4(\eta, 1)$. Similar estimates for $y_i$ lead to $\mathbb{E}\big[x_j + \beta y_j \bigm| \mathcal{W}_i\big] \le O(1)\epsilon^2\eta^{-2}\alpha_4(\eta, 1)$, hence

$$\begin{aligned} |\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)1_{\mathcal{W}_i}\big]| &\le O_\alpha(1)\epsilon^4\eta^{-4}\frac{\alpha_4(\eta,1)^2}{\alpha_4(l,1)}\left((2\epsilon/\gamma)^{k(\alpha)} + \frac{\eta}{\epsilon}\right) \\ &\le C(\alpha)\mathbb{E}\big[x_i x_j\big]\left((2\epsilon/\gamma)^{k(\alpha)} + \frac{\eta}{\epsilon}\right), \end{aligned}$$

for some fixed constant $C(\alpha) > 0$. The last expression combined with (5.11) and (5.14) gives

$$\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big] \leq O(1)\mathbb{E}\big[x_i x_j\big] \left( \frac{\alpha_5(\gamma, l)}{\alpha_4(\gamma, l)} + h(\alpha) + C(\alpha) \left(\frac{2\epsilon}{\gamma}\right)^{k(\alpha)} + C(\alpha)\frac{\eta}{\epsilon} \right).$$

Let us then fix the "quality threshold" $\alpha = \alpha(\delta)$ so that $O(1)h(\alpha) < \delta/8$. Recall we have already fixed $r = r(\delta)$ so that the diagonal term was less than $\delta/2\mathbb{E}\big[X^2\big]$. So we have $\gamma < r(\delta) < l$. It is a standard fact (proved by the BK inequality) that there is some $d > 0$ such that, for any $\gamma < l$, $\frac{\alpha_5(\gamma,l)}{\alpha_4(\gamma,l)} < (\gamma/l)^d$. So, we fix $\gamma$ so that $O(1)\frac{\alpha_5(\gamma,l)}{\alpha_4(\gamma,l)} < \delta/8$. Now by taking $\epsilon$ and $\eta/\epsilon$ small enough one obtains $\mathbb{E}\big[(x_i - \beta y_i)(x_j - \beta y_j)\big] \leq \delta/2\mathbb{E}\big[x_i x_j\big]$, which ends the proof of Lemma 5.4 □

We proved that we could guess $X/\mathbb{E}\big[X\big]$ from macroscopic observations $(Y/\mathbb{E}\big[Y\big])$, but the quantity we really need to guess is rather $\mu_\eta^A(B) = \frac{X}{\eta^2\alpha_4(\eta,1)}$. One can rewrite Lemma 5.4 in the following way:

$$\mathbb{E}\left[ \left( X - \frac{Y}{\mathbb{E}\big[Y\big]}\mathbb{E}\big[X\big] \right)^2 \right] = o(\mathbb{E}\big[X^2\big]),$$

when $\epsilon$ and $\eta/\epsilon$ go to zero. But since $\mathbb{E}\big[X^2\big] \asymp \eta^{-4}\alpha_4(\eta, 1)^2$, this implies

$$\mathbb{E}\left[ \left( \frac{X}{\eta^{-2}\alpha_4(\eta, 1)} - \frac{Y}{\mathbb{E}\big[Y\big]} \frac{\mathbb{E}\big[X\big]}{\eta^{-2}\alpha_4(\eta, 1)} \right)^2 \right] = o(1),$$

that is,

$$\mathbb{E}\left[ \left( \mu_\eta^A(B) - \frac{Y}{\mathbb{E}\big[Y\big]} \frac{\mathbb{E}\big[X\big]}{\eta^{-2}\alpha_4(\eta, 1)} \right)^2 \right] = o(1). \tag{5.22}$$

Thus, we need the following lemma.

**Lemma 5.5.** *For any parameters $a, \theta$ of the grid $G$,*

$$\frac{\mathbb{E}\big[X_\eta\big]}{\eta^{-2}\alpha_4(\eta, 1)} = (1 + o(1)) \frac{\mathbb{E}\big[Y_\eta^\epsilon\big]}{(2\epsilon)^{-2}\alpha_4(2\epsilon, 1)}$$

*as $\epsilon \to 0$, uniformly in $\eta \in (0, \epsilon/10)$ and $a, \theta$. Moreover, the limits*

$$\lim_{\eta \to 0} \frac{\mathbb{E}\big[X_\eta\big]}{\eta^{-2}\alpha_4(\eta, 1)} = m(A, B) > 0\,,$$

$$\lim_{\epsilon \to 0} \sup_{\eta < \epsilon/10} \left| \frac{\mathbb{E}\big[Y_\eta^\epsilon\big]}{(2\epsilon)^{-2}\alpha_4(2\epsilon, 1)} - m(A, B) \right| = 0$$

*exist, uniformly in $a, \theta$.*

**Proof.** We will use the same subdivision of the ball $B$ into $\epsilon$-squares $Q_1, \ldots, Q_p$ as in the previous proof. The boundary terms will be treated in the same way and we will make an extensive use of the other coupling, Proposition 4.5. Note that here, again, we apply the proposition in a slightly more general form than as it is stated, since our grid of squares is rotated by $e^{i\theta}$, but it is easy to check that Proposition 4.5 applies to this setting, with an exponent $k > 0$ that can be chosen independently of $\theta$.

Using the same notations as in the previous proof, we have $X = x_1 + \ldots + x_p + x_{\text{ext}}$. We have already seen that $\mathbb{E}\big[x_{\text{ext}}^2\big]$ is $o(\mathbb{E}\big[X^2\big])$ when $\epsilon \to 0$, uniformly in $\eta < \epsilon$, therefore, by Jensen's inequality, the second moment estimate for $\mathbb{E}\big[X^2\big]$ (i.e., $\mathbb{E}\big[X^2\big] < C\mathbb{E}\big[X\big]^2$), and $\mathbb{E}\big[X\big] \asymp \eta^{-2}\alpha_4(\eta, 1)$, we obtain that $\mathbb{E}\big[x_{\text{ext}}\big]$ is $o(\eta^{-2}\alpha_4(\eta, 1))$. Also, as in the previous proof, in order to use the coupling argument (Proposition 4.5), we will need to be careful with issues coming from the discrete lattice; but it will be somewhat simpler here, so we will not need to use $\hat{x}_1, \ldots, \hat{x}_p$ as in (5.17). Let $\bar{X} := x_1 + \ldots + x_p$; it remains to show that

$$\frac{\mathbb{E}\big[\bar{X}\big]}{\eta^{-2}\alpha_4(\eta, 1)} = \frac{\mathbb{E}\big[Y\big]}{(2\epsilon)^{-2}\alpha_4(2\epsilon, 1)} + o_{\epsilon \to 0}(1)\,,$$

where $o_{\epsilon \to 0}(1)$ goes to zero when $\epsilon$ goes to zero, uniformly in $\eta < \epsilon/10$. For any square $Q_i$, let $q_i$ be the closest $\eta$-tile (closest in any reasonable sense) to the center of $Q_i$. For any $\eta$-tile $x$ and $r > 0$, let $\mathcal{A}(x, r)$ be the event that there are four (alternate) arms from $x$ to $\partial B(x, r)$; also, let $\mathcal{A}(x, \partial_2 A)$ be the event that $x$ is $A$-important, i.e., that there are four arms from $x$ to the exterior boundary of the annulus $A$. In particular, by definition, for any $\eta$-tile $x$, $\mathbb{P}\big[\mathcal{A}(x, 1)\big] = \alpha_4(\eta, 1)$. For any $x \in Q_i$, let $\tilde{A}_x$ the annulus $A$ shifted by $x - q_i$ (so that the pair $(q_i, A)$ is a translate of the pair $(x, \tilde{A}_x)$). Notice that $\tilde{A}_x$ is shifted by less than $2\epsilon$, since $q_i, x \in Q_i$, a square of radius $\epsilon$.

$$
\begin{aligned}
\frac{\mathbb{E}\big[\bar{X}\big]}{\eta^{-2}\alpha_4(\eta,1)} &= \sum_i \eta^2 \sum_{x\in Q_i} \frac{\mathbb{P}\big[x \text{ is } A\text{-important}\big]}{\alpha_4(\eta,1)} \qquad (5.23)\\
&= \sum_i \eta^2 \sum_{x\in Q_i} \frac{\mathbb{P}\big[\mathcal{A}(x,\partial_2 A)\big]}{\mathbb{P}\big[\mathcal{A}(x,1)\big]}\\
&= \sum_i \eta^2 \sum_{x\in Q_i} \frac{\mathbb{P}\big[\mathcal{A}(x,\partial_2 A)\bigm|\mathcal{A}(B(x,2\epsilon),\partial_2 A)\big]}{\mathbb{P}\big[\mathcal{A}(x,1)\bigm|\mathcal{A}(B(x,2\epsilon),1)\big]}\frac{\alpha_4(B(x,2\epsilon),\partial_2 A)}{\alpha_4(2\epsilon,1)}.
\end{aligned}
$$

Now, if one applies Proposition 4.5 to the events $\mathcal{A}(B(x,2\epsilon),\partial_2 A)$ and $\mathcal{A}(B(x,2\epsilon),1)$, i.e., if one couples these two conditional probability measures around the same square $B(x,2\epsilon)$, one obtains

$$
\frac{\mathbb{P}\big[\mathcal{A}(x,\partial_2 A)\bigm|\mathcal{A}(B(x,2\epsilon),\partial_2 A)\big]}{\mathbb{P}\big[\mathcal{A}(x,1)\bigm|\mathcal{A}(B(x,2\epsilon),1)\big]} = 1+o_{\epsilon\to 0}(1)\,. \qquad (5.24)
$$

Now notice that

$$
\begin{aligned}
\mathbb{P}\big[\mathcal{A}(B(x,2\epsilon),\partial_2 A))\big] &= \mathbb{P}\big[\mathcal{A}(B(x,2\epsilon),\partial_2 \tilde{A}_x)\big](1+o_{\epsilon\to 0}(1))\\
&= \mathbb{P}\big[\mathcal{A}(B(q_i,2\epsilon),\partial_2 A)\big](1+o_{\epsilon\to 0}(1))\,. \qquad (5.25)
\end{aligned}
$$

Indeed,

$$
\mathcal{A}(B(x,2\epsilon),\partial_2 A))\Delta\mathcal{A}(B(x,2\epsilon),\partial_2\tilde{A}_x)
$$

holds only if there are four arms from $B(x,2\epsilon)$ to $B(x,d/2)$ (where $d$ is the distance between $\partial_1 A$ and $\partial_2 A$), and if there is some ball of radius $2\epsilon$ on $\partial_2 A$ which has three arms in $A\cup\tilde{A}_x$ up to distance $d/2$. Using the fact that the three arms exponent in $\mathbb{H}$ is two (also known for $\mathbb{Z}^2$), it is easy to show that

$$
\mathbb{P}\big[\mathcal{A}(B(x,2\epsilon),\partial_2 A))\Delta\mathcal{A}(B(x,2\epsilon),\partial_2\tilde{A}_x)\big] = o_{\epsilon\to 0}(1)\mathbb{P}\big[\mathcal{A}(B(x,2\epsilon),\partial_2\tilde{A}_x)\big]\,,
$$

implying the first line of (5.25). The second line follows by translation invariance. Now, writing (5.24) and (5.25) into (5.23) gives

$$
\begin{aligned}
\frac{\mathbb{E}\big[\bar{X}\big]}{\eta^{-2}\alpha_4(\eta,1)} &= \sum_i \eta^2 \sum_{x\in Q_i} \frac{\mathbb{P}\big[\mathcal{A}(B(q_i,2\epsilon),\partial_2 A)\big]}{\alpha_4(2\epsilon,1)}(1+o_{\epsilon\to 0}(1))\\
&= \sum_i 4\epsilon^2 \frac{\mathbb{P}\big[\mathcal{A}(B(q_i,2\epsilon),\partial_2 A)\big]}{\alpha_4(2\epsilon,1)}(1+o_{\epsilon\to 0}(1)).
\end{aligned}
$$

Now, since there are $O(1)\epsilon^{-2}$ squares, and since the constants involved in the various $o_{\epsilon\to 0}(1)$ are uniform in $\eta < \epsilon/10$ and can be chosen independently of $Q_i$, we get

$$\begin{aligned}\frac{\mathbb{E}\big[\bar{X}\big]}{\eta^{-2}\alpha_4(\eta,1)} &= \frac{1}{(2\epsilon)^{-2}\alpha_4(2\epsilon,1)}\sum_i \mathbb{P}\big[\mathcal{A}(B(q_i,2\epsilon),\partial_2 A)\big] + o_{\epsilon\to 0}(1)\\ &= \frac{\mathbb{E}\big[Y\big]}{(2\epsilon)^{-2}\alpha_4(2\epsilon,1)} + o_{\epsilon\to 1}(1),\end{aligned}$$

which is the first statement of the Lemma. Note that the term on the right in the above equation still depends on the mesh $\eta$, but it has a limit (given by $SLE$ computation) when $\eta$ goes to zero. This and the last equation together prove the Cauchy criterion for $\mathbb{E}\big[X\big]/(\eta^{-2}\alpha_4(\eta,1))$, when $\eta\to 0$, hence the existence of a limit $m(A,B)>0$ for $\mathbb{E}\big[X\big]/(\eta^{-2}\alpha_4(\eta,1))$, and therefore also for $\mathbb{E}\big[Y\big]/((2\epsilon)^{-2}\alpha_4(2\epsilon,1)))$. □

*Remark* 5.6. We proved Lemmas 5.4 and 5.5 for any angle $\theta$ and any translation parameter $a$ for the grid $G$ of $\epsilon$-squares. For more accurate notations, one should have used $Y_\eta^{\epsilon,a,\theta}$ instead of just $Y_\eta^\epsilon$. Moreover, in Lemma 5.5, the 4-arm probabilities $\alpha_4(2\epsilon,1)$ are with respect to a rotated and translated $\epsilon$-grid, with $\eta$ mesh, so they should be denoted by $\alpha_4^{\eta,a,\theta}$. Summarizing, (5.22) and Lemma 5.5 together imply that for any $a,\theta$,

$$\left\| \mu_\eta^A(B) - \frac{Y_\eta^{\epsilon,a,\theta}}{(2\epsilon)^{-2}\alpha_4^{\eta,a,\theta}(2\epsilon,1)}\right\|_2 = o(1), \tag{5.26}$$

when $\epsilon$ and $\eta/\epsilon$ go to 0, uniformly in the parameters $a,\theta$ and the ball $B\subset\Delta$.

**Proof of Theorem 5.3, continued.** The variable $Y = Y_\eta^{\epsilon,a,\theta}$ counts how many of certain macroscopic 4-arm events hold. Therefore, Remark 3.2 implies that, as $\eta$ goes to zero, $Y_\eta^{\epsilon,a,\theta}$ converges in law to a random variable $Y^{\epsilon,a,\theta}$ that is measurable with respect to the scaling limit $\omega$ of critical percolation. Since $\omega_\eta$ is a sequence of random variables on the separable space $(\mathscr{H}_D,\mathcal{T}_D)$, with weak limit $\omega$, there exists a coupling of all the $\omega_\eta$ and $\omega$ such that $\omega_\eta\to\omega$ almost surely. In this coupling, we then have, for any $\epsilon>0$, proper annulus $A$ and $B\subset\Delta$, almost surely $Y_\eta^{\epsilon,a,\theta}\to Y^{\epsilon,a,\theta}$.

Let us introduce the notations $F_\eta^{a,\theta}(\epsilon) := (2\epsilon)^{-2}\alpha_4^{\eta,a,\theta}(2\epsilon,1)$ and $F(\epsilon) := (2\epsilon)^{-2}\alpha_4(2\epsilon,1)$ for discrete and continuum percolation, so that $F_\eta^{a,\theta}(\epsilon)\to$

$F(\epsilon)$ as $\eta \to 0$. The dependence on $a, \theta$ is not shown here in the continuum version on purpose: our proof of Lemma 5.5 shows that the translation by $a$ always plays a negligible role, while the dependence on the rotation disappears in the limit because of the rotational invariance of the scaling limit (proved only for the triangular lattice). Hence $Y_\eta^{\epsilon,a,\theta}/F_\eta^{a,\theta}(\epsilon) \to Y^\epsilon/F(\epsilon)$ a.s. This and having a uniform upper bound on the second moments of $Y_\eta^{\epsilon,a,\theta}/F_\eta^{a,\theta}(\epsilon)$ and $Y^\epsilon/F(\epsilon)$ imply that

$$\lim_{\eta\to 0} \left\| \frac{Y_\eta^{\epsilon,a,\theta}}{F_\eta^{a,\theta}(\epsilon)} - \frac{Y^{\epsilon,a,\theta}}{F(\epsilon)} \right\|_2 = 0\,.$$

Combining with (5.26), this gives that $Y^\epsilon/F(\epsilon)$ is a Cauchy sequence in $L^2$ as $\epsilon \to 0$. Hence it has a limit:

$$\mu^A(B) := \lim_{\epsilon\to 0} \frac{Y^{\epsilon,a,\theta}}{(2\epsilon)^{-2}\alpha_4(2\epsilon,1)} \qquad \text{in } L^2. \tag{5.27}$$

Clearly, this $\mu^A(B) = \mu_A(B,\omega)$ is the distributional limit of the discrete variables $\mu_\eta^A(B,\omega_\eta)$, and Theorem 5.3 is proved. ☐

*Remark* 5.7. In (5.27) we have only convergence in $L^2$, not almost sure. We expect here that the stronger version does hold. However, our proof does not seem to yield this result, at least not without some additional work.

Note that it was convenient to introduce the measures $\mu_\eta^A$ for some proper annulus $A \subset \mathbb{C}$ instead of working directly with $\mu_\eta^\rho$, the counting measure on $\rho$-important points. Indeed, suppose that in the previous proof we were working with $\rho$-important points instead of $A$-important points. Then, for different points $x, y$ inside some $\epsilon$-square $Q_i$, given some configuration of faces around $Q_i$, we might need quite different information about where these faces are connected outside $Q_i$ if we want to know how $x$ and how $y$ has to be connected to these faces from the inside in order to be $\rho$-important. Indeed, if one of the four arms emanating from the faces around $Q_i$ goes to distance $\rho - \epsilon$ but not to $\rho$, then it might happen that some points in $Q_i$ that are connected "pivotally" to the four arms will be $\rho$-important while others will not. $A$-important points are simpler to handle in this respect.

Nevertheless, Section 2 deals with the concept of $\epsilon$-important points, so we need to relate in some way the measures $\mu_\eta^A$ with the measures $\mu_\eta^\rho$. The following partial order between enhanced tilings will handle this issue.

*Definition* 5.8. Let us say that an enhanced tiling $\mathcal{H}$ **refines** another, $\mathcal{H}'$, denoted by $\mathcal{H} \leq \mathcal{H}'$, if the following holds: for any pair of annuli $A = B_2 \setminus B_1 \in \mathcal{H}$ and $A' = B_2' \setminus B_1' \in \mathcal{H}'$, if the inner faces $B_1$ and $B_1'$ intersect each other, then $B_2 \subset B_2'$.

For example, one can consider the enhanced tiling

$$\mathcal{H}_\eta^\rho := \big\{ B(x,\rho) \setminus \{x\} : \text{all } \eta\text{-tiles } x \text{ of } D \big\}$$

in a domain $D$. Now, if $\mathcal{H}$ is an enhanced tiling with $\operatorname{diam}_2 \mathcal{H} < \rho$, then $\mathcal{H} \leq \mathcal{H}_\eta^\rho$. On the other hand, if $d(\partial_1 A, \partial_2 A) > 2\rho$ for all $A \in \mathcal{H}$, then $\mathcal{H} \geq \mathcal{H}_\eta^\rho$.

The point of this definition is that if $\mathcal{H} \leq \mathcal{H}'$, then we have the reversed domination between the associated annulus-pivotal measures: $\mu_\eta^{\mathcal{H}} \geq \mu_\eta^{\mathcal{H}'}$. Therefore, there is a coupled pair $(\mathcal{P}, \mathcal{P}')$ of Poisson samples from these measures such that $\mathcal{P} \supset \mathcal{P}'$. In particular, if $\big\{\mathcal{H}(\epsilon) : \epsilon \in I\big\}$ is an ordered family of enhanced tilings, i.e., $\mathcal{H}(\delta) \leq \mathcal{H}(\epsilon)$ whenever $\delta < \epsilon$, for $\delta, \epsilon \in I \subset \mathbb{R}_+$, then for each $\eta > 0$ we get a family $\big\{\mu_\eta^{\mathcal{H}(\epsilon)} : \epsilon \in I\big\}$ of increasing measures (as $\epsilon$ decreases), called a **filtered measure**, and there is an associated increasing family of Poisson samples $\big\{\mathcal{P}(\mathcal{H}(\epsilon)) : \epsilon \in I\big\}$.

If $\big\{\mathcal{H}(\epsilon)) : \epsilon \in I\big\}$ is an ordered family of enhanced tilings with $\operatorname{diam}_2 \mathcal{H}(\epsilon) = \epsilon$ and $d(\partial_1 A, \partial_2 A) > \epsilon/2$ for all $A \in \mathcal{H}(\epsilon)$, then we have the following dominations between the associated filtered measure and the $\rho$-pivotal measures. For any $\rho > 0$, if $\eta > 0$ is small enough, then

$$\mu_\eta^{\mathcal{H}(\rho)} \geq \mu_\eta^{\mathcal{H}_\eta^\rho} \geq \mu_\eta^{\mathcal{H}(\rho/4)} \,. \tag{5.28}$$

This will be our main tool for comparing annulus-pivotality to $\rho$-pivotality.

# 6 Conformal covariance of the “counting” measure on the pivotal points

Let $\Omega, \widetilde{\Omega}$ be two simply connected domains of the plane, and let $f : \Omega \to \widetilde{\Omega}$ be some conformal map. By conformal invariance, the image $\tilde{\omega} := f(\omega)$ is also a realization of continuum percolation in $\widetilde{\Omega}$. Consider some proper annulus $A \subset \bar{A} \subset \Omega$. Since $f$ is conformal on $\Omega$, we have that $f(A)$ is again a proper annulus, and $\mu^{f(A)}$ is the scaling limit of $\mu_\eta^{f(A)}$. We will prove the following:

**Theorem 6.1.** *Let $f_*(\mu^A(\omega))$ be the pushforward measure of $\mu^A$. Then, for almost all $\omega$, the Borel measures $\mu^{f(A)}(\tilde{\omega})$ and $f_*(\mu^A(\omega))$ on $f(\Delta)$ are absolutely continuous w.r.t. each other, and their Radon-Nikodym derivative satisfies, for any $w = f(z) \in \widetilde{\Omega}$,*

$$\frac{d\mu^{f(A)}(\tilde{\omega})}{df_*(\mu^A(\omega))}(w) = |f'(z))|^{3/4},$$

*or equivalently, for any Borel set $B \subset \Delta$,*

$$\mu^{f(A)}(f(B))(\tilde{\omega}) = \int_B |f'|^{3/4} d\mu^A(\omega)\,.$$

*Remark* 6.2. Using the conformal invariance of the scaling limit, from the almost sure equality in the theorem we get that $\mu^{f(A)}(f(B))$ has the same law as $\int_B |f'|^{3/4} d\mu^A$.

On a heuristical level, the scaling exponent $3/4$ comes from the fact that $\mu^A$ is a "natural" measure supported on the set of pivotal points, which is known to be of Hausdorff-dimension $3/4$; see [Bef04]. Of course, to make this explanation more grounded, one would also need to prove that the support of the measure $\mu^A(\omega)$ is exactly the set of $A$-important points of $\omega$.

Since any conformal map is locally a rotation times a dilatation, we will, as a warm-up, first check the theorem on these particular cases. This will be much easier than the general case, mainly because the grid of $\epsilon$-squares that we used in defining the approximating macroscopic quantities $Y^\epsilon$ is preserved quite nicely under rotations and dilatations, while distorted by a general conformal map $f$.

## 6.1 Rotational invariance

Let us consider some proper annulus $A$ of the plane, some ball $B \subset \Delta$, and the rotation $T: z \mapsto \exp(i\theta)\, z$ by an angle $\theta$. We need to show that

$$\mu^A(B, \omega) = \mu^{TA}(TB, T\omega)\,. \tag{6.1}$$

By (5.27), the right hand side equals $\lim_\epsilon Y^{G(\epsilon),TA}(TB, T\omega)/F(\epsilon)$, an $L^2$-limit, where $G(\epsilon)$ is our usual grid of $\epsilon$-squares. Or, by using a rotated grid, it is also equal to $\lim_\epsilon Y^{TG(\epsilon),TA}(TB, T\omega)/F(\epsilon)$, which (by rotating back the entire universe) is trivially the same as $\lim_\epsilon Y^{G(\epsilon),A}(B, \omega)/F(\epsilon)$, giving the left hand side of (6.1), as desired.

*Remark* 6.3. One might speculate that this type of rotational invariance should hold even if we have a scaling limit $\omega$ without (or unproved) rotational invariance (such as a subsequential limit of critical percolation on $\mathbb{Z}^2$), since the backbone of the above argument seems to be the following: $\mu^A$ is a function of $\omega$, and the definition of this function in (5.27) does not depend on any special orientation $\theta$, hence if one rotates $\omega$, then $\mu^A$ should get rotated, as well. (Of course, rotational invariance would still be essential for the equality in law, discussed in Remark 6.2.)

However, the rotational invariance of the scaling limit is in fact used here, in a somewhat implicit way, through the fact that the normalization factor $F(\epsilon)$ in (5.27) does not depend on $\theta$. Indeed, without rotational invariance, from (5.26) one can still prove a version of the approximation (5.27) for the $\epsilon$-grid rotated with any angle $\theta$, where the normalization is not always the same function $F(\epsilon)$, rather $F^\theta(\epsilon)$, coming from the four-arm probabilities in a $\theta$-rotated square annulus. But then, in the argument proving (6.1), one would need

$$\lim_{\epsilon\to 0} \frac{Y^{TG(\epsilon),TA}(TB,T\omega)}{F^\theta(\epsilon)} = \lim_{\epsilon\to 0} \frac{Y^{G(\epsilon),A}(B,\omega)}{F(\epsilon)}\,,$$

which is not at all clear: for a fixed $\epsilon > 0$, the numerators themselves are again trivially equal (the annulus, the ball and the configuration are all rotated), but the denominators are equal only in the presence of rotational invariance. There might be equality in the $\epsilon \to 0$ limit, but we are not trying to show this here.

We will see the same phenomenon in the proofs below: the conformal covariance comes in some sense from the fact that the normalization factor cannot be changed when applying a conformal map.

## 6.2 Scaling covariance

We show here the following special case of Theorem 6.1:

**Proposition 6.4.** *Let $A$ be some proper annulus of the plane and $\lambda > 0$ some scaling factor. Then, for any $B \subset \Delta$:*

$$\mu^{\lambda A}(\lambda B, \lambda\omega) = \lambda^{3/4}\mu^A(B,\omega)\,. \tag{6.2}$$

**Proof.** By (5.27), we have the $L^2$-limits

$$\mu^{\lambda A}(\lambda B, \lambda\omega) = \lim_{\epsilon\to 0} \frac{Y^{\lambda\epsilon,\lambda A}(\lambda B, \lambda\omega)}{F(\lambda\epsilon)} = \lim_{\epsilon\to 0} \frac{Y^{\epsilon,A}(B,\omega)}{F(\lambda\epsilon)}\,,$$

where the second equality is just a tautology even for fixed $\epsilon > 0$. So, in order to get the right side of (6.2), we need that $\lim_{\epsilon\to 0} \alpha_4(\lambda\epsilon, 1)/\alpha_4(\epsilon, 1) = \lambda^{5/4}$. By the scale invariance of the scaling limit, $\alpha_4(\lambda\epsilon, 1) = \alpha_4(\epsilon, 1/\lambda)$. Therefore, Proposition 6.4 follows from the next lemma.

**Lemma 6.5.** *For any fixed $r > 0$,*

$$\lim_{\eta\to 0} \frac{\alpha_4^\eta(\eta, r)}{\alpha_4^\eta(\eta, 1)} = \lim_{\epsilon\to 0} \frac{\alpha_4(\epsilon, r)}{\alpha_4(\epsilon, 1)} = r^{-5/4} \,.$$

*Remark* 6.6. This lemma might appear obvious knowing the critical exponent, but the probability $\alpha_4^\eta(\eta, 1)$ is only known to be $\eta^{5/4+o(1)}$, so there could be large but sub-polynomial factors, while $\alpha_4(\epsilon, 1)$ is known to be $\epsilon^{5/4}$ only up to constant factors. Therefore, Lemma 6.5 is not a direct consequence of the determination of the exponent, and, to our knowledge, is a new result in itself.

**Proof.** We will use the coupling argument from Proposition 4.5. First we prove that both limits exist, by showing that the sequences satisfy the Cauchy criterion. Then we identify that the limit is $r^{-5/4}$.

Suppose $r < 1$; the case $r \geq 1$ is symmetric. Let $10\eta < \gamma < \frac{r}{10}$. Let $\mathcal{A}(\gamma, r)$ and $\mathcal{A}(\gamma, 1)$ be the events that there are four alternate arms from radius $\gamma$ to radius $r$ and from radius $\gamma$ to radius 1. Applying Proposition 4.5, one easily gets

$$\begin{aligned} \frac{\alpha_4^\eta(\eta, r)}{\alpha_4^\eta(\eta, 1)} &= \frac{\mathbb{P}_\eta\big[\mathcal{A}(\eta, r) \mid \mathcal{A}(\gamma, r)\big]}{\mathbb{P}_\eta\big[\mathcal{A}(\eta, 1) \mid \mathcal{A}(\gamma, 1)\big]} \frac{\alpha_4^\eta(\gamma, r)}{\alpha_4^\eta(\gamma, 1)} \\ &= (1 + O(1)(\gamma/r)^d) \frac{\alpha_4^\eta(\gamma, r)}{\alpha_4^\eta(\gamma, 1)} \,. \end{aligned}$$

But for a fixed $\gamma > 0$, by $SLE$ computation, $\alpha_4^\eta(\gamma, r)/\alpha_4^\eta(\gamma, 1)$ has a limit when $\eta$ goes to zero, the limit being the ratio between the "macroscopic" probabilities $\alpha_4(\gamma, r)/\alpha_4(\gamma, 1)$. Therefore, one can rewrite the above equation as

$$\frac{\alpha_4^\eta(\eta, r)}{\alpha_4^\eta(\eta, 1)} = \frac{\alpha_4(\gamma, r)}{\alpha_4(\gamma, 1)}(1 + o(1)),$$

where $o(1)$ goes to zero when $\gamma$ goes to zero, uniformly in $\eta < \gamma/10$. This proves the Cauchy criterion for $\big(\alpha_4^\eta(\eta, r)/\alpha_4^\eta(\eta, 1)\big)_\eta$. So, there is some $\ell$ (which is easily seen to be positive and finite) so that $\alpha_4^\eta(\eta, r)/\alpha_4^\eta(\eta, 1)$ goes to $\ell$ when $\eta \to 0$. Furthermore, this also shows that the ratio of the macroscopic probabilities $\alpha_4(\gamma, r)/\alpha_4(\gamma, 1)$ have a limit when $\gamma \to 0$, and that they converge to the same limit $\ell$. It is easier to work with events at the scaling limit (since at the scaling limit we can use scale invariance), so we identify $\ell$ using $\lim_{\gamma\to 0} \alpha_4(\gamma, r)/\alpha_4(\gamma, 1) = \ell$.

On the triangular grid it is known that

$$\lim_{n\to\infty} \frac{\log \alpha_4(r^n, 1)}{n} = \log(r^{5/4}). \tag{6.3}$$

But one can write $\alpha_4(r^n, 1)$ in the following way:

$$\alpha_4(r^n, 1) = \frac{\alpha_4(r^n, 1)}{\alpha_4(r^n, r)} \frac{\alpha_4(r^{n-1}, 1)}{\alpha_4(r^{n-1}, r)} \cdots \frac{\alpha_4(r, 1)}{1}.$$

Therefore,

$$\frac{\log \alpha_4(r^n, 1)}{n} = \frac{1}{n} \sum_{k=1}^{n} \log \frac{\alpha_4(r^k, 1)}{\alpha_4(r^k, r)}. \tag{6.4}$$

But, since $r^k$ goes to zero with $k$, we have that $\lim_{k\to\infty} \log \frac{\alpha_4(r^k,1)}{\alpha_4(r^k,r)} = \log \frac{1}{\ell}$. By the convergence of the Cesàro mean, the right hand side of (6.4) converges to $\log \frac{1}{\ell}$, hence comparing with (6.3) gives that $\log \frac{1}{\ell} = \log(r^{5/4})$, which concludes the proof. □

## 6.3 Proof of the conformal covariance (Theorem 6.1)

The key step in the proof will be the following:

**Lemma 6.7.** *There are absolute constants $\delta_0 = \delta_0(A, \Omega, \widetilde{\Omega}) > 0$ and $K = K(A, \Omega, \widetilde{\Omega}) > 0$ so that almost surely (with respect to $\omega$), for any ball or square $B \subset \Delta$ centered at $z$ of radius $\delta \le \delta_0$, we have*

$$(1 - K\delta)\mu^A(B, \omega)|f'(z)|^{3/4} \le \mu^{f(A)}(f(B), \tilde{\omega}) \le (1 + K\delta)\mu^A(B, \omega)|f'(z)|^{3/4}.$$

Indeed, let us briefly show that Lemma 6.7 implies that a.s. $\mu^{f(A)} \ll f_*(\mu^A)$. Take any Borel set $U$ such that $f_*(\mu^A)(U) = 0$. For any $\epsilon > 0$ there is some finite cover $\bigcup_i B_i$ of $f^{-1}(U)$ (also a Borel set) by disjoint open squares $B_i$, each of radius less than $\delta_0$, so that $\mu^A(\bigcup_i B_i) \le \epsilon$. Therefore, by Lemma 6.7,

$$\begin{aligned}\mu^{f(A)}(U) &\le (1+K\delta_0)\sup_{z\in\Delta}|f'(z)|^{3/4}\,\mu^A\big(\bigcup_i B_i\big)\\ &\le (1+K\delta_0)\,H^{3/4}\,\epsilon\,,\end{aligned}$$

where $H := \sup_\Delta |f'|$ is finite, since $\bar{\Delta}$ is a compact set inside $\Omega$ where $f$ is conformal. By letting $\epsilon$ go to zero, this proves $\mu^{f(A)} \ll f_*(\mu^A)$. The other direction is proved in the same way. Therefore, the two measures are absolutely continuous, and it is straightforward from Lemma 6.7 that their Radon-Nikodym derivative is indeed as in Theorem 6.1.

**Proof of Lemma 6.7.** Since $\bar{A}$ is a compact subset of $\Omega$, we can define

$$\begin{array}{rcl} H_f &:=& \sup_{z\in A}|f'(z)| < \infty \\ L_f &:=& \sup_{z\in A}|f''(z)| < \infty. \end{array} \tag{6.5}$$

Since also $\overline{f(A)}$ is a compact subset of $\widetilde{\Omega}$, we have

$$\begin{array}{rcl} H_g &:=& \sup_{w\in f(A)}|g'(w)| < \infty \\ L_g &:=& \sup_{x\in f(A)}|g''(w)| < \infty. \end{array} \tag{6.6}$$

We will fix the value of $\delta_0 > 0$ later on. Let $B = B(z_0, \delta)$ some ball of radius $\delta \le \delta_0$ centered at $z_0$ and satisfying $B \subset \Delta$.

For any parameters $a \in \mathbb{C}, \theta \in [0, 2\pi)$, let $G$ be the grid of $\epsilon$-squares centered at $a$ and rotated by $e^{i\theta}$. As in the previous sections, $Y^{\epsilon,a,\theta} = Y^{\epsilon,a,\theta}(\omega)$ will be the random variable corresponding to the number of $G$-squares $Q$ inside $B$ for which $2Q$ is $A$-important for the configuration $\omega$. Recall that (5.27) gives an approximation to $\mu^A(B)$ using $Y^{\epsilon,a,\theta}$, with a speed of convergence that is independent of $a, \theta$ and also of the ball $B$.

In particular, for any $\epsilon > 0$, if $\pi^\epsilon$ is any probability measure on the parameters $a, \theta$, we obtain the $L^2$-limit

$$\mu^A(B) = \lim_{\epsilon\to 0}\int_{a,\theta}\frac{Y^{\epsilon,a,\theta}}{(2\epsilon)^{-2}\alpha_4(2\epsilon,1)}\,d\pi^\epsilon(a,\theta). \tag{6.7}$$

In our setup we make the natural choice to define $\pi^\epsilon$ as the product measure of the normalized uniform measure on $[-\epsilon, \epsilon]^2$ for $a$, times the normalized uniform measure on $[0, 2\pi]$ for $\theta$. With this particular choice, it turns out that one can rewrite (6.7) in a nicer way. First let us define for any $z \in B$ and any $\epsilon, \theta$ the random variable $X^\epsilon_\theta(z)$ to be the indicator function of the event that the square of radius $2\epsilon$ centered at $z$ and rotated by $e^{i\theta}$ is $A$-important. We will show the following lemma:

**Lemma 6.8.** *We have the $L^2$-limit*

$$\mu^A(B) = \lim_{\epsilon \to 0} \int_{B \times [0,2\pi)} \frac{X^\epsilon_\theta(z)}{\alpha_4(2\epsilon, 1)} \, d\mathcal{A}(z) \, d\mathcal{L}(\theta) \,,$$

*where $d\mathcal{A}$ is the (non-renormalized) area measure on $B$ and $d\mathcal{L}$ is the normalized Lebesgue measure on $[0, 2\pi]$.*

**Proof.** Recall that $B = B(z_0, \delta)$. It is straightforward to check that, by the definition of $Y^{\epsilon,a,\theta}$, we have

$$\int_{\substack{B(z_0,\,\delta-4\epsilon)\\(0,\,2\pi)}} X^\epsilon_\theta(z) \, d\mathcal{A}(z) \, d\mathcal{L}(\theta) \leq \int_{\substack{[-\epsilon,\,\epsilon]^2\\(0,\,2\pi)}} Y^{\epsilon,a,\theta} \, d\mathcal{A}(a) \, d\mathcal{L}(\theta)$$
$$\leq \int_{\substack{B(z_0,\,\delta)\\(0,\,2\pi)}} X^\epsilon_\theta(z) \, d\mathcal{A}(z) \, d\mathcal{L}(\theta).$$

Since we have $d\pi^\epsilon(a, \theta) = \frac{1}{4\epsilon^4} \, d\mathcal{A}(a) d\mathcal{L}(\theta)$ with the above choice of the measure $\pi^\epsilon$, the above inequalities can be rewritten as

$$\int_{\substack{B(z_0,\,\delta-4\epsilon)\\(0,\,2\pi)}} \frac{X^\epsilon_\theta(z)}{\alpha_4(2\epsilon, 1)} \, d\mathcal{A}(z) \, d\mathcal{L}(\theta) \leq \int_{\substack{[-\epsilon,\,\epsilon]^2\\(0,\,2\pi)}} \frac{Y^{\epsilon,a,\theta}}{(2\epsilon)^{-2}\alpha_4(2\epsilon, 1)} \, d\pi^\epsilon(a, \theta)$$
$$\leq \int_{\substack{B(z_0,\,\delta)\\(0,\,2\pi)}} \frac{X^\epsilon_\theta(z)}{\alpha_4(2\epsilon, 1)} \, d\mathcal{A}(z) \, d\mathcal{L}(\theta).$$

So, is it enough to prove that the boundary effect

$$\mathcal{E} := \int_{\substack{B(z_0,\,\delta)\,\setminus\, B(z_0,\,\delta-4\epsilon)\\(0,\,2\pi)}} \frac{X^\epsilon_\theta(z)}{\alpha_4(2\epsilon, 1)} \, d\mathcal{A}(z) \, d\mathcal{L}(\theta)$$

is negligible when $\epsilon$ goes to zero. For each $z \in \Delta$, the probability that $X^\epsilon_\theta(z)$ equals 1 is of order $O(1)\alpha_4(\epsilon, 1)$ (where $O(1)$ only depends on $A$). Since the area of $B(z_0, \delta) \setminus B(z_0, \delta - 4\epsilon)$ is of order $\delta\epsilon$, altogether we have

$$\mathbb{E}\big[\mathcal{E}\big] \leq O(1)\delta\epsilon,$$

which completes the proof of Lemma 6.8. □

For any $z, \theta$ and $\epsilon > 0$, $B^\theta(z,\epsilon)$ will denote the square of radius $2\epsilon$, centered at $z$ and rotated by $e^{i\theta}$; in particular $X_\theta^\epsilon(z) = 1_{B^\theta(z,\epsilon) \text{ is } A\text{-important}}$.

Lemma 6.8 says that

$$\mu^A(B) = \lim_{\epsilon \to 0} \frac{1}{\alpha_4(2\epsilon, 1)} \int_{B\times[0,2\pi)} 1_{B^\theta(z,\epsilon) \text{ is } A\text{-important}}\, d\mathcal{A}(z)\, d\mathcal{L}(\theta).$$

Let us change variables in the following way:

$$\begin{cases} \tilde z &= f(z) \\ \tilde\theta &= \theta + \mathrm{Im}\,(\log f'(z)) \end{cases} \quad \text{or equivalently} \quad \begin{cases} z &= g(\tilde z) \\ \theta &= \tilde\theta + \mathrm{Im}\,(\log g'(\tilde z)). \end{cases}$$

The Jacobian of the change of variables $(\tilde z, \tilde\theta) \mapsto (z, \theta)$, from $\mathbb{R}^3$ to $\mathbb{R}^3$, is $|g'(\tilde z)|^2$, one therefore has

$$\begin{aligned} &\int_{B\times[0,2\pi)} 1_{B^\theta(z,\epsilon) \text{ is } A\text{-important for } \omega}\; d\mathcal{A}(z)\, d\mathcal{L}(\theta) \qquad (6.8) \\ &\quad = \int_{f(B)\times[0.2\pi)} 1_{f(B^\theta(z,\epsilon)) \text{ is } f(A)\text{-important for } \tilde\omega}\; |g'(\tilde z)|^2\, d\mathcal{A}(\tilde z)\, d\mathcal{L}(\tilde\theta)\,, \end{aligned}$$

since $\tilde\omega$ is the continuum percolation satisfying $\tilde\omega = f(\omega)$.

Now, for any $z \in B = B(z_0, \delta)$, by the definition of $H_f$, we have that $|f(z) - f(z_0)| < H_f|z - z_0| \le H_f\delta$; hence, if $\tilde{z_0} := f(z_0)$, then $f(B) \subset B(\tilde{z_0}, H_f\delta)$. Now, by the definition of $L_g$, for any $\tilde z \in f(B)$, we have that $|g'(\tilde z)| \le |g'(\tilde{z_0})| + L_g H_f \delta$; here one needs to take $\delta$ small enough so that $B(\tilde{z_0}, H_f\delta)$ is still included in $f(A)$. This gives, for any $\tilde z \in f(B)$,

$$\begin{aligned} |g'(\tilde z)|^2 &\le |g'(\tilde{z_0})|^2 + 2L_g H_g H_f \delta + L_g^2 H_f^2 \delta^2 \\ &\le |g'(\tilde{z_0})|^2 + O(1)\delta\,. \end{aligned} \qquad (6.9)$$

Similarly, we have that $|g'(\tilde z)|^2 \ge |g'(\tilde{z_0})|^2 - O(1)\delta$.

Now notice that the $2\epsilon$-squares are very little distorted by $f$. Indeed, consider some square $B^\theta(z,\epsilon)$ (recall that it is the square of radius $2\epsilon$, centered at $z$, rotated by $\theta$); for any point $u \in B^\theta(z,\epsilon)$ we have

$$|f(u) - f(z) - f'(z)(z-u)| \le L_f \frac{(4\epsilon)^2}{2},$$

since $|u-z| \leq \sqrt{2}\,2\epsilon \leq 4\epsilon$. Therefore, if

$$\begin{cases} \tilde{\epsilon_1} = \tilde{\epsilon_1}(z) = |f'(z)|\epsilon - 4\epsilon^2 L_f \\ \tilde{\epsilon_2} = \tilde{\epsilon_2}(z) = |f'(z)|\epsilon + 4\epsilon^2 L_f\,, \end{cases}$$

then

$$B^{\tilde{\theta}}(\tilde{z},\tilde{\epsilon_1}) \subset f(B^{\theta}(z,\epsilon)) \subset B^{\tilde{\theta}}(\tilde{z},\tilde{\epsilon_2}). \tag{6.10}$$

This and (6.9) imply the following upper bound (and the lower bound would work in a similar way)

$$\begin{aligned} &\int_{f(B)\times[0.2\pi)} 1_{f(B^{\theta}(z,\epsilon)) \text{ is } f(A)\text{-important for } \tilde{\omega}}\ |g'(\tilde{z})|^2\, d\mathcal{A}(\tilde{z})\, d\mathcal{L}(\tilde{\theta}) \\ &\qquad \leq \int_{f(B)\times[0,2\pi)} 1_{B^{\tilde{\theta}}(\tilde{z},\tilde{\epsilon_2}) \text{ is } f(A)\text{-important for } \tilde{\omega}}\ |g'(\tilde{z})|^2\, d\mathcal{A}(\tilde{z})\, d\mathcal{L}(\tilde{\theta}) \\ &\qquad \leq \int_{f(B)\times[0,2\pi)} X_{\tilde{\theta}}^{\tilde{\epsilon_2}}(\tilde{z})\, \big(|g'(\tilde{z_0})|^2 + O(1)\delta\big)\, d\mathcal{A}(\tilde{z})\, d\mathcal{L}(\tilde{\theta}). \end{aligned}$$

Combined with (6.8), and using that there is a uniform lower bound on $|g'(\tilde{z_0})|^2$ , this leads to

$$\begin{aligned} &\int_{B\times[0,2\pi)} \frac{X_{\theta}^{\epsilon}(z)}{\alpha_4(2\epsilon,1)}\, d\mathcal{A}(z)\, d\mathcal{L}(\theta) \qquad\qquad (6.11) \\ &\qquad \leq |g'(\tilde{z_0})|^2\big(1+O(1)\delta\big)\, \frac{\alpha_4(2\tilde{\epsilon_2},1)}{\alpha_4(2\epsilon,1)} \int_{f(B)\times[0,2\pi)} \frac{X_{\tilde{\theta}}^{\tilde{\epsilon_2}}(\tilde{z})}{\alpha_4(2\tilde{\epsilon_2},1)}\, d\mathcal{A}(\tilde{z})\, d\mathcal{L}(\tilde{\theta}). \end{aligned}$$

Lemma 6.5 and the scale invariance of $\alpha_4(\cdot,\cdot)$ imply that

$$\frac{\alpha_4(2\tilde{\epsilon_2},1)}{\alpha_4(2\epsilon,1)} = \frac{\alpha_4\big(2|f'(z)|\epsilon + 8\epsilon^2 L_f, 1\big)}{\alpha_4(2\epsilon,1)} \xrightarrow{\epsilon\to 0} |f'(z)|^{5/4}.$$

Therefore, by letting the mesh $\epsilon$ (hence also the mesh $\tilde{\epsilon_2}$) go to 0 in (6.11), and using Lemma 6.8 in both domains $\Omega, \widetilde{\Omega}$, one ends up with

$$\begin{aligned} \mu^A(B) &\leq |g'(\tilde{z_0})|^2 |f'(z)|^{5/4}\big(1+O(\delta)\big)\, \mu^{f(A)}(f(B)) \\ &\leq |g'(\tilde{z_0})|^{3/4}\big(1+O(\delta)\big)\, \mu^{f(A)}(f(B)), \end{aligned}$$

since we have a uniform control on how $|f'(z)|$ is close to $|f'(z_0)| = |g'(\tilde{z_0})|^{-1}$ on the ball $B(z,\delta)$. This, together with the lower bound that is proved in the same way, completes the proof of Lemma 6.7.

Note that we finally choose the threshold radius $\delta_0$ to be small enough so that for any $z \in \Delta$, the ball $B(z, H_f\delta_0)$ is still inside $f(A)$. $\square$

# 7 Scaling limit of the counting measure on the percolation cluters, interfaces and exterior boundaries

The proof of the convergence of the counting measure on the set of pivotal points (normalized by $\eta^2\alpha_4(\epsilon,1)^{-1}$) works in the exact same way in the following situation. Consider the exploration interface $\gamma_\eta$ in $\mathbb{H}$ as well as its natural parametrization (or length) that can be defined as a measure in the following way

$$\mu_\eta := \eta^2\alpha_2(\eta,1)^{-1}\sum_{e\in\gamma_\eta}\delta_e\,,$$

where we sum over the edges $e$ (of the honeycomb lattice) which are along the curve $\gamma_\eta$. Note the we normalize the measure with the two-arms probability rather than the four arms, which gives a normalization in $\eta^2\alpha_2(\eta,1)^{-1} = \eta^{7/4+o(1)}$. This is natural since in average the interface $\gamma_\eta\cap\mathbb{D}$ is $O(1)\eta^{-2}\alpha_2(\eta,1) = \eta^{-7/4+o(1)}$ long. Using the same proof we obtain

**Theorem 7.1.** *When $\eta\to 0$, the random variable $(\gamma_\eta,\mu_\eta)$ converges in law to some $(\gamma,\mu)$, where $\gamma$ has the law of* $\mathrm{SLE}_6$*, and the Borel measure $\mu=\mu(\gamma)$ is a measurable function of the* $\mathrm{SLE}_6$ $\gamma$.

The adaptations are straightforward: one uses a coupling argument with two faces instead of four, and so on.

One can define this measure (or parametrization) in arbitrary simply connected domains. Following section 6, one can prove that this natural parametrization of the $\mathrm{SLE}_6$ has the following conformal covariance properties.

**Theorem 7.2.** *Let $f:\Omega\to\widetilde{\Omega}$ be some conformal map. Let $\gamma$ be some chordal* $\mathrm{SLE}_6$ *in $\Omega$ and let $\mu$ denotes its parametrization measure; by conformal invariance $\widetilde{\gamma} := f(\gamma)$ is also an* $\mathrm{SLE}_6$ *in $\widetilde{\Omega}$ with a parametrization measure $\widetilde{\mu}$.*

*Let $f_*(\mu(\gamma))$ be the pushforward measure of $\mu=\mu(\gamma)$. Then for almost all realization of $\gamma$, the Borel measures $\widetilde{\mu}(\widetilde{\gamma})$ and $f_*(\mu(\gamma))$ are absolutely continuous on the domain $\widetilde{\Omega}$, and their Radon-Nykodym derivative satisfies for any $\tilde{z}=f(z)\in\widetilde{\Omega}$,*

$$\frac{d\widetilde{\mu}(\widetilde{\gamma})}{df_*(\mu(\gamma))}(\tilde{z}) = |f'(z)|^{7/4}\,.$$

As well, one can consider the scaling limit of the counting measure on, say, the largest cluster in the disk, or on the clusters of diameter larger than $\epsilon$. This gives a limiting measure whose covariance cluster is in $|f'|^{91/48}$. The same proof applies in this case as well, and is actually quite simpler since one can use FKG.

Finally, we can consider the scaling limit of the counting measure on the exterior boundaries of clusters. This gives at the limit a natural parametrization of the $\mathrm{SLE}_{8/3}$ curve. Here the proof needs some non trivial arrangements, which will be detailed in our ongoing project. The reason being that the three-arm event does not "disconnect" the information from one scale to the other as well as in the case of the two and four arms events. Indeed, if one detects a three arm event with two interfaces, on one side the two interfaces will never touch each other, in particular we cannot use this notion of "faces" which was convenient with the four arms case. Nevertheless it is feasible and only requires some additional technicalities.

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