N° 140

Sur la topologie du complémentaire d'une hypersurface dans P^{n+1}

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SUR LA TOPOLOGIE DU COMPLEMENTAIRE D'UNE HYPERSURFACE DANS $\mathbb{P}^{n+1}$.

INTRODUCTION.

Soit $f(z_o,z_1,\ldots,z_{n+1})$ un polynôme homogène réduit et soit V l'hypersurface dans $\mathbb{P}^{n+1}$ définie par f . Pour étudier l'homotopie du complémentaire de V dans $\mathbb{P}^{n+1}$, nous considérons les deux fibrations :

(i) La fibration de Milnor

$$f : \mathbb{C}^{n+2} - f^{-1}(0) \longrightarrow \mathbb{C}^* \text{ , la fibre } f^{-1}(1) \text{ est notée } F \text{ .}$$

(ii) La fibration de Hopf.

$$\varphi : \mathbb{C}^{n+2} - f^{-1}(0) \longrightarrow \mathbb{P}^{n+1} - V \text{ , la fibre est } \mathbb{C}^* \text{ .}$$

L'inclusion $F \longrightarrow \mathbb{C}^{n+2} - f^{-1}(0)$ et la projection φ induisent les idomorphismes :

$$\pi_j(F) \simeq \pi_j(\mathbb{C}^{n+2} - f^{-1}(0)) \simeq \pi_j(\mathbb{P}^{n+1} - V), \ j \geqq 2 \ .$$

Ce travail se divise en quatre chapitres. Les chapitres I et II sont consacrés à l'étude du groupe fondamental du complémentaire d'une courbe dans $\mathbb{P}^2$. Dans le chapitre III nous étudions les groupes d'homotopie $\pi_j(\mathbb{P}^{n+1} - V)$, où V est une hypersurface dans $\mathbb{P}^{n+1}$.

Le résultat principal du chapitre I est le Théorème : Soit V une courbe dans $\mathbb{P}^2$. On suppose que les points singuliers de V sont des points doubles ordinaires. Alors la monodromie de la fibration de Milnor agit trivialement sur $H_1(F ; Q)$.

Ce théorème est motivé par la proposition :

<u>Proposition</u> : Soit V une courbe dans $\mathbb{P}^2$. Alors les deux conditions suivantes sont équivalentes.

(i) $\pi_1(\mathbb{P}^2 - V)$ est abélien.

(ii) $\pi_1(F)$ est abélien et la monodromie $h* : H_1(F ; \mathbb{Z}) \longrightarrow H_1(F ; \mathbb{Z})$ est l'identité.

Dans le chapitre II, nous considérons des courbes irréductibles V_j, $1 \leq j \leq r$, en position générale dans $\mathbb{P}^2$. Soit $V = V_1 \cup V_2 \cup \ldots \cup V_r$.

<u>Théorème</u> : Le groupe fondamental $\pi_1(\mathbb{P}^2 - V)$ est abélien si et seulement si les groupes fondamentaux $\pi_1(\mathbb{P}^2 - V_j)$, $1 \leq j \leq r$, sont abéliens. En particulier on obtient le

<u>Corollaire</u> : Le groupe fondamental $\pi_1(\mathbb{P}^2 - V)$ est abélien, si l'on suppose que les courbes V_j, $1 \leq j \leq r$, sont régulières.

Au chapitre III, nous étendons au cas des hypersurfaces, le résultat suivant de A. Hattori. Théorème (Hattori [7]). Soient $L_j (j = 1,2,\ldots,r)$ des hyperplans dans $\mathbb{P}^{n+1}$ en position générale et soit $L = L_1 \cup L_2 \cup \ldots \cup L_r$. Alors le groupe fondamental $\pi_1(\mathbb{P}^{n+1} - L)$ est abélien et le revêtement universel de $\mathbb{P}^{n+1} - L$ est n-connexe.

Notre résultat est :

<u>Théorème</u> : Soient $V_j (j = 1,2,\ldots,r)$ des hypersurfaces régulières en position générale dans $\mathbb{P}^{n+1}$. Soit $V = V_1 \cup V_2 \cup \ldots \cup V_r$. Alors

(i) Le groupe fondamental $\pi_1(\mathbb{P}^{n+1} - V)$ est abélien.

(ii) Le revêtement universel de $\mathbb{P}^{n+1} - V$ est n-connexe.

(iii) $H_j(F ; \mathbb{Z})$ est isomorphe à $\mathbb{Z}^k$ où $k = \binom{r-1}{j}$ et la monodromie $h*$ agit trivialement sur $H_j(F ; \mathbb{Z})$ pour $j \leq n$.

Dans le chapitre IV, nous donnerons un example de la courbe V dans $\mathbb{P}^2$ tel que le groupe fondamental $\pi_1(\mathbb{P}^2 - V)$ est isomorphe à $\mathbb{Z}_p * \mathbb{Z}_q$ où $\mathbb{Z}_n$ est $\mathbb{Z}/n\mathbb{Z}$.

C'est une extension du résultat de Zariski [19] .

Le chapitre I est publié dans Inventiones Math. 27, 1974. Le chapitre II sera publié dans Journal of the London Math. Scociety. Les chapitres III et IV sont soumis au journal Topology et Math. Annalen respectivement.

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Contents

Chapter I THE MONODROMY OF A CURVE WITH ORDINARY DOUBLE POINTS

§1. Introduction

Let $f(x,y,z)$ be a square-free homogeneous polynomial of degree d and let C be the projective curve in $\mathbb{P}^2$ which is defined by $C = f^{-1}(0)$. We want to study $\pi_1(\mathbb{P}^2 - C)$. For this we consider the Milnor fibering of f : $f/|f| = \arg(f) : S^5 - K \to S^1$ where $K = f^{-1}(0) \cap S^5$. The fiber F of this fibering is naturally diffeomorphic to any affine hypersurface $X_o = f^{-1}(t) \subset \mathbb{C}^3$ $(t \neq 0)$. Let $h : F \to F$ be the monodromy map which is defined by

$$h(x,y,z) = (x\cdot\xi_d, y\cdot\xi_d, z\cdot\xi_d)$$

where $\xi_d = \exp\frac{2\pi i}{d}$. The first monodromy $h_* : H_1(F) \to H_1(F)$ is deeply related to $\pi_1(\mathbb{P}^1 - C)$. In fact, we have that h_* is equal to the identity map if $\pi_1(\mathbb{P}^2 - C)$ is abelian (Proposition 5).

The main purpose of this paper is to prove that h_* is equal to I (identity map) modulo torsion if C admits only ordinaly double points as singularities (Theorem 1).

This is an important step to Zariski's conjecture that $\pi_1(\mathbb{P}^2 - C)$ should be abelian if C admits only ordinaly double points as singularities ([19], [20]).

This result is also true if C admits only a certain type of singularities (admissible singularities) (Theorem 2, in §4).

§2. Preliminaries.

Let $\arg(f) : S^5 - K \to S^1$ be the Milnor fibering as above. There is a cannonical $\mathbb{Z}_d$-action on F by the monodromy map h which is compatible with the natural S^1-action on $S^5 - K$.

Proposition 1. We have the following exact sequences and commutative diagrams.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \pi_1(S^1) & \searrow^{d} & & & \\
 & 0 \searrow & & & \downarrow j & & & & \\
0 \to & \pi_1(F) & \xrightarrow{i} & & \pi_1(S^5 - K) & \xrightarrow{\phi} & \pi_1(S^1) & \to 0 & \\
 & & \searrow^{k} & & \downarrow \psi & & & & \\
 & & & & \pi_1(\mathbb{P}^2 - C) & \searrow & & & \\
 & & & & \downarrow & & \mathbb{Z}_d & \searrow 0 & \\
 & & & & 0 & & & &
\end{array}
$$

Proof : The law sequence is obtained from the Milnor fibering and the column is a result of the Hopf-bundle : $S^5 - K \to \mathbb{P}^2 - C$ and of the fact that j is injective.

Proposition 2. Image(j) is contained in the Center of $\pi_1(S^5 - K)$.

Proof : Let $a = (x_o, y_o, z_o) \in F$ be a fixed base point. Then the generator of Image(j) can be represented by the orbit loop $S : I \to S^5 - K$ defined by $s(t) = (x_o \exp 2\pi i t, y_o \exp 2\pi i t, z_o \exp 2\pi i t)$. Let $[\omega] \in \pi_1(S^5 - K; a)$ be any element represented by a loop ω . Then $S^{-1}\omega S$ is naturally homotopic to ω by pulling back along the orbit of S^1-action. Therefore we have $[S]^{-1}[\omega][S] = [\omega]$. This completes the proof.

Let G be a group. By $Z(G)$ and $D(G)$, we mean the center of G and the commutator group of G respectively. Then the following proposition is an immediate corollary of Propositions 1 and 2.

Proposition 3. (i) $D(\pi_1(S^5 - K))$ is a normal subgroup of $\pi_1(F)$ and we have $D(\pi_1(S^5 - K)) = D(\pi_1(\mathbb{P}^2 - C))$

(ii) $Z(\pi_1(\mathbb{P}^2 - C)) = \psi(Z(\pi_1(S^5 - K))$.

Now we consider the condition for $\pi_1(\mathbb{P}^2 - C)$ to be abelian. Let $\ell : I \to F$ be any fixed path from a to $h(a) = (x_o \exp\frac{2\pi i}{d},\ y_o \exp\frac{2\pi i}{d},\ z_o \exp\frac{2\pi i}{d})$. Then in the sequence of Proposition 1, we can define a cross-section τ of $\emptyset$ by the following loop :

$$\tau(t) = \begin{cases} (x_o \exp\frac{4\pi i t}{d},\ y_o \exp\frac{4\pi i t}{d},\ z_o \exp\frac{4\pi i t}{d}) & 0 \le t \le \frac{1}{2} \\ \ell^{-1}(2t-1) = \ell(2-2t) & \frac{1}{2} \le t \le 1 \end{cases} .$$

Because $\pi_1(F,a)$ is a normal subgroup of $\pi_1(S^5 - K,a)$, we can define an automorphism $\tau_\# : \pi_1(F;a) \to \pi_1(F;a)$ by $\tau_\#([\omega]) = [\tau]^{-1}[\omega]\cdot[\tau]$ for $[\omega] \in \pi_1(S^5-K;a)$. It is easy to see that $\tau_\#$ is well-defined in $\mathrm{Aut}(\pi_1(F;a))/\mathrm{Int}(\pi_1(F;a))$ where $\mathrm{Aut}\ \pi_1(F;a)$ is the group of automorphisms and $\mathrm{Int}(\pi_1(F,a))$ is the group of inner-automorphisms. It is also easy to see that $\tau_\#([\omega])$ is represented by $\ell\cdot h(\omega)\cdot\ell^{-1}$ where $h(\omega)$ is a loop defined by $h(\omega)(t) = h(\omega(t))$. Since $\tau_\#$ preserves $D(\pi_1(F,a))$, it induces an isomorphism h_τ of $H_1(F)$. By the above consideration, we have

Proposition 4. h_τ is equal to the monodromy

$$h_* : H_1(F) \to H_1(F) .$$

Now we can state a fundamental criterien for $\pi_1(\mathbb{P}^2 - C)$ to be abelian.

Proposition 5. The following three conditions are equivalent.

(i) $\pi_1(\mathbb{P}^2 - C)$ is an abelian group.

(ii) $\pi_1(S^5 - K)$ is an abelian group.

(iii) $\pi_1(F)$ is an abelian group and $h_* : H_1(F) \to H_1(F)$ is the identity map

Proof : (i) ⇔ (ii) is the result of Propositions 1,2 and 3. (ii) ⇔ (iii) can be obtained from the fact that $\pi_1(S^5 - K)$ is a semi-direct product of $\pi_1(F)$ and $\mathbb{Z}$ using the cross-section τ .

Proposition 6. Assume that the curve C is irreducible. Then we have :

(i) $D(\pi_1(\mathbb{P}^2 - C)) = \pi_1(F)$

(ii) $\pi_1(\mathbb{P}^2 - C)$ is abelian if and only if $\pi_1(F)$ is trivial.

This is an immediate consequence of Proposition 1 and the fact that $H_1(\mathbb{P}^2 - C) = \mathbb{Z}_d$.

§3. Main results about the monodromy.

Let $C = C_1 \cup C_2 \cup \ldots \cup C_r$ be a curve in $\mathbb{P}^2$ which has only ordinary double points as singularities. Then we will prove the following theorems which are fundamental steps for $\pi_1(\mathbb{P}^2 - C)$ to be abelian. We use the same notations as before.

Theorem 1. (i) The first homology group $H_1(F;Q)$ is equal to $Q \oplus Q \oplus \ldots \oplus Q$ ((r-1)-copies)

(ii) The monodromy $h_* : H_1(F;Q) \to H_1(F;Q)$ is equal to the identity map.

Proof of Theorem 1. Let $f(x.y.z)$ be the fixed square-free homogeneous polynomial defining C . We consider a homogeneous polynomial $g(x.y.z,w) = f(x,y,z) + w^d$ and let V be the projective hypersurface of complex dimension 2 defined by $V = g^{-1}(0) \subset \mathbb{P}^3$. Then we can see easily that $V \cap \{w = 0\} = C$ and $V - C$ is isomorphic to F . Moreover we have that the singular set ΣV of V is equal to the singular points ΣC of C . Therefore V has only isolated singularities. Now we want to compute $H_1(F)$. By the Lefschetz duality, $H_1(F)$ is isomorphic to $H^3(V,C)$.

From the exact sequence

$$\cdots \to H^2(V) \xrightarrow{\phi} H^2(C) \to H^3(V,C) \to H^3(V) \to 0$$

we have a short exact sequence

$$\text{(A)} \qquad 0 \to \text{Coker } \phi \to H^3(V,C) \to H^3(V) \to 0$$

First we assume the following lemmas.

Lemma 1. $H^3(V;\mathbb{Z})$ is a finite group.

Lemma 2. The rank of $H^3(V,C)$ is equal to or greater than $r - 1$.

Now by the sequence (A), we have that rank (Coker ϕ) is less or equal to $r - 1$ because $H^2(C;Q)$ is $Q \oplus Q \oplus \cdots \oplus Q$ (r-copies) and the image of ϕ contains the Euler class τ of the Hopf-bundle $K \to C$ and τ is non-zero. ([14]). Therefore by Lemmas 1 and 2 we have that

$$H^3(V,C;Q) \cong Q \oplus \cdots \oplus Q \quad ((r - 1)\text{-copies}) \quad .$$

Now we consider the Wang sequence of the Milnor fibering of f :

$$\cdots \to H_1(F;Q) \xrightarrow{h_*-I} H_1(F;Q) \to H_1(S^5 - K;Q) \to Q \to 0 \quad .$$

We know that $H_1(S^5 - K) \cong H^3(K)$ by the Alexander duality and therefore we have that $H_1(S^5 - K;Q)$ is isomorphic to $Q \oplus \cdots \oplus Q$ (r - copies)

Thus we have that $\operatorname{coker}(h_*-I) = H_1(F;Q)$. This implies that $h_* = I$ (identity map), completing the proof of (ii) of Theorem 1.

Proof of Lemma 1. At each singular point $p \in \Sigma V = \Sigma C$, let g_p be a defining polynomial of V in a neighborhood of p and take a small disk $D^6_{\epsilon,p}$ centered at P. Let $K_p = V \cap S^5_{\epsilon,p}$ and $C_p = V \cap D^6_{\epsilon,p}$ which is a cone of K_p. Take $\eta > 0$ small enough and let $V_{p,\eta} = g^{-1}(\eta) \cap D^6_{\epsilon,p}$.

Since $\partial V_{p,\eta}$ is diffeomorphic to K_p, we can replace C_p by $V_{p,\eta}$ at each singular point p. Then we have a non-singular surface $\tilde{V}$ and it is easy to see that $\tilde{V}$ is diffeomorphic to a non-singular projective hypersurface of degree d. Let $V_c = V - \Sigma \operatorname{Int} C_p$ where Σ means the disjoint sum at every singular point p. Then we have two Meyer-Vietories exact sequences :

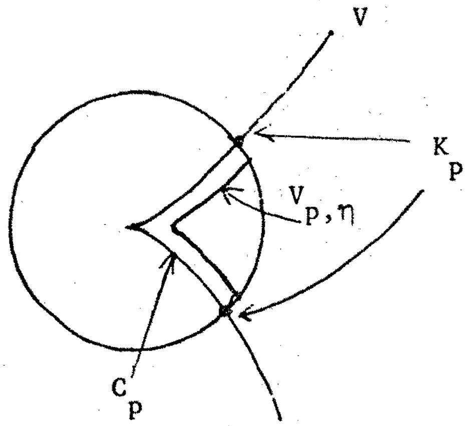


Figure 1

(B) $\cdots \to H^2(\Sigma K_p) \to H^3(V) \to H^3(V_c) \oplus H^3(\Sigma C_p) \to H^3(\Sigma K_p) \to \cdots\cdot$

(C) $\cdots \to H^2(\Sigma K_p) \to H^3(\tilde{V}) \to H^3(V_c) \oplus H^3(\Sigma V_{\eta,p}) \to H^3(\Sigma K_p) \to \cdots\cdot$

Because $V_{\eta,p}$ has a homotopy type of a 2-dimensional CW-complex, $H^3(\Sigma V_{\eta,p}) = \Sigma H^3(V_{\eta,p}) = 0$. Therefore, in the sequence (C) $H^3(V_c) \to H^3(\Sigma K_p)$ is injective because $H^3(\tilde{V}) = 0$. This means that $\{H^3(\Sigma K_p) \to H^3(V) \to 0\}$ is exact. Thus to prove Lemma 1 it is sufficient to prove that $H^3(K_p)$ is a torsion group. Now by the assumption, at each singular point p we can take $x^2+y^2+w^d$ as a defining polynomial g_p. Identifying $V_{p,\eta}$ as the fibre of the Milnor fibering of g_p at p, we have a Wang sequence :

$$\cdots \to H_2(V_{p,\eta}) \xrightarrow[\bar{h}_{p*}-I_*]{} H_2(V_{p,\eta}) \to H_2(S^5_{\varepsilon,p} - K_p) \to 0$$

By the join theorem of Brieskorn-Pham ([11]), we have $H_2(V_{p,\eta}) \simeq \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ ((d-1)-copies) and $\bar{h}_{p*}$ is represented by the matrix

$$\left.\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & -1 \end{pmatrix}\right\} (d-1)$$

Thus $H_2(S^5_{\varepsilon,p} - K_p) = \mathbb{Z}_d$ by a slight computation. This means $H^2(K_p) = \mathbb{Z}_d$ by the Alexander duality. Thus $H^2(\Sigma K_p) \cong \Sigma \mathbb{Z}_d$ and this completes the proof.

Proof of Lemma 2. Consider the Wang sequence of the Milnor fibering of f

$$\cdots \to H_1(F) \xrightarrow[h_* - I]{} H_1(F) \to H_1(S^5 - K) \to \mathbb{Z} \to 0 .$$

We know that $H_1(S^5 - K;Q) \cong H^3(K;Q) \cong Q \oplus \cdots \oplus Q$ (r-copies). Therefore the above exact sequence says that rank $(H_1(F;Q)) \geq r-1$. This completes the proof of Theorem 1.

§4. Generalization of the results in §3.

Let C be any curve of degree d and let p be a singular point of C. Let f_p be a local defining polynomial of C. Then we can consider the Milnor fibering of f_p at p : $\arg(f_p) : S^3_{\epsilon,p} - K_\epsilon \to S^1$ where $K_\epsilon = S^3_{\epsilon,p} \cap C$. Let F_p be the fibre of this fibering and let $\Delta_p(t)$ be the characteristic polynomial defined by the determinant of $t\cdot I - h_{p*} : H_1(F_p;Q) \to H_1(F_p;Q)$ where h_{p*} is the monodromy map of the fibering.

Definition. A singular point $p \in C$ is admissible if and only if the roots of $\Delta_p(t)$ are distinct from $\xi_d, \xi_d^2, \ldots, \xi_d^{d-1}$ where $\xi_d = \exp \frac{2\pi i}{d}$. Ordinary double points are clearly admissible. Now we can generalize Theorem 1 as follows.

Theorem 3. Let C be a projective curve which admits only admissible singularities. Then we have (i) $H_1(F;Q) \cong \underbrace{Q \oplus \cdots \oplus Q}_{r-1}$ where r is the number of irreducible components of C.

(ii) The monodromy $h_* : H_1(F;Q) \to H_1(F;Q)$ is equal to the identity map.

Proof of Theorem 3. The proof is essentially the same as that of Theorem 1. We used the fact that C has only ordinary double points to prove that $H^2(K_p)$ is a torsion group in the proof of Lemma 1. This is also the case if P is an admissible singularity of C because the local monodromy $\bar{h}_{p*}$ in the proof of Lemma 1 is the tensor product of the local monodromy h_{p*} of the curve C and the matrix.

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -1 & \cdots & \cdots & -1 \end{pmatrix}$$

by the join theorem of Thom-Sebastiani, ([13]) Therefore $h_{p*} - I : H_2(V_{p,\eta}) \to H_2(V_{p,\eta})$ has only a finite cokernel, because $\bar{h}_{p*}$ has not 1 as eignevalue. This completes the proof.

Example. Let $C = \{x^d + y^{d-q} z^q = 0\}$ $(d \geq 0)$.

Case 1. Assume that $q = 1$. Then C has only one singular point $p = [0,0,1]$. As p, C is defined by $x^d + y^{d-1} = 0$ and we have

$$\Delta_p(t) = \frac{(t^{d(d-1)} - 1)\ (t-1)}{(t^d-1)\quad (t^{d-1}-1)}$$

Therefore p is admissible. In fact we have that $\pi_1(F) = 0$ by the join theorem ([13]).

Case 2. Assume that $d-2 \geq q \geq 2$ and $(d,q) = 1$. C has two singular points $p = [\ 0;0;1]$, $q = [\ 0;1;0]$ and we have that

$$\Delta_p(t) = \frac{(t^{d(d-q)}-1)\ (t-1)}{(t^d-1)\quad (t^{d-q}-1)}$$

and

$$\Delta_q(t) = \frac{(t^{dq}-1)\ (t-1)}{(t^d-1)(t^q-1)}$$

Thus p and q are admissible. Similarly we have that $\pi_1(F) = 0$.

Case 3. Assume that $d-2 \geq q \geq 2$ and $r = (d,q) > 1$. Then C has the same singular points p,q but we have

$$\Delta_p(t) = \frac{(t^{\mu}-1)^r\ (t-1)}{(t^d-1)\ (t^{d-q}-1)} \quad , \quad \mu = \frac{d(d-q)}{r}$$

and

$$\Delta_q(t) = \frac{(t^{\lambda}-1)^r\ (t-1)}{(t^d-1)\ (t^q-1)} \quad , \quad \lambda = \frac{dq}{r}$$

Thus neither p nor q are admissible. In this case we have that $\pi_1(F) = F((d-1)(r-1))$ and not abelian. (The right side means a free group of rank $(d-1)(r-1)$.)

Remark. Assume that a curve $C = C_1 \cup C_2 \ldots \cup C_r$ admits only admissible singularities. Let μ_p be the multiplicity at a singular point p . As for the

Euler number $\chi(C)$ of C , we have a formula,

$$\chi(C) = 3d - d^2 + \Sigma \mu_p$$

where d is the degree of C and Σ means the sum at each singular point p Then by [14], we have that

$$\frac{\chi(F)}{d} = \chi(\mathbb{P}^2) - \chi(C)$$

$$= (3-3d+d^2) - \Sigma \mu_p \quad .$$

We consider the zeta function $\zeta(t)$ of the monodromy map $h : F \to F$. Then we have

$$\zeta(t) = (1-t^d)^{\frac{-\chi(F)}{d}}$$

$$= P_o(t)^{-1} \, P_1(t) \, P_2(t)^{-1}$$

where $P_i(t)$ is the determinant of the linear map

$$h_* - tI : H_i(F;Q) \to H_i(F;Q). \quad ([14]) \; .$$

By theorem 3 we have that $P_1(t) = (1-t)^{r-1}$. Therefore we have that $P_2(t) = (1-t^d)^k \; (1-t)^{r-2}$ where $k = 3 - 3d + d^2 - \Sigma \mu_p$. This implies that (i) $h_2(F;Q) \cong \{d(3-3d+d^2 - \Sigma \mu_p) + r-2\} \; Q$ and (ii) the rank of the kernel of the map

$$h_* - I : H_2(F) \to H_2(F)$$

is equal to $1+r - 3d + d^2 - \Sigma \mu_p$. From this we can see that the total multiplicity $\Sigma \mu_p$ has a upper-bound $(d-1)(d-2)$ if C is an admissible, irreducible curve. The curve of the above example is one of the such curves.

Chapter II On the fundamental group of the complement of a reducible curve in $\mathbb{P}^2$

§ 1. Statement of results

Let $C = C_1 \cup C_2 \cup \ldots \cup C_r$ be an algebraic curve in $\mathbb{P}^2$ such that its irreducible components $\{C_j\}$ are in general position i.e. C_i and C_j meet transversely for each i, j $(i \neq j)$ and $C_i \cap C_j \cap C_k = \emptyset$ for each mutually distinct i, j and k. How can we decide the fundamental group $\pi_1(\mathbb{P}^2 - C)$ in the words of $\pi_1(\mathbb{P}^2 - C_j)$ $(j=1,2,\ldots,r)$?

Zariski's conjecture says that $\pi_1(\mathbb{P}^2-C)$ should be abelian if each irreducible component C_j has only ordinary double points as singularities. ([20]). Our results are partial answers to this question.

<u>Theorem 1.</u> Let C' be any curve in $\mathbb{P}^2$ and let C be an irreducible curve such that C meets transversely with C' and $\pi_1(\mathbb{P}^2-C)$ is abelian. Then we have the following central extension.

$$1 \to \mathbb{Z} \xrightarrow{i} \pi_1(\mathbb{P}^2 - C \cup C') \to \pi_1(\mathbb{P}^2 - C') \to 1$$

Moreover the composition homomorphism of i with the Hurewicz homomorphism is also injective.

$$\mathbb{Z} \xrightarrow{i} \pi_1(\mathbb{P}^2 - C \cup C') \to H_1(\mathbb{P}^2 - C \cup C')$$

(By 1 we mean the trivial group.)

In this paper, every homomorphism is induced by the respective inclusion map, unless otherwise stated. In [16], we have proved this theorem assuming that C is non-singular. As an immediate corollary, we have:

<u>Corollary 1.</u> Under the same assumption, $\pi_1(\mathbb{P}^2 - C \cup C')$ is abelian if and only if $\pi_1(\mathbb{P}^2 - C')$ is abelian.

Using Corollary 1 inductively, we have the following reduction theorem.

Corollary 2 (Reduction Theorem). Let $C = C_1 \cup C_2 \cup \dots \cup C_r$ be a curve such that its irreducible components $\{C_j\}$ are in the general position. Then $\pi_1(\mathbb{P}^2 - C)$ is abelian if and only if $\pi_1(\mathbb{P}^2 - C_j)$ is abelian for each $j = 1,2,\dots,r$.

The only if part is the result of the general position property i.e. $\pi_1(\mathbb{P}^2 - C) \to \pi_1(\mathbb{P}^2 - C_j)$ is surjective. This implies, for example, that Zariski's conjecture is true if it is true for irreducible curves.

§ 2. A reduction lemma.

In many cases, it is more convenient to study $\pi_1(\mathbb{C}^2 - C)$ rather than $\pi_1(\mathbb{P}^2 - C)$. One of the reasons is that $H_1(\mathbb{P}^2 - C)$ has a torsion $\mathbb{Z}/d_o\mathbb{Z}$ if, assuming that C has r-components $\{C_j\}$ $(j=1,2,\dots,r)$, the greatest common divisor d_o of their degrees $\{d_j\}$ is greater than 1.

For this, we prove the following lemma. (See also [16]).

Lemma 1. Let C be a curve in $\mathbb{P}^2$ and let L be a general line to C . Then we have a central extension

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{P}^2 - C \cup L) \to \pi_1(\mathbb{P}^2 - C) \to 1$$

such that the composition map

$$\mathbb{Z} \to \pi_1(\mathbb{P}^2 - C \cup L) \to H_1(\mathbb{P}^2 - C \cup L)$$

is also injective. This implies that $\pi_1(\mathbb{P}^2 - C \cup L)$ is abelian if and only if $\pi_1(\mathbb{P}^2 - C)$ is abelian.

Proof. Let L_∞ be another line which is general to $C \cup L$. Without losing generality, we can assume that L_∞ is defined by $Z = 0$ and L is defined by $Y = 0$. Let L_η be the line $Y - \eta Z = 0$. This is a pencil centered at $\infty = [1; 0; 0]$. We can take a positive number ε so that L_η is general to C for each η $(|\eta| \leq \varepsilon)$. Let $N = \bigcup_{|\eta| \leq \varepsilon} L_\eta$ and take a base point $*$ on $L_\varepsilon - C \cup L_\infty$. Then we have a following Van Kampen diagram.

$$\begin{array}{ccccc} & & \pi_1(N-C, *) & & \\ & \overset{\varphi}{\nearrow} & & \searrow & \\ \pi_1(N-C \cup L, *) & & & & \pi_1(\mathbb{P}^2-C, *) \\ & \underset{\psi}{\searrow} & & \nearrow & \\ & & \pi_1(\mathbb{P}^2 - C \cup L, *) & & \end{array}$$

Considering the fibering map $h : N - L_\infty \to D_\varepsilon^2 = \{\eta \in \mathbb{C}, |\eta| \leq \varepsilon\}$ which is defined by $h[X; Y; Z] = Y/Z$, we have that $N - C \cup L$ is diffeomorphic to $(L_\varepsilon - C \cup \{\infty\}) \times (D_\varepsilon^2 - \{0\})$ and $N-C$ is homeomorphic to the quotient space of $(L_\varepsilon - C) \times D_\varepsilon^2$ identified $\{\infty\} \times D_\varepsilon^2$ to a point. Therefore $L_\varepsilon - C$ is a deformation retract of $N-C$. We can take generators $\{\bar{g}_j\}$ $(j = 1,2,\dots,d)$ of $\pi_1(N-C, *)$ so that their generating relation is only $\bar{g}_1 \circ \bar{g}_2 \dots \bar{g}_d = 1$ (d is the degree of C .) (See Figure 1)

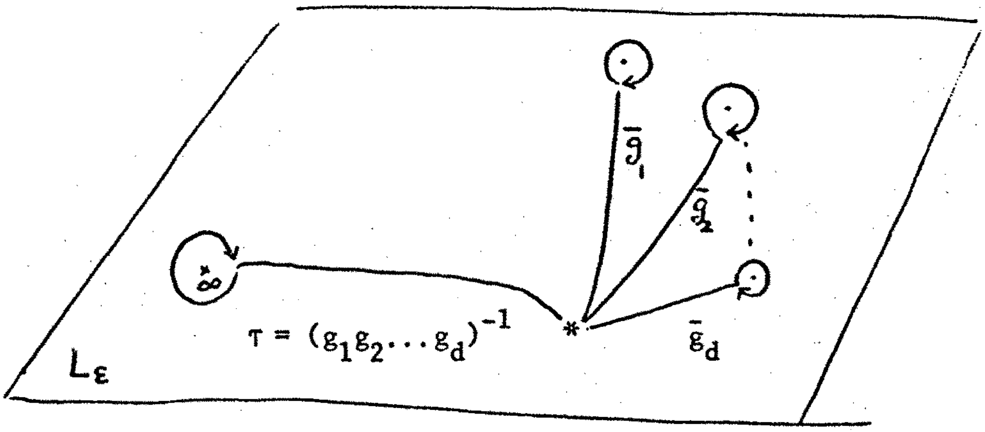


Figure 1.

$\pi_1(N - C \cup L, *)$ is isomorphic to $F(g_1,\dots,g_d) \times \mathbb{Z}$. The first part $F(g_1,\dots,g_d)$ is the free group generated by $\{g_i\}$ which corresponds to $\pi_1(L_\epsilon - C \cup \{\infty\}, *)$ and each generator g_j is mapped to $\bar{g}_j$ by φ . The generator of $\mathbb{Z}$ (say t) is expressed by $[\ell^{-1}. v_p . \ell]$ where v_p is a small loop which revolves round L starting at $p \in L_\epsilon - C \cup \{\infty\}$ and ℓ is a path in $N - C \cup L$ connecting p to $*$. Because t is contained in the center of $\pi_1(N - C \cup L, *)$, we can take p and ℓ arbitrarily. Thus in the above diagram we have that φ is surjective and Ker φ is the minimal normal subgroup containing t and $g_1 g_2 \dots g_d$. Therefore we obtain the following exact sequence.

$$\text{(A)} \quad 1 \to N(\psi(t), \psi(g_1 g_2 \dots g_d)) \to \pi_1(\mathbb{P}^2 - C \cup L, *) \to \pi_1(\mathbb{P}^2 - C, *) \to 1$$

where $N(\psi(t), \psi(g_1 g_2 \dots g_d))$ is the minimal normal subgroup containing $\psi(t)$ and $\psi(g_1 g_2 \dots g_d)$. First we assert that $\psi(t) = \psi(g_1 g_2 \dots g_d)^{-1}$ (under a suitable orientation of t). We can represent t by a loop sufficiently near ∞ . Projecting t on L_ϵ in the direction parallel to L , we have that $\psi(t) = \tau = \psi(g_1 g_2 \dots g_d)^{-1}$ (See Figure 2).

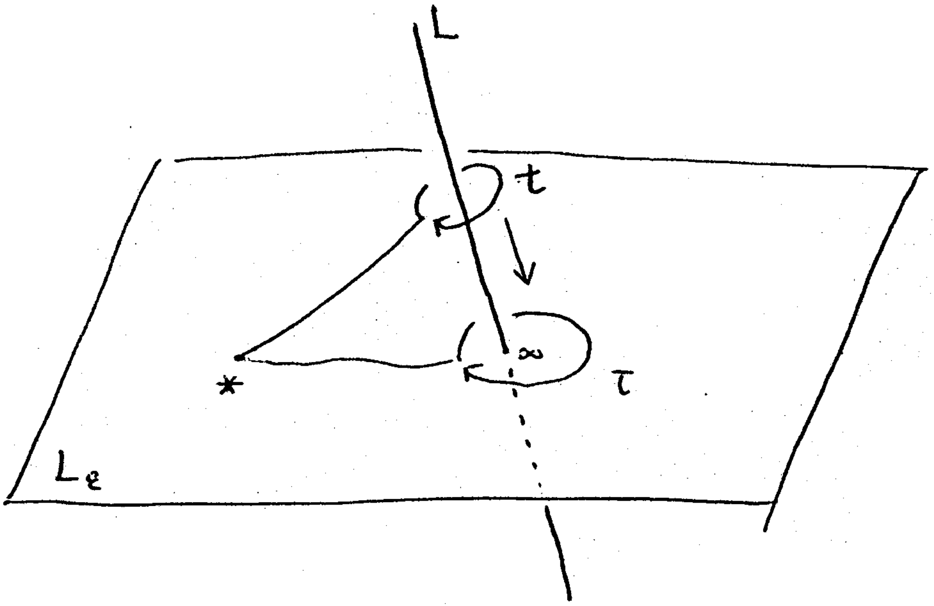


Figure 2.

Now we prove that ψ is surjective. By the general position property, $\pi_1(\mathbb{P}^2 - C \cup L \cup L_\infty, *) \to \pi_1(\mathbb{P}^2 - C \cup L, *)$ is surjective. Therefore we need only prove that $\tilde{\psi} : \pi_1(N - C \cup L, *) \to \pi_1(\mathbb{P}^2 - C \cup L \cup L_\infty, *)$ is surjective. Let Σ be the set defined by $\{\eta \in \mathbb{C};\ L_\eta$ and C are not in the general position$\}$. By the elimination theory, we can see that Σ is a finite set. Let $\Sigma = \{\rho_1, \rho_2, \dots, \rho_\mu\}$ and let $h : \mathbb{P}^2 - C \cup L_\infty \to \mathbb{C}$ be defined by $h[X; Y; Z] = Y/Z$. Then, using a controlled vector field near $C \cup L_\infty$, we have that $h : h^{-1}(\mathbb{C} - \Sigma) \to \mathbb{C} - \Sigma$ is a fiber bundle. Take a positive number δ so that $\{D^2_\delta(\rho_j)\}$ are mutually disjoint and included in $\mathbb{C} - D^2_\epsilon$ where $D^2_\delta(\rho_j)$ is the disk defined by $\{\rho \in \mathbb{C};\ |\rho - \rho_j| \le \delta\}$. Take paths $\{\ell_j\}$ $(j=1,2,\dots,\mu)$ which satisfy the following conditions.

(i) $\ell_j(0) = \epsilon$ and $\ell_j(1)$ is a point of the boundary of $D^2_\delta(\rho_j)$

(ii) $\ell_j \cap D^2_i(\rho_i) = \ell_j(1)$ or $\emptyset$ for $j=i$ or $j \ne i$ respectively.

(iii) $\ell_j \cap \ell_j = \{\epsilon\}$ for each i,j $(i \ne j)$.

Let $\Gamma_i = \ell_j \cup D^2_\delta(\rho_j)$ and $W_j = (D^2_\epsilon - \{0\}) \cup \bigcup_{k \le j} \Gamma_k$. (See Figure 3.)

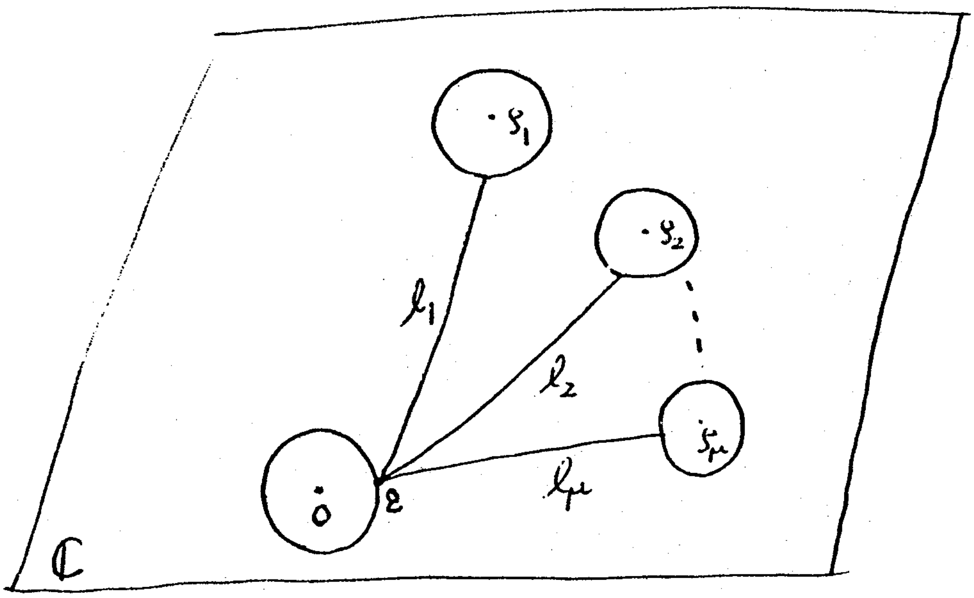
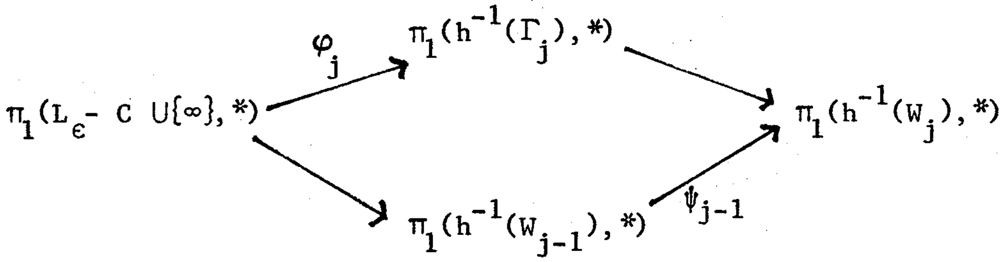


Figure 3.

Then one can see that $h^{-1}(W_\mu)$ is a deformation retract of $\mathbb{P}^2 - C \cup L \cup L_\infty$ using the above fibering. Now we consider the following exact sequence.

$$1 \to \pi_1(L_\epsilon - C \cup \{\infty\}, *) \to \pi_1(h^{-1}(\Gamma_j) - L_{\rho_j}, *) \xrightarrow{h_\#} \pi_1(\Gamma_j - \{\rho_j\}, \epsilon) \to 1$$

Take an element τ_j of $\pi_1(h^{-1}(\Gamma_j)-L_{\rho_j}, *)$ such that $h_\#(\tau_j)$ is a generator of $\pi_1(\Gamma_j - \{\rho_j\}, \epsilon) \cong \mathbb{Z}$ and $a_j(\tau_j) = 1$ where a_j is the homomorphism $\pi_1(h^{-1}(\Gamma_j)-L_{\rho_j}, *) \xrightarrow{a_j} \pi_1(h^{-1}(\Gamma_j), *)$. We can define a cross-section σ_j of $h_\#$ using τ_j so that $\pi_1(h^{-1}(\Gamma_j)-L_{\rho_j}, *)$ is a semi-product of $\pi_1(L_\epsilon - C \cup \{\infty\}, *)$ and $\mathbb{Z}$. Because a_j is surjective by the general position property, it is clear that $\varphi_j : \pi_1(L_\epsilon - C \cup \{\infty\}, *) \to \pi_1(h^{-1}(\Gamma_j), *)$ is surjective. Now consider the following Van Kampen diagram



Because φ_j is surjective, we have that ψ_{j-1} is also surjective. Thus by the induction on j we obtain that

$$\psi_{\mu-1} \circ \psi_{\mu-2} \circ \dots \circ \psi_o : \pi_1(N - C \cup L, *) \to \pi_1(h^{-1}(W_\mu), *)$$

is surjective. This implies that $\tilde{\psi}$ (and therefore ψ) is surjective. Going back to the exact sequence (A), we have proved that $N(\psi(t), \psi(g_1 g_2 \dots g_d))$ is the cyclic group generated by $\psi(t)$ because the surjectivity of ψ implies that $\psi(t)$ is contained in the center of $\pi_1(\mathbb{P}^2 - C \cup L, *)$.

Let $\xi : \pi_1(\mathbb{P}^2 - C \cup L, *) \to H_1(\mathbb{P}^2 - C \cup L)$ be the Hurewicz homomorphism. Then by Lefschetz duality we have $H_1(\mathbb{P}^2 - C \cup L) \cong H^3(\mathbb{P}^2, C \cup L)$. By the exact sequence of the couple $(C \cup L, \mathbb{P}^2)$

$$\cdots \to H^2(\mathbb{P}^2) \underset{\pi}{\to} H^2(C \cup L) \to H^3(\mathbb{P}^2, C \cup L) \to 0$$

we have that $H_1(\mathbb{P}^2 - C \cup L) \cong H^3(\mathbb{P}^2, C \cup L)$ is isomorphic to coker π which is clearly isomorphic to the quotient group $\mathbb{Z}(t_0) \oplus \mathbb{Z}(t_1) \oplus \cdots \mathbb{Z}(t_r) / t_0 + d_1 t_1 + \cdots + d_r t_r$ where $\mathbb{Z}(t_j)$ is the infinite cyclic group generated by t_j $(j = 0, \cdots, r)$ and $d_j =$ degree(C_j), assuming that $\{C_j\}$ $(j = 1, \cdots, r)$ are irreducible components of C. Using this isomorphism, $\xi \circ \psi(t)$ corresponds to t_0. Thus $\xi \circ \psi(t)$ is not a torsion element. Therefore by (A) we obtain a central extension with the desired property.

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{P}^2 - C \cup L, *) \to \pi_1(\mathbb{P}^2 - C, *) \to 1$$

This completes the proof of Lemma 1.

§ 3. Preliminaries.

Let C be a curve in $\mathbb{P}^2$. Taking a general line L_∞, we identify $\mathbb{P}^2 - L_\infty$ with $\mathbb{C}^2$. Let $V_o = \mathbb{C}^2 \cap C$ and let $f(x, y)$ be a square-free polynomial which defines V_o. Let $\tilde{\Sigma}$ be the set of critical values of f. It is clear that $\tilde{\Sigma}$ is a finite set. Therefore we put $\Sigma = \tilde{\Sigma} - \{0\} = \{\rho_1, \ldots, \rho_\mu\}$. Let ϵ be a positive number so that $D^2_\epsilon \cap \Sigma = \emptyset$. Let $N = f^{-1}(D^2_\epsilon)$ and take a base point $*$ on $f^{-1}(\epsilon)$.

<u>Lemma 2.</u> (M. Kato) The following homomorphism is surjective.

$$\pi_1(N - V_o, *) \to \pi_1(\mathbb{C}^2 - V_o, *)$$

Proof. The proof is parallel to that of ψ in § 2. Let $V_p = f^{-1}(p)$. Then we have $\bar{V}_p \cap L_\infty = \bar{V}_0 \cap L_\infty = C \cap L_\infty$ where $\bar{V}_p$ is the closure curve in $\mathbb{P}^2$. Thus $\bar{V}_p$ is in the general position to L_∞. Therefore using a controlled vector field near L_∞, $f : f^{-1}(\mathbb{C} - \Sigma) \to \mathbb{C} - \Sigma$ is a fiber bundle. Take a positive number δ and paths $\{\ell_j\}$ in the exact same way as in the proof of lemma 1 and let $\Gamma_j = \ell_j \cup D^2_\delta(P_j)$ similarly.

Let $\varphi_j : \pi_1(V_\varepsilon, *) \to \pi_1(f^{-1}(\Gamma_j), *)$ be the natural homomorphism and consider the exact sequence:

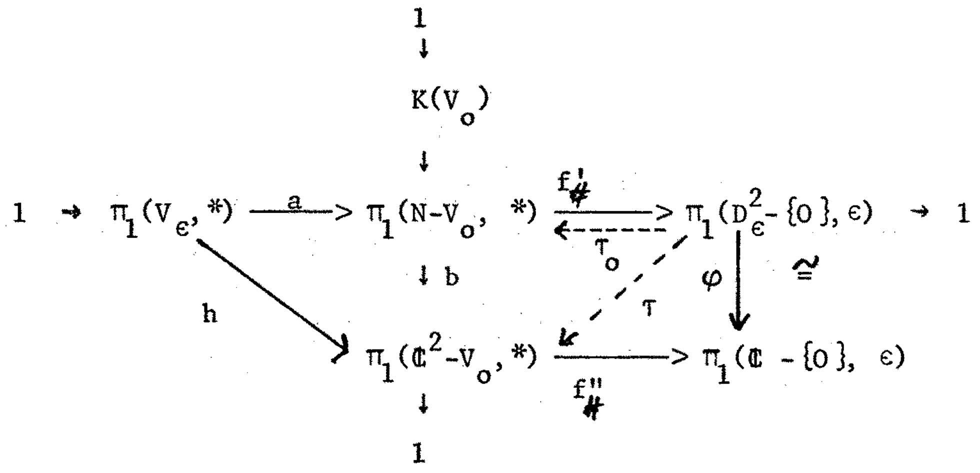
$$1 \to \pi_1(V_\varepsilon, *) \to \pi_1(f^{-1}(\Gamma_j) - V_{\rho_j}, *) \xrightarrow{f_\#} \pi_1(\Gamma_j - \{\rho_j\}, \varepsilon) \to 1$$

First observe that f has finite critical points on V_{ρ_j}. Otherwise $f(x, y) - \rho_j$ should have a square divisor which implies $\bar{V}_{\rho_i} \cap L_\infty$ contains strictly less than d points by Bezout's Theorem. This is a contradiction. Thus we can take an element τ_j such that $f_\# \tau_j$ is a generator of $\pi_1(\Gamma_j - \{\rho_j\}, \varepsilon) \cong \mathbb{Z}$ and τ_j is of the form $[\ell^{-1}.v.\ell]$ where v is a small loop revolving round V_{ρ_j} in the normal plane of a non-singular point of V_{ρ_j} and ℓ is a path in V_ε which connects $v(0)$ to the base point $*$. Define a cross-section σ_j of $f_\#$ naturally using τ_j. Then $\pi_1(f^{-1}(\Gamma_j) - V_{\rho_j}, *)$ is a semi-product of $\pi_1(V_\varepsilon, *)$ and $\mathbb{Z}$. It is clear that τ_j is mapped to the unit element 1 of $\pi_1(f^{-1}(\Gamma_j), *)$. Thus by the above argument, we can see that φ_j is surjective. Then the proof is done by the exact same way as that of surjectivity of ψ in Lemma 1.

Let $K(V_o)$ be the kernel of $\{\pi_1(N - V_o, *) \to \pi_1(\mathbb{C}^2 - V_o, *)\}$.

Lemma 3. Assume that V_o is irreducible. Then $\pi_1(\mathbb{C}^2 - V_o, *)$ is abelian if and only if $K(V_o)$ is equal to $\pi_1(V_\varepsilon, *)$ considering $\pi_1(V_\varepsilon, *)$ to be a subgroup of $\pi_1(N - V_o, *)$.

Proof. Consider the following diagrams.



Take a cross-section τ_o of $f'_{\#}$ and let $\tau = b \circ \tau_o$. Because V_o is irreducible, $\pi_1(\mathbb{C}^2-V_o,*)$ is abelian if and only if $\pi_1(\mathbb{C}^2-V_o,*)$ is isomorphic to $\mathbb{Z}$. Therefore, by the diagram, $\pi_1(\mathbb{C}^2-V_o,*)$ is abelian if and only if $f''_{\#}$ is isomorphism.

Assume that $\pi_1(\mathbb{C}^2-V_o,*)$ is abelian. Then we have $f''_{\#} \circ h = \varphi \circ f'_{\#} \circ a =$ the trivial map. This implies that h is trivial i.e. $\pi_1(V_\varepsilon,*) = K(V_o)$. On the contrary, assuming $\pi_1(V_\varepsilon,*) = K(V_o)$, we have that τ is isomorphic which implies $\pi_1(\mathbb{C}^2-V_o,*)$ is isomorphic to $\mathbb{Z}$. This completes the proof.

§ 4. Proof of Theorem 1.

Let C be an irreducible curve in $\mathbb{P}^2$ such that $\pi_1(\mathbb{P}^2-C)$ is abelian and let C' be any curve which is in the general position to C i.e. C and C' meet transversely. Take a general line L_∞ to $C \cup C'$. Identifying $\mathbb{P}^2-L_\infty$ with $\mathbb{C}^2$, let V and V' be the corresponding affine curves $C \cap \mathbb{C}^2$ and $C' \cap \mathbb{C}^2$ respectively. Actually we are going to prove the following theorem.

<u>Theorem 2.</u> $\pi_1(\mathbb{C}^2-V \cup V')$ is naturally isomorphic to $\pi_1(\mathbb{C}^2-V) \times \pi_1(\mathbb{C}^2-V')$ i.e. we have the following central extension which splits by the natural homomorphism: $\pi_1(\mathbb{C}^2-V \cup V', *) \to \pi_1(\mathbb{C}^2-V, *)$.

$$1 \to \pi_1(\mathbb{C}^2-V, *) \to \pi_1(\mathbb{C}^2-V \cup V', *) \to \pi_1(\mathbb{C}^2-V', *) \to 1$$

Assuming this theorem, we can prove Theorem 1 as follows. Consider the following commutative diagrams where the vertical sequences are obtained by Lemma 1.

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \uparrow & & \uparrow & & \\
1 & \to & \text{Ker } a & \xrightarrow{i} & \pi_1(\mathbb{P}^2-C \cup C', *) & \xrightarrow{a} & \pi_1(\mathbb{P}^2-C', *) & \to & 1 \\
 & & \Uparrow h & & \uparrow b & & \uparrow d & & \\
1 & \to & \mathbb{Z} & \xrightarrow[j]{} & \pi_1(\mathbb{P}^2-C \cup C' \cup L_\infty, *) & \xrightarrow{c} & \pi_1(\mathbb{P}^2-C' \cup L_\infty, *) & \to & 1 \\
 & & & & \uparrow k & & \uparrow \ell & & \\
 & & & & \mathbb{Z} & \xrightarrow{id.} & \mathbb{Z} & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

Let $h : \mathbb{Z} \to \text{Ker } a$ be the canonical homomorphism induced by the above diagram. We assert that h is isomorphic. Take $m \in \mathbb{Z}$ and assume that $h(m) = 1$. Then we can take an element m' of $\mathbb{Z}$ such that $j(m) = k(m')$. Then we have $c.j(m) = \ell(m') = 1$ which implies that $m' = 0$ and therefore $m = 0$. (We consider $\mathbb{Z}$ as an additive group.) Thus we have that h is injective. Take an element ω in Ker a . Then we can take an element ω' of $\pi_1(\mathbb{P}^2-C \cup C' \cup L_\infty, *)$ such that $b(\omega') = i(\omega)$. Because $d\, c(\omega') = 1$, we have an element m of $\mathbb{Z}$ such that $\ell(m) = c(\omega')$. Now letting $\omega'' = \omega'\, k(m)^{-1}$, we have that $b(\omega'') = i(\omega)$ and $c(\omega'') = 1$. Therefore we can find an element n of $\mathbb{Z}$ so that $j(n) = \omega''$ which implies $h(n) = \omega$. Thus we obtain that h is surjective. Now it is clear that Ker a is included in the center of $\pi_1(\mathbb{P}^2-C \cup C', *)$. This completes the proof of Theorem 1 modulo Theorem 2.

Let $f(x, y)$ and $g(x, y)$ be square-free polynomials which define V and V' respectively.

Let $\tilde{\Sigma}$ be the set of critical values of f and let $\Sigma = \tilde{\Sigma} - \{0\} = \{\rho_1, \rho_2, \ldots, \rho_\mu\}$.

Let D be a large disk which includes $\Sigma \cup \{0\}$. We can assume that $\infty = [1; 0; 0]$ is contained in $L_\infty - C \cup C'$. Consider pencil lines L_η centered at ∞ where L_η is defined by $y = \eta$. (In $\mathbb{P}^2, \bar{L}_\eta$ is defined by $Y = \eta Z$ because $x = X/Z$ and $y = Y/Z$).

We can take a positive number α large enough so that $V_\rho = f^{-1}(p)$ and L_η meet transversely for each $\rho \in D$ and $\eta(|\eta| \geq \alpha)$. Let $\tilde{D}$ be $f^{-1}(D) \cap \bigcup_{|\eta| \leq \alpha} L_\eta$. Then $\tilde{D}$ is a compact subset of $\mathbb{C}^2$ satisfying the following properties.

(i) $\tilde{D}$ is a deformation retract of $f^{-1}(D)$ and therefore it is also a deformation retract of $\mathbb{C}^2$.

(ii) $f : \tilde{D} - f^{-1}(\Sigma \cup \{0\}) \to D - \Sigma \cup \{0\}$ is a fiber bundle which is homotopically equivalent to the fibering

$$f : f^{-1}(D - \Sigma \cup \{0\} \to D - \Sigma \cup \{0\}.$$

Take a point $P_o = (x_o, y_o)$ in $\mathbb{C}^2 - V \cup V'$. Let $U(p_o)$ be a neighborhood of P_o in $\mathbb{C}^2 - V \cup V'$. Now we consider radical deformations of V' centered at P_o. More precisely, let $V'(\eta)$ be the affine curve defined by the polynomial equation $g_\eta(x,y) = g(\eta(x-x_o)+x_o, \eta(y-y_o)+y_o) = 0$.

Let h_η be the liner transformation of $\mathbb{C}^2$ defined by $h_\eta(x, y) = (\eta(x-x_o)+x_o, \eta(y-y_o)+y_o)$. Then we have that (i) $h_\eta(x_o, y_o) = (x_o, y_o)$ for each $\eta \in \mathbb{C}$ and (ii) $V'=V'(1)$ and $V'(\eta) = h_\eta^{-1}(V')$ for each η, $(\eta \neq 0)$.

Let A be the set defined by $\{\eta \in \mathbb{C} - \{0\} ; \overline{V'(\eta)}$ and C are not in the general position$\}$. We consider that 0 is contained in A.

Then we have the following lemma.

Lemma 4. A is a 0-dimensional analytic subset of $\mathbb{C}$.

Proof. $\overline{V'(\eta)}$ is defined by the homogeneous polynomial $G_\eta(X, Y, Z) = Z^{d_2} g(\eta(X/Z-x_o)+x_o, \eta(Y/Z-y_o)+y_o)$ where d_2 is the degree of $g(x,y)$ $(\eta \neq 0)$. Expressing $G_\eta(X,Y,Z)$ as $G_\eta(X,Y,0)+Z\cdot\widetilde{G}_\eta(X,Y,Z)$, we can see that $G_\eta(X,Y,0)/\eta^{d_2}$ does not depend on η $(\eta \neq 0)$. This implies that $\overline{V'(\eta)} \cap L_\infty = V' \cap L_\infty$. Thus each curve $V'(\eta)$ $(\eta \neq 0)$ is controlled by L_∞ . Let $F(X,Y,Z)$ be the homogeneous polynomial which corresponds to $f(x,y)$.

We consider an algebraic set B of $\mathbb{P}^2 \times \mathbb{C}$ by the following polynomial equations.

$$F(X, Y, Z) = 0 \quad . \quad G_\eta(X, Y, Z) = 0 \quad \text{and}$$

$$\operatorname{rank}\begin{pmatrix} \frac{\partial F}{\partial X}, & \frac{\partial F}{\partial Y}, & \frac{\partial F}{\partial Z} \\ \frac{\partial G_\eta}{\partial X}, & \frac{\partial G_\eta}{\partial Y}, & \frac{\partial G_\eta}{\partial Z} \end{pmatrix} \leq 1$$

Here η is considered to be the variable of $\mathbb{C}$. Let $\pi : \mathbb{P}^2 \times \mathbb{C} \to \mathbb{C}$ be the projection map. Then by the proper mapping theorem (p.162, [4]) , $\pi(B)$ is an analytic set of $\mathbb{C}$ and $\pi(B) = A$. Because $V'(1) = V$, we have that 1 is not contained in A . This means that A is a 0-dimensional analytic subset of $\mathbb{C}$ completing the proof.

Now we can take a number η_b in $\mathbb{C}-A$ ($|\eta_b|$ small enough) so that $V'(\eta_b) \cap \widetilde{D} = \emptyset$. This is done by taking η_b so that $h_{\eta_b}^{-1}(U(p_o)) \supset \widetilde{D}$ Take a smooth path p in $\mathbb{C}-A$ such that $p(0) = 1$ and $p(1) = \eta_b$. We can assume that p is an embedding of the unit interval $I = [0, 1]$.

Then we can prove the following lemma.

Lemma 5. There is a diffeomorphism $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\psi(V) = V$ and $\psi(V'(\eta_b)) = V'$. Therefore in particular we have a diffeomorphism $\psi : \mathbb{C}^2 - V \cup V'(\eta_b) \to \mathbb{C}^2 - V \cup V'$.

Proof: Let $W = \bigcup_{t \in I} \{V \cup V'(p(t)) \times t\}$ and $W_1 = \bigcup_{t \in I} (V \cap V'(p(t)) \times t)$ which are subsets of $\mathbb{C}^2 \times I$. Let $q : \mathbb{C}^2 \times I \to I$ be the projection map. By $\partial/\partial t$, we mean the unit vector field with positive direction on I. We can construct a connection vector field $\tilde{v}(x,y,t) = v(x,y,t) + \partial/\partial t$ for q, where $v(x,y,t)$ is the $\mathbb{C}^2$-component of $\tilde{v}(x,y,t)$, satisfying the following conditions. Let ϵ be a small number so that $V_\rho = f^{-1}(\rho)$ and $V'(\eta)$ meet transversely for each $\rho(|\rho| \le \epsilon)$ and η which is contained in the ϵ-neighborhood of $p(I)$ in $\mathbb{C}-A$.

(i) For any point (x,y,t) such that $|g_{p(t)}(x,y)| \ge \epsilon$, $v(x,y,t) = 0$.

(ii) For any point (x,y,t) such that $|g_{p(t)}(x,y)| \le \epsilon$ and $|f(x,y)| \le \epsilon/2$, $v(x,y,t)$ is tangent to $V_{f(x,y)}$ and in particular, if $g_{p(t)}(x,y) = 0$, $v(x,y,t)$ is tangent to the curve $w(s)$ which is defined by the corresponding component of $V_{f(x,y)} \cap V'(p(s))$. $v(x,y,t)$ is normalized so that the integral curves of $\tilde{v}$ are stable in W and W_1.

(iii) For any point (x,y,t) such that $g_{p(t)}(x,y) = 0$ and $|f(x,y)| \ge \epsilon/2$, $v(x,y,t)$ is taken so that the integral curves are stable in W. If $|f(x,y)| \ge \epsilon$, we can take $v(x,y,t)$ so that its integral curve $w(s)$ is $h_{p(s)}^{-1} \cdot h_{p(t)}(x,y)$ except near $L_\infty \cap \overline{V'(p(t))}$.

(iv) We can consider that $\infty = [1; 0; 0]$ is contained in $L_\infty - C \cup C'$. Considering the pencil lines $L_\eta = \{y=\eta\}$ centered at ∞ ($|\eta|$ is sufficiently large so that L_η and $V'(p(t))$ $(t \in I)$ meet transversely), we can construct v so that $v(x,y,t)$ is controlled by $\{L_\eta\}$ near $L_\infty \cap \overline{V'(p(t))}$ i.e. $v(x',y',t)$ is tangent to $L_{y'}$ and if $g_{p(t)}(x',y') = 0$, v is tangent to the curve $L_{y'} \cap V'(p(s))$ and normalized so that W is integrably stable.

$\tilde{v}$ is integrable and integral curves are stable in W and W_1 . Using the integral curves of $\tilde{v}$ we obtain a desired diffeomorphism ψ of $\mathbb{C}^2$. This completes the proof.

We are ready to prove Theorem 2. Take a positive number ϵ and δ so that the following conditions are satisfied.

(i) $D^2_\epsilon \cap \Sigma = \emptyset$ and V_ρ meets transversely with $V'(\eta_0)$ for each $\rho \in D^2_\epsilon$.

(ii) Let $P_1, P_2, \ldots, P_m$ be the singular points of V and let $D_\delta(P_j)$ be the 4-disk of radius δ centered at P_j which is included in $\tilde{D}$ For each $\rho \in D^2_\epsilon$, V_ρ meets transversely with the sphere $\partial D_\delta(P_j)$. and $f: E_j - V \to D^2_\epsilon - \{0\}$ is a Milnor fibering where $E_j = f^{-1}(D^2_\epsilon) \cap D_\delta(P_j)$. Let F_j be the fiber $V_\epsilon \cap D_\delta(P_j)$. (See Figure 4).

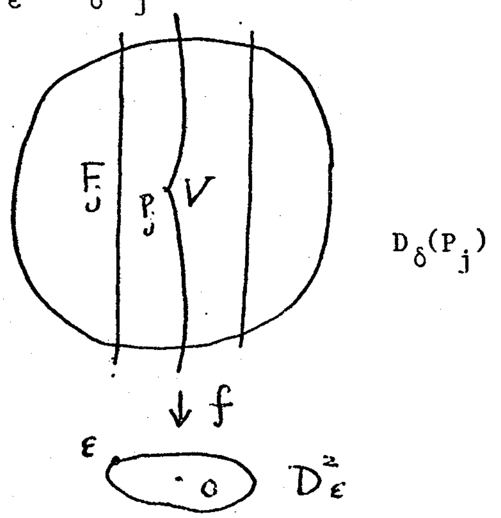


Figure 4.

Let $N = f^{-1}(D^2_\epsilon)$ and consider the following Van Kampen diagram.

$$\text{(V.K)}\qquad \begin{array}{ccccc} & & \pi_1(N-V'(\eta_b),*) & & \\ & \overset{\varphi_1}{\nearrow} & & \searrow & \\ \pi_1(N-V\cup V'(\eta_b),*) & & & & \pi_1(\mathbb{C}^2-V'(\eta_b),*) \\ & \underset{\varphi_2}{\searrow} & & \nearrow & \\ & & \pi_1(\mathbb{C}^2-V\cup V'(\eta_b),*) & & \end{array}$$

Consider the following fibering: $f: N-V\cup V'(\eta_b) \to D^2_\epsilon-\{0\}$. Using the fact that $f : N-(V'(\eta_b) \cup \bigcup_{j=1}^{m} \overset{o}{E}_j) \to D^2_\epsilon$ is trivial fibering, we have a family of characteristic diffeomorphisms $\{T_s\} : V_\epsilon-V'(\eta_b) \to V_{\epsilon(s)}-V'(\eta_b)$ ($\epsilon(s) = \epsilon.\exp(2\pi\, is)$, $0 \le s \le 1$) such that (i) T_o is the identity map and (ii) $T_1 | V_\epsilon-V'(\eta_b) \cup \bigcup_{j=1}^{m} \overset{o}{E}_j$ is the identy map. ($\overset{o}{E}_j$ is the interior of E_j). We can assume that the base point $*$ is contained in $V_\epsilon \cap \tilde{D} - \bigcup_{j=1}^{m} E_j$. Now consider the following exact sequence.

$$\text{(A)}\quad 1 \to \pi_1(V_\epsilon-V'(\eta_b),*) \to \pi_1(N-V\cup V'(\eta_b),*) \xrightarrow{f_\#} \pi_1(D^2_\epsilon-\{0\},\epsilon) \to 1$$

Let τ be the element of $\pi_1(N-V\cup V'(\eta_b),*)$ which is represented by the loop $w(s) = T_s(*)$. We can define a cross-section σ of $f_\#$ using τ . Using this cross-section σ , $\pi_1(N-V\cup V'(\eta_b),*)$ is a semi-product of $\pi_1(V_\epsilon-V'(\eta_b),*)$ and $\pi_1(D^2_\epsilon-\{0\}, \epsilon) \cong \mathbb{Z}$. By the above consideration, $(V_\epsilon-V'(\eta_b)) \cup \bigcup_{j=1}^{m} E_j$ is a deformation retract of $N-V'(\eta_b)$. Let K_o be the kernel of $\{\pi_1(V_\epsilon-V'(\eta_b),*) \to \pi_1(N-V'(\eta_b),*)\}$. First we prove the next lemma.

<u>Lemma 6.</u> K_o is generated by elements of the form $[\ell^{-1}.v.\ell]$ where v is a loop contained in some F_j and ℓ is a path in $V_\epsilon-V'(\eta_b)$ such that $\ell(0) = v(0)$ and $\ell(1) = *$.

Proof: Let $\Gamma_j = (V_\epsilon - V'(\eta_b)) \cup \bigcup_{i \leq j} E_i$ and consider tha following Van Kampen diagram.

$$(B_{j+1}) : \quad \begin{array}{ccccc} & & \pi_1(E_{j+1} \cup \ell_{j+1}, *) & & \\ & \overset{a_{j+1}}{\nearrow} & & \searrow & \\ \pi_1(F_{j+1} \cup \ell_{j+1}, *) & & & & \pi_1(\Gamma_{j+1}, *) \\ & \underset{b_{j+1}}{\searrow} & & \nearrow_{\pi_{j+1}} & \\ & & \pi_1(\Gamma_j, *) & & \end{array}$$

where ℓ_{j+1} is a path such that (i) $\ell_{j+1}(0) = *$ and $\ell_{j+1}(1)$ is a point of ∂F_{j+1}. (ii) The inclusion $F_{j+1} \hookrightarrow F_{j+1} \cup \ell_{j+1}$ is a homotopy equivalence. This means ℓ_{j+1} makes no cycles. By the induction on j, we prove that Kernel $[\pi_1(V_\epsilon - V'(\eta_b), *) \to \pi_1(\Gamma_j, *)]$ is generated by elements of the form $[\ell^{-1}.\nu.\ell]$ where ν is a loop contained in some $F_i (i \leq j)$ and ℓ is a path in $V_\epsilon - V'(\eta_b)$ which connects $\nu(0)$ to $*$. Let $K(\Gamma_j)$ be the latter group. Because E_{j+1} is contractible, a_{j+1} is the trivial homomorphism. Thus we have an exact sequence from (B_{j+1}).

$$(B'_{j+1}) : \quad 1 \to N(\mathrm{Image}(b_{j+1})) \to \pi_1(\Gamma_j, *) \to \pi_1(\Gamma_{j+1}, *) \to 1 .$$

where $N(\mathrm{Image}(b_{j+1}))$ is the normal closure of $\mathrm{Image}(b_{j+1})$.

Putting $j = 0$, we have

$$1 \to K(\Gamma_1) \to \pi_1(V_\epsilon - V'(\eta_b), *) \to \pi_1(\Gamma_1, *) \to 1 .$$

Assume the exact sequence

$$1 \to K(\Gamma_j) \to \pi_1(V_\epsilon - V'(\eta_b), *) \to \pi_1(\Gamma_j, *) \to 1 .$$

Then using (B_{j+1}'), we have that the sequence

$$1 \to K(\Gamma_{j+1}) \to \pi_1(V_\epsilon - V'(\eta_b), *) \to \pi_1(\Gamma_{j+1}, *) \to 1$$

is exact, completing the proof.

Now we return to the diagram (V.K) . By the above argument, we have that φ_1 is surjective and Ker φ_1 is normally generated by τ and K_o . Therefore we obtain the following exact sequence.

$$(E) : 1 \to N(\varphi_2(\tau), \varphi_2(K_o)) \to \pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *) \to \pi_1(\mathbb{C}^2 - V'(\eta_b), *) \to 1$$

where $N(\varphi_2(\tau), \varphi_2(K_o))$ is the minimal normal subgroup which contains $\varphi_2(\tau)$ and every element of $\varphi_2(K_o)$.

Assertion 1. $\varphi_2(K_o)$ is the trivial group.

For this, we consider the following diagrams

$$\begin{array}{ccccc}
\pi_1(V_\varepsilon \cap \tilde{D}, *) & \xrightarrow{c} & \pi_1(V_\varepsilon - V'(\eta_b), *) & \hookrightarrow & \pi_1(N - V \cup V'(\eta_b), *) \\
\downarrow a & & \downarrow b & & \downarrow \varphi_2 \\
\pi_1(\tilde{D} - V, *) & \longrightarrow & \pi_1(f^{-1}(D) - V'(\eta_b), *) & \longrightarrow & \pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *)
\end{array}$$

By Lemma 3 and the definition of $\tilde{D}$, we have that a is the trivial homomorphism. On the other hand, by Lemma 6, K_o is included in the normal closure of Image c . Thus we have that $b(K_o)$ is the trivial group which implies $\varphi_2(K_o)$ is also the trivial group.

Assertion 2. In (V.K), φ_2 is surjective.

For this, let Σ' be the set defined by $\{\eta \in \mathbb{C};\ V_\eta = f^{-1}(\eta)$ and $V'(\eta_b)$ are not in the general position$\}$. By the elimination theory, this is a finite set. Let $\Sigma_o = \Sigma \cup \Sigma' \cup \{0\}$ and consider $f : f^{-1}(\mathbb{C} - \Sigma_o) - V'(\eta_b) \to \mathbb{C} - \Sigma_o$. Using a controlled vector field near L_∞ and $V'(\eta_b)$, this is a fiber bundle. Then the proof is completely parallel to that of Lemma 2.

Assertion 3. $\varphi_2(\tau)$ is contained in the center of $\pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *)$.

For this, we consider the geometric picture of $V_\epsilon \cap \tilde{D}$ and $V_\epsilon - V'(\eta_b)$. Let d_1 and d_2 be the respective degrees of V_ϵ and $V'(\eta_b)$. Then V_ϵ is a Riemann surface punctured at d_1-points. By the definition of $\tilde{D}$, $V_\epsilon - V_\epsilon \cap \tilde{D}$ has d_1-connected components each of which is diffeomorphic to a punctured disk. By Bezout's theorem, $V_\epsilon \cap V'(\eta_b)$ contains exactly $d_1 d_2$ points. It is easy to see that $\hat{G}_o(X, Y, Z) \equiv \lim_{\eta \to 0} G_\eta(X, Y, Z) =$ $= Z^{d_2} \cdot g(x_o, y_o)$ (see Lemma 4).

This implies that $\lim_{\eta \to 0} \overline{V'(\eta)}$ is d_2-fold L_∞ . Thus we have that, in each component of $V_\epsilon - V_\epsilon \cap \tilde{D}$, there are exactly d_2-points which are contained in $V_\epsilon \cap V'(\eta_b)$.(See Figure 5.)

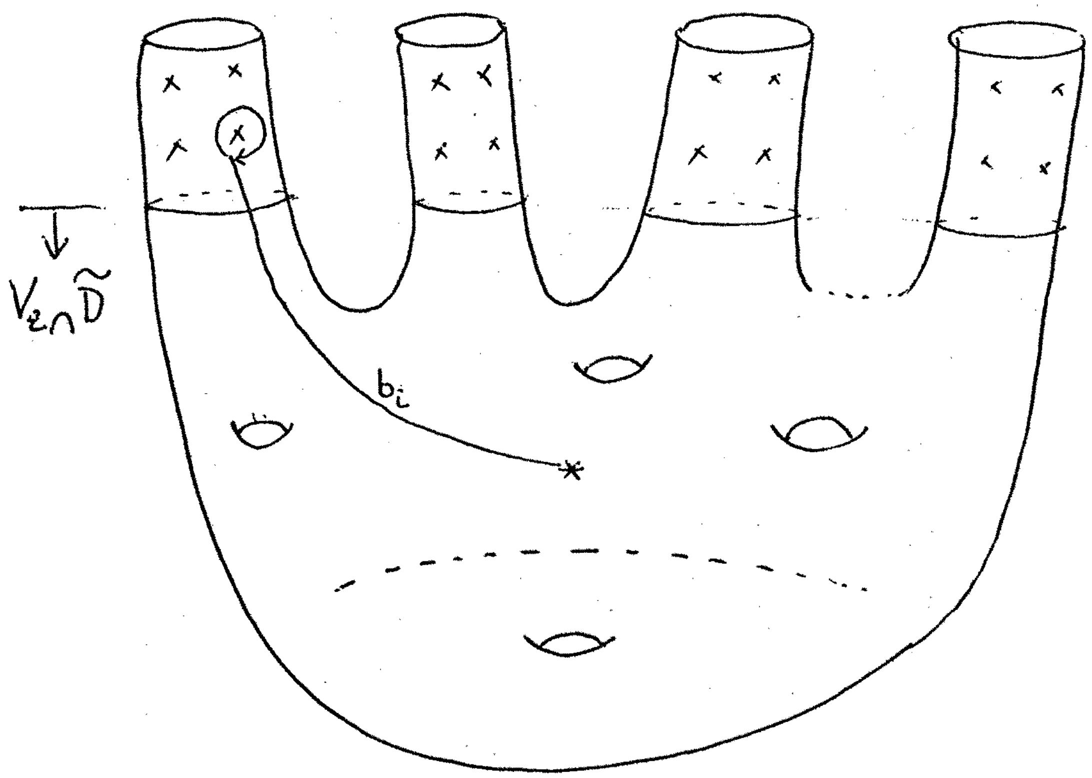


Figure 5

Let $V_\epsilon \cap V'(\eta_b) = \{a_1, a_2, \dots, a_{d_1d_2}\}$. By Van Kampen's theorem, we can take loops $\{b_j\}$ $(j = 1, 2, \dots, d_1d_2)$ so that the following conditions are satisfied.

(i) b_j is of the form $\ell_j^{-1} \nu_j \ell_j$ where ν_j is a small loop revolving round a_j and ℓ_j is a path such that $\ell_j(0) = \nu(0)$ and $\ell_j(1) = *$.

(ii) $\pi_1(V_\epsilon - V'(\eta_b), *)$ is generated by $\{[b_j]\}$ $(j=1,2,\dots,d_1d_2)$ and Image $[\pi_1(V_\epsilon \cap \tilde{D}, *) \to \pi_1(V_\epsilon - V'(\eta_b), *)]$.

(iii) Because V is irreducible, $V_\epsilon - \bigcup_i F_j$ is connected. Therefore we can also assume that ℓ_j is a path in $V_\epsilon - \bigcup_j F_j$.

Recall the exact sequence:

$$(F) : \quad 1 \to \pi_1(V_\epsilon - V'(\eta_b), *) \to \pi_1(N - V \cup V'(\eta_b), *) \to \pi_1(D_\epsilon^2 - \{0\}, \epsilon) \to 1$$

Take any element $[\omega]$ of $\pi_1(V_\epsilon - V'(\eta_b), *)$. By pulling back by characteristic diffeomorphisms $\{T_s\}$, $\tau^{-1}[\omega]\tau$ is nothing but $[T_1(\omega)]$. Therefore $\tau^{-1}[b_j]\tau = [b_j]$ in $\pi_1(N - V \cup V'(\eta_b), *)$. Using the diagrams of the proof of Assertion 1, it is easy to see that Image (φ_2) is generated by $\varphi_2([b_j])$ $(j = 1,2,\dots,d_1d_2)$ and $\varphi_2(\tau)$. Thus we obtain that $\varphi_2(\tau)$ is contained in the center of Image φ_2 which is equal to $\pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *)$ by Assertion 2. This completes the proof of Assertion 3. Returning to the sequence (E), we have just proved that $N(\varphi_2(\tau), \varphi_2(K_o))$ is isomorphic to the cyclic group generated by $\varphi_2(\tau)$. By the following diagram, it is clear that $\varphi_2(\tau)$ is not a torsion element.

$$\begin{array}{ccc} \pi_1(N - V \cup V'(\eta_b), *) & \xrightarrow{f_\#} & \pi_1(D_\epsilon^2 - \{0\}, \epsilon) \\ \downarrow & & \wr\| \\ \pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *) & \xrightarrow[f_\#]{} & \pi_1(\mathbb{C} - \{0\}, \epsilon) \end{array}$$

Thus we can reduce (E) as follows.

$$1 \to \mathbb{Z} \to \pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *) \to \pi_1(\mathbb{C}^2 - V'(\eta_b), *) \to 1$$

Identifying $\pi_1(\mathbb{C}^2 - V, *)$ with $\mathbb{Z}$, one obtains, that $\pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *) \to \pi_1(\mathbb{C}^2 - V, *)$ is a splitting of the above sequence. Since $\mathbb{Z}$ is included in the center of $\pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *)$, this gives us a natural isomorphism.

$$\pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *) \cong \pi_1(\mathbb{C}^2 - V, *) \times \pi_1(\mathbb{C}^2 - V'(\eta_b), *)$$

Now by Lemma 5, we have the following diagrams.

$$\begin{array}{ccccccccc} 1 \to & \pi_1(\mathbb{C}^2 - V, *) & \to & \pi_1(\mathbb{C}^2 - V \cup V'(\eta_b), *) & \to & \pi_1(\mathbb{C}^2 - V'(\eta_b), *) & \to 1 \\ & \Big\downarrow \psi'_\# \cong & & \Big\downarrow \psi_\# \cong & & \Big\downarrow \psi''_\# \cong & \\ 1 \to & \pi_1(\mathbb{C}^2 - V, *') & \to & \pi_1(\mathbb{C}^2 - V \cup V', *') & \to & \pi_1(\mathbb{C}^2 - V', *') & \to 1 \end{array}$$

This completes the proof of Theorem 2.

Remark. Let C be a non-irreducible curve. It is not always necessary that its irreducible components are in the general position for $\pi_1(\mathbb{P}^2 - C)$ to be abelian.

Example. Let C_1 be the non-singular curve $X^d + Y^d - Z^d = 0$ and let L be the line $Y-Z = 0$. $(d \geq 2)$. Then $C_1 \cap L = \{[0; 1; 1]\}$ and the intersection multiplicity is d. We can see that $\pi_1(\mathbb{P}^2 - C_1 \cup L)$ is isomorphic to $\mathbb{Z}$ as follows. Let $L_\infty = \{Z = 0\}$ and consider the map $\varphi : \mathbb{P}^2 - L_\infty \cup C_1 \cup L \to \mathbb{C} - \{0\}$ defined by $\varphi([X; Y; Z]) = Y/Z$. Then φ has $(d-1)$-critical values $\Sigma = \{\zeta, \zeta^2, \ldots, \zeta^{d-1}\}$ where $\zeta = \exp(2\pi i/d)$.

$\varphi : \varphi^{-1}(\mathbb{C} - \Sigma \cup \{0\}) \to \mathbb{C} - \Sigma \cup \{0\}$ is a fiber bundle. At each critical value, we have topologically the same situation. The general fiber F is diffeomorphic to $\mathbb{C} - \{\eta;\ \eta^d = 1\}$ and the characteristic map T_j around $\zeta^j (T_j : F \to F)$ can be considered to be the rotation of the angle $2\pi/d$. Therefore we have that $\pi_1(\mathbb{P}^2 - L_\infty \cup C_1 \cup L)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. This implies by Lemma 1 that $\pi_1(\mathbb{P}^2 - C_1 \cup L) = \mathbb{Z}$. This example is essentially due to Zariski ([19]).

Chapter III:

On the topology of the complement of a hypersurface in $\mathbb{P}^{n+1}$

§ 0. Introduction.

The purpose of this paper is to describe the similarity of $\mathbb{P}^{n+1}-V$ to $K(\pi, 1)$ where V is a hypersurface in $\mathbb{P}^{n+1}$ and π is the fundamental group of $\mathbb{P}^{n+1}-V$ in the case that π is abelian. This paper is organized as follows.

§ 1. Statement of results

Let $f_j(z_o, z_1, \ldots, z_{n+1})$ $(j=1,2,\ldots,r)$ be mutually distinct irreducible homogeneous polynomials and let V_j be the projective hypersurface defined by $V_j = \{[z] \in \mathbb{P}^{n+1} ; f_j(z) = 0\}$ $(j=1,\ldots,r)$. Let V be $V_1 \cup V_2 \cup \ldots \cup V_r$ and let F be the affine hypersurface defined by $F = \{z \in \mathbb{C}^{n+2};$ $f_1(z).f_2(z) \ldots f_r(z) = 1\}$. Then F is a d-fold cyclic covering space of $\mathbb{P}^{n+1}-V$ where $d = \sum_{j=1}^{r}$ degree (f_j). We have that $\pi_1(F)$ is a free abelian group of rank $r - 1$ if $\pi_1(\mathbb{P}^{n+1}-V)$ is abelian. (See § 4). We assume that $r \geq 2$. (For $r=1$, see Example 1 in § 8.)

We define a map $\xi : F \to (\mathbb{C}^*)^{r-1}$ by

$$\xi(z) = (f_2(z), f_3(z), \ldots, f_r(z)) .$$

Then we can express our results as follows.

<u>Theorem 3</u>. Assume that $V_1 \cap V_2 \ldots \cap V_r$ is non-singular and complete (i.e. $\dim_{\mathbb{C}} V_1 \cap V_2 \cap \ldots \cap V_r = n-r+1$) and that $\pi_1(\mathbb{P}^{n+1}-V)$ is abelian. Then ξ is an $(n-r+2)$-equivalence. (Actually it is not necessary to assume that $\pi_1(\mathbb{P}^{n+1}-V)$ is abelian if $\{V_j\}$ $(j=1,2,\ldots,r)$ are in a general position. For the assumption that $V_1 \cap V_2 \cap \ldots \cap V_r$ is non-singular implies that $\dim_{\mathbb{C}} \Sigma V_j \leq n-2$ $(r \leq n)$ and we know that $\pi_1(\mathbb{P}^{n+1}-V)$ is abelian by Theorem 1 and Corollary 1 of Theorem 2 in § 5.)

By the Whitehead theorem, we have the following;

<u>Corollary 1</u>. $\pi_j(\mathbb{P}^{n+1}-V) = \pi_j(F) = 0$ for $2 \leq j \leq n-r+1$.

<u>Corollary 2</u>. $H^j(F;\mathbb{Z})$ is isomorphic to $\binom{r-1}{j}\mathbb{Z}$ and the monodromy map $h^* : H^j(F;\mathbb{Z}) \to H^j(F;\mathbb{Z})$ is equal to the identity map for $j \leq n-r+1$. Here $k\mathbb{Z}$ means the direct sum $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ (k-copies) and the monodromy map $h : F \to F$ is defined by

$$h(z) = (z_o \exp \frac{2\pi i}{d}, z_1 \exp \frac{2\pi i}{d}, \ldots, z_{n+1} \exp \frac{2\pi i}{d}) .$$

Let $V_1, V_2, \ldots, V_r$ be non-singular hypersurfaces. We assume that $V_{i_1} \cap V_{i_2} \cap \ldots \cap V_{i_s}$ is non-singular and complete for each sequence $i_1 < i_2 < \ldots < i_s$ $(s \leq r)$. Then we say briefly that $\{V_j\}$ $(j=1,2,\ldots,r)$ <u>meet transversely in the strict sense</u>.

<u>Theorem 4.</u> Assume that $\{V_j\}$ $(j=1,\ldots,r)$ are non-singular and meet transversely in the strict sense. Then ξ is an $(n+1)$-equivalence.

As a corollary, we have the following.

<u>Corollary</u>. (i) $\pi_j(\mathbb{P}^{n+1}-V) = \pi_j(F) = 0$ for $2 \leq j \leq n$

(ii) $H^j(F;\mathbb{Z}) \cong \binom{r-1}{j}\mathbb{Z}$ and the monodromy map

$h^* : H^j(F; \mathbb{Z}) \longrightarrow H^j(F; \mathbb{Z})$ is the identity map for $j \leq n$.

Theorem 4 was essentially proved by Hattori-Kimura ([8]) and Hattori ([7]) in the case of each V_j being a hyperplane.

§ 2. A Zariski type theorem.

Let $f(z_o, z_1, \ldots, z_{n+1})$ be a square-free polynomial such that $f(0) = 0$. Let H_o be the affine hypersurface in $\mathbb{C}^{n+2}$ defined by $H_o = \{z \in \mathbb{C}^{n+2}; f(z)=0\}$ and let K be $H_o \cap S_\epsilon^{2n+3}$ where S_ϵ^{2n+3} is the $(2n+3)$-dimensional sphere of radius ϵ centered at the origin and ϵ is a small positive number which is a stable radius of the Milnor fibering of f at the origin. Let L be a general hyperplane which contains the origin. Then we have the following theorem.

Theorem Z . (Hamm; Lê [6] . The homomorphism

$$\pi_j((S_\epsilon - K) \cap L , *) \to \pi_j(S_\epsilon - K, *)$$

defined by the inclusion map is

(i) bijective for $j \leq n-1$

(ii) surjective for $j = n$.

(Here $S_\epsilon = S_\epsilon^{2n+3}$ and the base point * is chosen on $(S_\epsilon - K) \cap L$.)

Roughly speaking, a plane L is general if L meets transversely for each stratum X of a good stratification $\mathcal{S}$ of H_o (or K) so that $\{L \cap X\}_{X \in \mathcal{S}}$ should be a good stratification of $H_o \cap L$. For the precise definition and the proof of Theorem Z, we refer to [6] .

The following corollary will be used to prove Theorem 4. Assume that $f(z)$ is a homogeneous polynomial and let V be the projective hypersurface defined by $\{[z] \in \mathbb{P}^{n+1};\ f(z) = 0\}$. Let $\tilde{L}$ be the corresponding projective hyperplane to L. Then we have:

Corollary. The natural homomorphism

$$\pi_j((\mathbb{P}^{n+1} - V) \cap \tilde{L}, *) \to \pi_j(\mathbb{P}^{n+1} - V, *)$$

is (i) bijective for $j \leq n-1$

and

(ii) surjective for $j = n$.

Proof. Let $\varphi : S^{2n+3} - K \to \mathbb{P}^{n+1} - V$ be the restriction of the Hopf fibering $\varphi : S^{2n+3} \to \mathbb{P}^{n+1}$. Put $S = S^{2n+3}$ and $P = \mathbb{P}^{n+1}$. Take base points x_o and $\tilde{x}_o$ respectively so that $x_o \in (P-V) \cap \tilde{L}$ and $\varphi(\tilde{x}_o) = x_o$. Using the homotopy exact sequence of a fibration, we obtain that

$$\varphi_\# : \pi_j(S - K, \tilde{x}_o) \to \pi_j(P - V, x_o)$$

is bijective for $j \geq 3$. For $j = 2$, we consider the Milnor fibering

$$\psi = f/|f| : S - K \to S^1 .$$

Identifying $\pi_1(\varphi^{-1}(x_o), \tilde{x}_o)$ and $\pi_1(S^1, *)$ with the infinite cyclic group $\mathbb{Z}$, we see that the composition homomorphism

$$\pi_1(\varphi^{-1}(x_o), \tilde{x}_o) \to \pi_1(S-K, \tilde{x}_o) \xrightarrow[\varphi_\#]{} \pi_1(S^1, *)$$

is the multiplication with d = degree (f) under a suitable orientation ($* = \psi(\tilde{x}_o)$.) This implies that the homomorphism

$$\pi_1(\varphi^{-1}(x_o), \tilde{x}_o) \to \pi_1(S - K, \tilde{x}_o)$$

is injective. Combining this and the homotopy exact sequence of the fibration $\varphi : S - K \to P - V$, we obtain that $\varphi_\# : \pi_2(S-K, \tilde{x}_o) \to \pi_2(P-V, x_o)$

is also bijective. Considering $f|_L$ in the case of $(S-K)\cap L$, the above Corollary is an immediate consequence of Theorem Z and the above arguments using the following commutative diagram and the five lemma:

$$\begin{array}{ccc} \pi_j(S-K,\tilde{x}_o) & \xrightarrow[\cong]{\varphi_\#} & \pi_j(\mathbb{P}^{n+1}-V,x_o) \\ \uparrow & & \uparrow \\ \pi_j((S-K)\cap L,\tilde{x}_o) & \xrightarrow[\varphi_\#]{\cong} & \pi_j((\mathbb{P}^{n+1}-V)\cap\tilde{L},x_o) \end{array} \qquad j \geq 2$$

$$\begin{array}{ccccccccc} 0 \to & \pi_1(\varphi^{-1}(x_o),\tilde{x}_o) & \to & \pi_1(S-K,\tilde{x}_o) & \to & \pi_1(\mathbb{P}^{n+1}-V,x_o) & \to & 0 \\ & \uparrow \text{id} & & \uparrow\cong & & \uparrow & & \\ 0 \to & \pi_1(\varphi^{-1}(x_o),\tilde{x}_o) & \to & \pi_1((S-K)\cap L,\tilde{x}_o) & \to & \pi_1((\mathbb{P}^{n+1}-V)\cap\tilde{L},x_o) & \to & 0 \end{array}$$

This completes the proof of the Corollary. (This corollary was proved by Zariski [21] for the fundamental groups.)

§ 3. A Lefschetz type theorem.

Let $f_1(z_o, z_1,\dots,z_{n+1}),\dots,f_r(z_o, z_1,\dots,z_{n+1})$ be square-free homogeneous polynomials and let X be the projective variety defined by $X = \{[z] \in \mathbb{P}^{n+1} ;\ f_1(z) = f_2(z) = \dots = f_r(z) = 0\}$. Let $a : X \to \mathbb{P}^{n+1}$ be the inclusion map. Then we have the following theorem (Kato, [9], Lemma 6.1 of § 6).

Theorem L. $a : X \to \mathbb{P}^{n+1}$ is $(n-r+1)$-equivalence i.e.

$$a_\# : \pi_j(X, *) \to \pi_j(\mathbb{P}^{n+1}, *)$$

is bijective for $j \leq n-r$ and surjective for $j = n-r+1$.

Proof: Let H be the affine variety $\{z \in \mathbb{C}^{n+2};\ f_1(z) = f_2(z) = \dots = f_r(z) = 0\}$

and let $K = H \cap S^{2n+3}$. Then we know that (S^{2n+3}, K) is $(n-r+1)$-connected by Hamm, Satz 2.9, [5]. Now considering the homotopy exact sequence of the S^1-bundle pair $\varphi : (S^{2n+3}, K) \to (\mathbb{P}^{n+1}, X)$, we obtain the desired result. By virtue of the Whitehead theorem, we have the following corollary.

<u>Corollary 1.</u> (Oka, [14]) $a_* : H_j(X) \to H_j(\mathbb{P}^{n+1})$ is bijective for $j \leq n-r+1$. (Unless otherwise stated, every homology is with $\mathbb{Z}$-coefficient.)

In the case of X being a non-singular, complete intersection variety (i.e. $\dim_{\mathbb{C}} X = n-r+1$), we can decide $H_*(X; Q)$ as follows.

<u>Corollary 2.</u> Assume that X is a non-singular and complete intersection variety. Then we have:

$$H_j(X;Q) \cong \begin{cases} Q & 0 \leq j \leq 2(n+1-r), \ j : \text{even}, \ j \neq n+1-r \\ (\mu_r(d_1,\dots,d_r) + \epsilon(n+1-r))Q & j = n+1-r \\ 0 & \text{otherwise} \end{cases}$$

where $\epsilon(j) = 1$ or 0 for j even or odd respectively and $d_j = \text{degree}\,(f_j)$ $(j = 1,2,\dots,r)$ and μ_r is the following polynomial.

$$\mu_r(d_1,d_2,\dots,d_r) = (-1)^{n-r+1}\left(\prod_{j=1}^{r} d_j\right) \sum_{j+j_1+\dots+j_r=n-r+1} \binom{n+2}{j} (-d_1)^{j_1}(-d_2)^{j_2}\cdots(-d_r)^{j_r}$$

$$- (-1)^{n-r+1}(n-r+2)$$

<u>Proof.</u> In the case of $j \neq n-r+1$, Corollary 2 is an immediate consequence of Corollary 1 and Poincaré duality. μ_r is computed by the adjunction formula of the normal bundle. For the algebra structure of $H^*(X; Q)$, see Oka, [14].

§4. Fundamental groups

Let $f(z_o, z_1, \ldots, z_{n+1})$ be a square-free homogeneous polynomial of degree d. Let $V = \{[z] \in \mathbb{P}^{n+1};\ f(z) = 0\}$ and $K = \{z \in \mathbb{C}^{n+2};\ f(z) = 0,\ \|z\| = 1\}$. Consider the Milnor fibering $\psi = f/|f| : S^{2n+3}-K \to S^1$ and let F' be the fiber $\psi^{-1}(1)$. F' is naturally diffeomorphic to the affine hypersurface $F = \{z \in \mathbb{C}^{n+2};\ f(z) = 1\}$ by the diffeomorphism $k : F \to F'$ defined by

$$k(z_o, z_1, \ldots, z_{n+1}) = (z_o/\|z_o\|,\ z_1/\|z\|, \ldots, z_{n+1}/\|z\|) .$$

The monodromy maps $h : F \to F$ and $h' : F' \to F'$ are defined by the coordinate-wise multiplication with $\exp \frac{2\pi i}{d}$. These maps define free $\mathbb{Z}/d\mathbb{Z}$-actions on F and F' so that k is $\mathbb{Z}/d\mathbb{Z}$-compatible (i.e. $h' \circ k = k \circ h$). The orbit space $F'/\mathbb{Z}/d\mathbb{Z}$ is clearly diffeomorphic to $\mathbb{P}^{n+1}-V$. Therefore we have:

<u>Proposition 1.</u> F is a d-fold cyclic covering space of $\mathbb{P}^{n+1}-V$.

Next we consider the case that $V = V_1 \cup V_2 \cup \ldots \cup V_r$ and $f(z) = f_1(z)\, f_2(z) \ldots f_r(z)$ where V_j is irreducible and defined by $\{f_j = 0\}$ for $j = 1,2,\ldots,r$. Assume that $\pi_1(\mathbb{P}^{n+1}-V, *)$ be abelian. Then $\pi_1(\mathbb{P}^{n+1}-V, *)$ is decided as follows.

$$\begin{aligned} \pi_1(\mathbb{P}^{n+1}-V,\ *) &\cong H_1(\mathbb{P}^{n+1}-V) \\ &\cong H^{2n+1}(\mathbb{P}^{n+1}, V) \qquad \text{(Lefschetz duality)} \end{aligned}$$

Considering the following exact sequence

$$\to H^{2n}(\mathbb{P}^{n+1}) \xrightarrow[\phi]{} H^{2n}(V) \to H^{2n+1}(\mathbb{P}^{n+1}, V) \to 0 ,$$

we have that $H^{2n+1}(\mathbb{P}^{n+1}, V) \cong \operatorname{Coker} \phi$. Using the canonical isomorphism:

$H^{2n}(\mathbb{P}^{n+1}) \cong \mathbb{Z}$ and $H^{2n}(V) \cong H^{2n}(V_1) \oplus \ldots \oplus H^{2n}(V_r) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, ϕ is defined by $\phi(1) = (d_1, d_2, \ldots, d_r)$ where d_j = degree (f_j) $(j = 1,\ldots,r)$. Therefore we can take canonical generators e_j $(j = 1,2,\ldots,r)$ of $\pi_1(\mathbb{P}^{n+1}-V, *)$ as follows. Take a non-singular point P_j of $V_j - \bigcup_{i\neq j} V_i$ and let s_j be a small loop defined by a S^1-fibre in the normal bundle of V_j at P_j . Let ℓ_j be a path in $\mathbb{P}^{n+1}-V$ such that $\ell_j(0) = *$ and $\ell_j(1) = s_j(0)$. Define e_j by $[\ell_j\, s_j\, \ell_j^{-1}]$. (Figure 1)

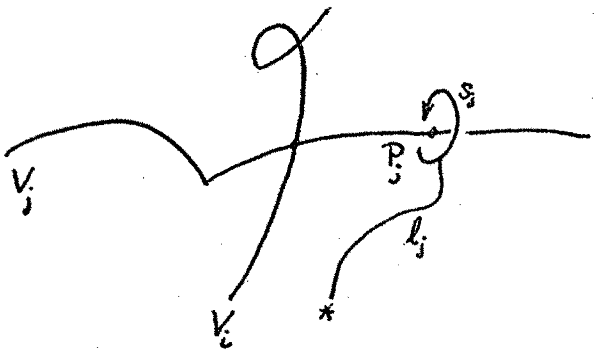


Figure 1.

By the above isomorphisms, e_j corresponds to $(0,\ldots,\overset{j}{1},\ldots,0)$. Note that $\{e_j\}$ $(j = 1,2,\ldots,r)$ have one generating relation

$$(G) \quad \sum_{j=1}^{r} d_j\, e_j = 0 \quad .$$

Let $P : F \to \mathbb{P}^{n+1} - V$ be the above covering map. Because $P_\# : \pi_1(F, \tilde{*}) \to \pi_1(\mathbb{P}^{n+1}-V, *)$ is an injection, we can consider $\pi_1(F, \tilde{*})$ to be a subgroup of $\pi_1(\mathbb{P}^{n+1}-V; *)$. $(P(\tilde{*}) = *)$.

<u>Lemma 1</u>. Assume that $\pi_1(\mathbb{P}^{n+1}-V, *)$ be abelian. Then $\pi_1(F, \tilde{*})$ is a free abelian group of rank $r-1$ and $P_\#(\pi_1(F, *))$ is generated by $\{e_1 - e_j\}$ $(j = 2,3,\ldots,r)$.

<u>Proof</u>. Let L be a general plane to V . Because e_j is independent of the choice of P_j , s_j and ℓ_j $(j=1,\ldots,r)$ we can assume that $\ell_j s_j \ell_j^{-1}$

is a loop in $\mathbb{P}^{n+1} - V \cup L$ for $j=1,2,\ldots,r$. If necessary, by a suitable transformation of coordinates, we can assume that L is defined by $\{z_o=0\}$. Let $\tilde{*}$ (the fixed base point) $= (\tilde{z}_o, \tilde{z}_1,\ldots,\tilde{z}_{n+1})$. Consider the canonical diffeomorphism $a : (\mathbb{C}^{n+2}-f^{-1}(0)) \cap \{z_o=\tilde{z}_o\} \to \mathbb{P}^{n+1}-V \cup L$ defined by $a(\tilde{z}_o, z_1,\ldots,z_{n+1}) = [\tilde{z}_o; z_1;\ldots;z_{n+1}]$. By virtue of a, we have a canonical element $\tilde{e}_j$ of e_j in $\pi_1(\mathbb{C}^{n+2}-f^{-1}(0), \tilde{*})$ $(j=1,2,\ldots,r)$. Consider the following exact sequence derived from the Milnor fibering:

$f : \mathbb{C}^{n+2}- f^{-1}(0) \to \mathbb{C}^*$

$$
\begin{array}{ccccccc}
0 \to \pi_1(F, \tilde{*}) & \xrightarrow{\alpha} & \pi_1(\mathbb{C}^{n+2}-f^{-1}(0), \tilde{*}) & \xrightarrow{f_\#} & \pi_1(\mathbb{C}^*, f(\tilde{*})) \to 0 \\
 & \searrow^{P_\#} \quad \circlearrowright & \downarrow \varphi_\# & & \wr\Vert \\
 & & \pi_1(\mathbb{P}^{n+1}-V, *) & & \mathbb{Z}
\end{array}
$$

Under the canonical orientation of s_j $(j=1,\ldots,r)$, we can assume that $f_\#(\tilde{e}_j) = 1$ (identifying $\pi_1(\mathbb{C}^*, f(\tilde{*}))$ with $\mathbb{Z}$) for each $j = 1, 2,\ldots,r$. This implies $f_\#(\tilde{e}_1 - \tilde{e}_j) = 0$ for $j=2,\ldots,3$ and therefore they are contained in the image of α. By the definition we have that $\varphi_\#(\tilde{e}_j) = e_j$. Thus by the commutability of the above diagram, we have that e_1-e_j $(j=2,\ldots,r)$ are contained in the image of $P_\#$. Let N be the subgroup of $\pi_1(\mathbb{P}^{n+1}-V, *)$ generated by $\{e_1-e_j\}$ $(j=2,\ldots,r)$. Using the generating relation (G), we have that $\pi_1(\mathbb{P}^{n+1}-V, *)/N$ is isomorphic to $\mathbb{Z}/d\mathbb{Z}$ which implies, by the fact that $\pi_1(\mathbb{P}^{n+1}-V,*)/P_\#(\pi_1(F, *)) \cong \mathbb{Z}/d\mathbb{Z}$, that $N = P_\#(\pi_1(F, \tilde{*}))$.

Now we prove that $\{e_1-e_j\}$ $(j=2,3,\ldots,r)$ are linearly independent. Assume that $a_2(e_1-e_2) + a_3(e_1-e_3)+\ldots a_r(e_1-e_r) = 0$ for some $a_j \in \mathbb{Z}$ $(j=2,\ldots,r)$. Eliminating e_1 using (G) and the above equation, we obtain the following equation using the independence of $e_2,\ldots,e_r$.

$$
r-1\left\{\begin{pmatrix} d_1+d_2, & d_2, \dots & , d_2 \\ d_3, & d_1+d_3, & d_3,\dots,d_3 \\ \dots\dots & & \dots\dots \\ d_r, & d_r,\dots & d_r, d_1+d_r \end{pmatrix}\right. \begin{pmatrix} a_2 \\ a_3 \\ \cdot \\ \cdot \\ a_r \end{pmatrix} = \begin{pmatrix} 0 \\ \cdot \\ 0 \\ 0 \end{pmatrix}
$$

This implies that $a_j = 0$ by the next sublemma, completing the proof.

Sublemma. Let A_n be the following matrix.

$$
\begin{pmatrix} 1+x_1, & 1,\dots,1 \\ 1, & 1+x_2, & 1,\dots,1 \\ \\ 1,\dots,1 & , & 1+x_n \end{pmatrix} \qquad (x_j > 0 \text{ for } j=1,2,\dots,n)
$$

Then the determinant of A_n is always positive.

Proof. Let $f_n(x_1,\dots,x_n)$ be the determinant of A_n.
Then $f_n(x_1,\dots,x_n)$ is a symmetric polynomial of $\{x_j\}$. The coefficient of the monomial $x_1.x_2\dots x_j$ is clearly the constant term of $f_{n-j}(x_{j+1},\dots,x_n)$ i.e. $f_{n-j}(0)$. But $f_{n-j}(0)$ is 0 except $j = n$ or $n-1$. Thus we have

$$
f_n(x) = x_1x_2\dots x_n + \sum_{j=1}^{n} x_1\, x_2\dots\hat{x}_j\dots x_n
$$

Therefore $f_n(x) > 0$ if x_j is positive for each $j=1,\dots,n$.

Now recall that $\xi : F \to (\mathbb{C}^*)^{r-1}$ is defined by $\xi(z) = (f_2(z),\dots,f_r(z))$. Then we have:

<u>Lemma 2:</u> Assume that $\pi_1(\mathbb{P}^{n+1}-V, *)$ be abelian.

Then $\xi_\# : \pi_1(F, \tilde{*}) \to \pi_1((\mathbb{C}^*)^{r-1}, \xi(\tilde{*}))$ is bijective.

Proof. Let $\tilde{\xi} : \mathbb{C}^{n+1}-f^{-1}(0) \to (\mathbb{C}^*)^{r-1}$ be defined by $\tilde{\xi}(z)=(f_2(z),\ldots,f_r(z))$.
Then it is clear that $\tilde{\xi}|_F = \xi$.
Identifying $\pi_1((\mathbb{C}^*)^{r-1}, \xi(\tilde{*}))$ with $\underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}_{r-1}$ in a natural way, we put $\sigma_j = (0,\ldots,\overset{j+1}{1},\ldots 0)$. Then by definition of $\tilde{e}_j$, we have that $\tilde{\xi}_\#(\tilde{e}_j) = \sigma_j$ for $j=2,\ldots,r$ and $\tilde{\xi}_\#(\tilde{e}_1) = 0$. This implies that $\xi_\#(e_1-e_j) = -\sigma_j$ $(j=2,3,\ldots,r)$, completing the proof.

§ 5. Criterions for $\pi_1(\mathbb{P}^{n+1}-V)$ to be abelian.

Again assume that $V_1,\ldots,V_j$ be irreducible hypersurfaces in $\mathbb{P}^{n+1}$ and let $V = V_1 \cup V_2 \cup \ldots \cup V_r$. Generally V_j may have singularities.

Definition: $V_1, V_2,\ldots,V_r$ are said to be in a general position (in the weak sense) if they satisfy the following inductive conditions.

(C_1) If $n=1$, each two curves V_j and V_k $(j\neq k)$ meet transversely and $V_i \cap V_j \cap V_k = \emptyset$ for mutually distinct i, j, k .

(C_n) There is a hyperplane L which is general to V_j $(j=1,\ldots,r)$ and V in the sense of Theorem Z (§2) such that $V_1 \cap L$, $V_2 \cap L,\ldots,V_r \cap L$ satisfy (C_{n-1}) .

It is clear that if $\{V_j\}$ are non-singular and meet transversely in the strict sense, then $\{V_j\}$ are in a general position.

We have the following criterion for $\pi_1(\mathbb{P}^{n+1}-V, *)$ to be abelian.

Theorem 1. Assume that $V_1, V_2, \ldots, V_r$ are in a general position. Then $\pi_1(\mathbb{P}^{n+1}-V, *)$ is abelian if and only if $\pi_1(\mathbb{P}^{n+1}-V_j, *)$ is abelian for each $j = 1,2,\ldots,r$.

Proof: Applying the Corollary of Theorem Z inductively, we can take a general $\mathbb{P}^2$ for $V_1,\ldots,V_r$ and V which satisfies the following conditions. Let $C_j = V_j \cap \mathbb{P}^2$ $(j=1,\ldots,r)$ and $C = V \cap \mathbb{P}^2$.

(i) $\pi_1(\mathbb{P}^2 - C_j, *) \to \pi_1(\mathbb{P}^{n+1} - V_j, *)$ $(j=1,\ldots,r)$

and

$\pi_1(\mathbb{P}^2 - C, *) \to \pi_1(\mathbb{P}^{n+1} - V, *)$

are bijective.

Now by Corollary 2 of Theorem 1 in Oka [17], we know that $\pi_1(\mathbb{P}^2-C, *)$ is abelian if and only if $\pi_1(\mathbb{P}^2-C_j, *)$ is abelian for each $j=1,\ldots,r$. This completes the proof. As for the irreducible curves, we have the following criterion.

Theorem 2. Assume that V is irreducible (i.e. $r=1$). Then $\pi_1(\mathbb{P}^{n+1}-V, *)$ is abelian if and only if $\pi_1(F, \tilde{*}) = 0$.

Proof: Let $F \to \mathbb{P}^{n+1} - V$ be the covering map. Then we have that the quotient group $\pi_1(\mathbb{P}^{n+1}-V, *)/P_{\#}\pi_1(F_1 \tilde{*})$ is isomorphic to the cyclic group $\mathbb{Z}/d\mathbb{Z}$ (d = degree f), while $H_1(\mathbb{P}^{n+1}-V)$ is also isomorphic to $\mathbb{Z}/d\mathbb{Z}$ by the Lefschetz duality. This implies that $P_{\#}\pi_1(F, \tilde{*})$ is the commutator group of $\pi_1(\mathbb{P}^{n+1}-V, *)$, completing the proof.

Corollary 1. Let ΣV be the singular points of V . Assume that $\dim_{\mathbb{C}} \Sigma V \leq n-2$. Then $\pi_1(\mathbb{P}^{n+1}-V, *)$ is abelian.

Proof: This is an immediate consequence of Theorem 2 and the Theorem of Kato-Matsumoto [10] because F is $(n-s-1)$-connected where $s = \dim_{\mathbb{C}} \Sigma V$. (This can also be proved by the Corollary of Theorem Z.)

As a special case of Theorem 1 and Theorem 2, we have the following

Corollary 2. Assume that $\{V_j\}$ $(j=1,2,\ldots,r)$ are non-singular and meet transversely in the strict sense. Then $\pi_1(\mathbb{P}^{n+1}-V, *)$ is abelian.

§ 6. Proof of Theorem 3.

Assume that $\pi_1(\mathbb{P}^{n+1}-V, *)$ is abelian. Recall that $\xi : F \to (\mathbb{C}^*)^{r-1}$ is defined by $\xi(z) = (f_2(z),\ldots,f_r(z))$ where F is the affine hypersurface $\{z \in \mathbb{C}^{n+2};\ f_1(z).f_2(z)\ldots f_r(z) = 1\}$.

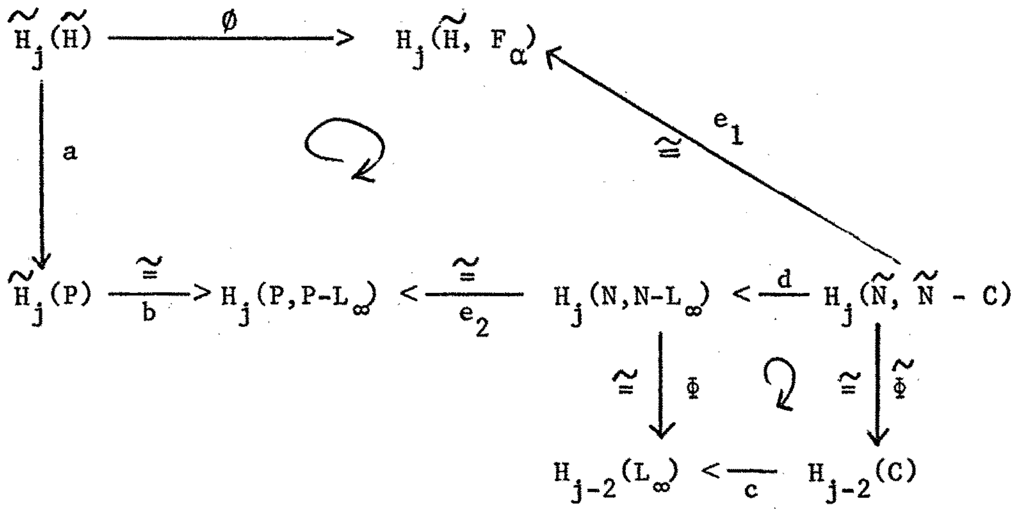
The following lemma is essential for the proof of Theorem 3.

Lemma.3. Under the same assumption as in Theorem 3, we have that $\tilde{H}_j(F_\alpha) = 0$ for $j \leq n-r+1$ and for each $\alpha \in (\mathbb{C}^*)^{r-1}$ where $F_\alpha = \xi^{-1}(\alpha)$.

Proof: Let $\alpha = (\alpha_2, \alpha_3,\ldots,\alpha_r)$. Then by the definition we can express $F_\alpha = H_1 \cap H_2 \cap \ldots \cap H_r$ where $\{H_j\}$ are affine hypersurfaces in $\mathbb{C}^{n+2}$ defined by $H_1 = \{z \in \mathbb{C}^{n+2};\ f_1(z) = (\alpha_2.\alpha_3\ldots\alpha_r)^{-1}\}$ and $H_j = \{z \in \mathbb{C}^{n+2};\ f_j(z) = \alpha_j\}$ for $j=2, 3,\ldots,r$. Consider the projective hypersurfaces $\tilde{H}_j$ in $\mathbb{P}^{n+2}$ defined by $\tilde{H}_1 = \{[z;\ w] \in \mathbb{P}^{n+2};\ f_1(z) = (\alpha_2\ldots\alpha_r)^{-1} w^{d_1}\}$ and $\tilde{H}_j = \{[z;\ w] \in \mathbb{P}^{n+2};\ f_j(z) = \alpha_j w^{d_j}\}$ for $j=2,\ldots,r$. (d_j = degree (f_j).) $\tilde{H}_j$ is the closure of H_j in $\mathbb{P}^{n+2}$ by the inclusion $H_j \subset \mathbb{C}^{n+2} \subset \mathbb{P}^{n+2}$. Let L_∞ be the hyperplane $\{w = 0\}$. Then we have natural homeomorphisms $F_\alpha \cong \tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r - \tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r \cap L_\infty$ and

$\tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r \cap L_\infty \cong V_1 \cap V_2 \cap \ldots \cap V_r$.

By the assumption, $V_1 \cap V_2 \cap \ldots \cap V_r$ is non-singular and complete. Let N be a tubular neighbourhood of L_∞ in $\mathbb{P}^{n+2}$. Because $\tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r \cap L_\infty$ is non-singular and complete, we can assume that $\tilde{N} = N \cap \tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r$ is a tubular neighbourhood of $\tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r \cap L_\infty$ in $\tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r$. Putting $P = \mathbb{P}^{n+2}$, $\tilde{H} = \tilde{H}_1 \cap \tilde{H}_2 \cap \ldots \cap \tilde{H}_r$ and $C = \tilde{H} \cap L$, we have the following commutative diagram .



Here $e_j (j=1, 2)$ are excision isomorphisms and Φ and $\tilde{\Phi}$ are Thom-isomorphisms. Because $P - L_\infty \cong \mathbb{C}^{n+2}$, b is bijective. By the corollary of Theorem L, a is bijective for $j \le n-r+1$ and surjective for $j = n-r+2$. Similarly c (therefore d) is bijective for $j \le n-r+2$ and surjective for $j = n-r+3$. Therefore we obtain from the diagram that $\emptyset$ is bijective for $j \le n-r+1$ and surjective for $j = n-r+2$. Considering the homology exact sequence of the pair $(\tilde{H}, F_\alpha)$, we have that $\tilde{H}_j(F_\alpha) = 0$ for $j \le n-r+1$. This completes the proof.

Now we are ready to prove Theorem 3.

Let $\pi : R \to (\mathbb{C}^*)^{r-1}$ be the universal covering map and let $\xi^{-1}R$ be the pull back of $\pi : R \to (\mathbb{C}^*)^{r-1}$ i.e. $\xi^{-1}R = \{(z,y) \in F \times R \ ; \ \xi(z) = \pi(y)\}$.

Let $p : \xi^{-1}R \to F$ and $\tilde{\xi} : \xi^{-1}R \to R$ be the respective projection maps. By Lemma 2 of § 4, $p : \xi^{-1}R \to F$ is the universal covering map i.e. $\xi^{-1}R$ is simply connected.

For each $y \in R$, we have that $\tilde{\xi}^{-1}(y) \cong \xi^{-1}(\pi(y)) = F_{\pi(y)} = \xi^{-1}(\pi(y))$. By the above lemma, we have that $H_j(\tilde{\xi}^{-1}(y)) = 0$ for each $j \leq n-r+1$. Now we consider the Leray's spectral sequence for $\tilde{\xi}$. (See for example VI, 6 of [3]. We have a convergent E_2 - spectral sequence:

$$E_2^{p,q} = H^p(R;\ \mathcal{H}^q(\tilde{\xi})) \Rightarrow H^{p+q}(\xi^{-1}R\ ;\ \mathbb{Z})$$

where $\mathcal{H}^q(\tilde{\xi})$ is the associated sheaf to the presheaf defined by $U \longmapsto H^q(\tilde{\xi}^{-1}(U);\ \mathbb{Z})$. Now note that $\tilde{\xi}$ is locally equivalent to ξ and that ξ can be considered to be a proper map. (For a given compact set $K \subset (\mathbb{C}^*)^{r-1}$, we can take a tubular neighbourhood N of L_∞ in the proof of Lemma 3 so that $\bar{F}_\alpha \cap N$ is a tubular neighbourhood of $\bar{F}_\alpha \cap L_\infty$ in $\bar{F}_\alpha$ for each $\alpha \in K$ where $\bar{F}_\alpha$ is the closure of F_α in $\mathbb{P}^{n+2}$. This implies that $F_\alpha - \mathring{N} \subset F_\alpha$ is a homotopy equivalence for each $\alpha \in K$, $\mathring{N}$ being the interior of N.) Therefore we have that $\mathcal{H}^q(\tilde{\xi})_x \cong H^q(\tilde{\xi}^{-1}(x);\ \mathbb{Z})$. Then Lemma 3 implies that $E_2^{p,q} = 0$ for $0 < q \leq n-r+1$ and $E_2^{0,n-r+2}$ is torsion-free. Thus we obtain that $\tilde{\xi}^* : H^j(R:\mathbb{Z}) \to H^j(\tilde{\xi}^{-1}R;\ \mathbb{Z})$ is bijective for $j \leq n-r+1$ and $H^{n-r+2}(\tilde{\xi}^{-1}R;\ \mathbb{Z})$ is torsion-free. By the universal coefficient theorem, we have that $\tilde{\xi}_* : H_j(\tilde{\xi}^{-1}R;\ \mathbb{Z}) \to H_j(R;\ \mathbb{Z})$ is bijective for $j \leq n-r+1$ which implies that $\tilde{\xi}$ (therefore ξ) is $(n+r+2)$-equivalence by the Whitehead theorem. This completes the proof of Theorem 3.

Proof of Corollary 2. The first part is clear. By the spectral sequence of a covering space (see [1]), $H^j(\mathbb{P}^{n+1}-V;\ Q)$ is isomorphic to $[H^j(F;\ Q)]^{\mathbb{Z}/d\mathbb{Z}}$ which is the kernel of $h^* - id : H^j(F;\ Q) \to H^j(F;\ Q)$. Because

$H^1(\mathbb{P}^{n+1}-V;\, Q) = (r-1)Q$, this implies that $h^*: H^1(F;\, Q) \to H^1(F;\, Q)$ is the identity map. Therefore $h^* = id : H^1(F;\, \mathbb{Z}) \to H^1(F;\, \mathbb{Z})$ by the universal coefficient theorem. By Theorem 3, $\Lambda^j H^1(F;\, \mathbb{Z}) \to H^j(F;\, \mathbb{Z})$ is bijective for $j \le n-r+1$. Therefore the multiplicative property of h^* implies the desired result, completing the proof.

§ 7. Proof of Theorem 4.

Let $\{V_j\}$ $(j=1,2,\ldots,r)$ be non-singular hypersurfaces, meeting transversely in the strict sense. Let $V = V_1 \cup V_2 \cup \ldots \cup V_r$

Lemma 4. The topology of $\mathbb{P}^{n+1}-V$ is decided by the respective degree d_j of V_j $(j=1,\ldots,r)$ and it does not depend on the particular choice of V_j's $(j=1,\ldots,\gamma)$

Proof. Let $\mathbb{P}^{N_j}$ be the parameter space of hypersurfaces of degree d_j where each point $t \in \mathbb{P}^{N_j}$ corresponds to a homogeneous polynomial $f_t(z)$ of degree d_j (or a hypersurface $V_t = \{f_t = 0\}$. $N_j = \binom{n+d_j+1}{d_j} - 1)$

Let $U = \{t = (t_1, t_2,\ldots,t_r) \in \mathbb{P}^{N_1} \times \mathbb{P}^{N_2} \times \ldots \times \mathbb{P}^{N_r}$; $\{V_{t_j}\}$ $(j=1,\ldots,\gamma)$ are non-singular and meet transversely in the strict sense. $\}$

Then we have that U is Zariski-open and therefore path-connected. Let $V' = V_1' \cup V_2' \cup \ldots \cup V_r'$ be another hypersurface satisfying the assumption of Theorem 4 such that degree V_i' = degree V_i $(i=1,\ldots,\gamma)$.

Then we can find a smooth family of hypersurfaces $\{V(t)\}$ $(0 \le t \le 1)$ such that $V(0) = V$ and $V(1) = V'$ and $V(t)$ can be written as $V(t) = V_1(t) \cup V_2(t) \cup \ldots \cup V_r(t)$ satisfying the assumption of Theorem 4. Therefore we can construct (using the technique of vector fields) an isotopy φ_t of $\mathbb{P}^{n+1}$ such that $\varphi_0 = id$ and $\varphi_1(V) = V'$. This completes the proof.

Proof of Theorem 4.

Take a positive integer N $(N-r+1 \geq n)$ and let $\tilde{V}_1, \tilde{V}_2, \dots, \tilde{V}_r$ be non-singular hypersurfaces in $\mathbb{P}^N$ such that degree $(\tilde{V}_j)$ = degree (V_j) and $\{\tilde{V}_j\}$ $(j=1,2,\dots,r)$ meet transversely in the strict sense. By Theorem 3 and Corollary 2 of Theorem 2 in § 5, putting $\tilde{V} = \tilde{V}_1 \cup \tilde{V}_2 \cup \dots \cup \tilde{V}_r$ we have that $\pi_j(\mathbb{P}^N - \tilde{V}) = 0$ for $2 \leq j \leq N-r+1$. Taking a sequence of general hyperplanes L_j $(j=1,2,\dots,N-n-1)$ where $L_j \cong \mathbb{P}^{N-j}$ and applying the Corollary of Theorem Z in § 2 inductively, we have that $\pi_j(L - L \cap \tilde{V}) \cong 0$ for $2 \leq j \leq n$ where $L = L_{N-n-1} \cong \mathbb{P}^{n+1}$. By Lemma 4 this implies that $\pi_j(\mathbb{P}^{n+1} - V) = 0$ for $2 \leq j \leq n$. This completes the proof of Theorem 4, combining Lemma 2 in § 5.

§ 8. The algebra structure and examples

In this section, we assume that $V_1, \dots, V_r$ are non-singular and meet transversely in the strict sense.

Because F is a non-singular affine hypersurface in $\mathbb{C}^{n+2}$, F has the homotopy type of a CW-complex of dimension $(n+1)$. Therefore we obtain the following theorem as a corollary of Theorem 4.

Theorem 5. $H^*(F; \mathbb{Z})$ is isomorphic as an algebra to the quotient algebra of the exterior algebra

$$E = \Lambda(x_1, x_2, \dots, x_{r-1}; y_1, \dots, y_\mu)$$

by the ideal $\mathfrak{a}_{n+2}$ which is generated by the monomials of degree $\geq n+2$, where degree $x_j = 1$ $(j=1,2,\dots,r-1)$ and degree $y_j = n+1$, $(j=1,\dots,\mu)$.

(μ is a polynomial of $d_1, d_2, \dots, d_r$. See Remark 1)

Using the Corollary of Theorem 4 and the fact that $H^*(\mathbb{P}^{n+1} - V; Q) \cong [H^*(F; Q)]^{\mathbb{Z}/d\mathbb{Z}}$, we have the following theorem.

Theorem 6. $H^*(\mathbb{P}^{n+1}-V; Q)$ is isomorphic to the quotient algebra of the exterior algebra $E' = \Lambda(x_1,\ldots,x_{r-1}; y'_1,\ldots,y'_\lambda)$ by the ideal $\mathfrak{a}'_{n+2}$ generated by the monomials of degree $\geq n+2$ where degree $x_j = 1$ $(j=1,\ldots,r-1)$ and degree $y'_j = n+1$ $(j=1,2,\ldots,\lambda)$.
(λ is a polynomial of $d_1, d_2,\ldots,d_r$. See Remark 1)

Example 1. Let V be a non-singular hypersurface of degree d in $\mathbb{P}^{n+1}$. Then F has the homotopy type of a bouquet $S^{n+1} \vee S^{n+1} \vee \ldots \vee S^{n+1}$ ($(d-1)^{n+2}$-copies) . Therefore $\pi_j(\mathbb{P}^{n+1}-V) \cong \pi_j(F) \cong \pi_j(S^{n+1}) \oplus \ldots \oplus \pi_j(S^{n+1})$ for $1 \leq j < 2n+1$.

Example 2. Assume that $\{L_j\}$ $(j=1,2,\ldots,r)$ are hyperplanes which meet transversely in the strict sense.

Case 1. $r \leq n+2$. In this case we have that ξ is an ∞-equivalence i.e. $\mathbb{P}^n-L$ is a $K((r-1)\mathbb{Z}, 1)$ space. $(L = L_1 \cup L_2 \cup \ldots \cup L_r)$.

Case 2. $r \geq n+3$ In this case $\mathbb{P}^n-L$ is not a $K((r-1)\mathbb{Z}, 1)$ space but Hattori prove that $H_j(\widetilde{\mathbb{P}^{n+1}-L}) = 0$ for $j \neq 0$, $n+1$ where $\widetilde{\mathbb{P}^{n+1}-L}$ is the universal covering space of $\mathbb{P}^{n+1}-L$ (See [9]).

Remark 1. In general, $H^*(\mathbb{P}^{n+1}-V; \mathbb{Z})$ has a torsion.

The number λ in Theorem 6 is decided by a direct computation of $H^*(\mathbb{P}^{n+1}-V; Q)$ as follows:

$$\lambda = \mu_r(d_1, d_2,\ldots,d_r) + \sum_{i=1}^{r} \mu_{r-1}(d_1,\ldots,\hat{d_i},\ldots,d_r) + \ldots + \sum_{i=1}^{r} \mu_1(d_i)$$

where $\{\mu_j\}$ are the polynomials defined in Corollary 2 of Theorem L in § 3. The number μ in Theorem 5 is decided by the following equation of the Euler-Poincaré characteristics.

$$\chi(F) = d\chi(\mathbb{P}^{n+1}-V) .$$

References.

[1] ABHYANKAR, S.S.; Tame coverings and fundamental groups of algebraic varieties. Amer. J. Math. I. 81 (1959); II 82 (1960)

[2] BOREL, A.; Seminar on Transformation Groups, Annals of Math. Studies, n° 46, Princeton Univ. Press, 1960.

[3] BREDON, G.E.; Sheaf theory, McGraw-Hill, 1967.

[4] GUNNING-ROSSI; Analytic function of several complex variables, Prentice-Hall, 1965.

[5] HAMM, H.; Lokal topologische Eigenschaften komplexer Räume, Math. Ann. 191, 1971.

[6] HAMM, H. et LÊ DUNG TRANG; Un théorème de Zariski du type de Lefschetz, Ann. Sci. de l'Ecole Normale Supérieure, fasc. 3, 1973.

[7] HATTORI, A.; Topology of $\mathbb{C}^n$ minus a finite number of affine hyperplanes in general position, to appear.

[8] HATTORI, A., and KIMURA, T., On the Euler integral representations of hypergeometric functions in several variables, J. Math. Soc. Japan, 26, 1974.

[9] KATO, M.; Partial Poincaré duality for K-regular spaces and complex algebraic sets, to appear.

[10] KATO, M. and MATSUMOTO, Y.; On the connectivity of Milnor fiber of a holomorphic function at a critical point, to appear in Proc. of Tokyo Manifold Conference, 1973.

[11] MILNOR, J.; Singular points of complex hypersurfaces, Annals of Math. Studies, n° 61, Princeton Univ. Press, 1968.

[12] MILNOR, J. and ORLIK, P.; Isolated singularities defined by weighted homogeneous polynomials, Topology 9 (1970).

[13] OKA, M.; On the homotopy types of hypersurfaces defined by weighted homogeneous polynomials, Topology, 12, 1973.

[14] OKA, M.; On the cohomology structure of projective varieties, to appear in the Proc. of Tokyo Manifold Conference.

[15] OKA, M.; The monodromy of a curve with ordinary double points, Inventiones Math. 27, 1974.

[16] OKA, M.; A central extension theorem of the fundamental group; Note, I.H.E.S.

[17] OKA, M.; On the fundamental group of the complement of a reducible curve, to appear.

[18] SPANIER, E.H.; Algebraic topology, McGraw-Hill, 1966.

[19] ZARISKI, O.; On the problem of existence of algebraic functions of two variables possessing a given branch curve. Amer. J. Math. Vol. 51 (1929).

[20] ZARISKI, O.; Algebraic surfaces; second edition, Springer, 1971.

[21] ZARISKI, O.; A theorem on the Poincaré group of an algebraic hypersurface, Annals of Math. 38, 1937.

Chapter IV: Non-trivial examples of projective curves

In [2], O. Zariski gave an example of a projective curve C of degree 6 such that the fundamental group $\pi_1(\mathbb{P}^2 - C)$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ where $\mathbb{Z}_n$ is a cyclic group of order n and $*$ is the free product. This curve C has six cusps on a conic. Each of them is locally described by the following equation (in the sense of topological equivalence).

$$x^2 + y^3 = 0 .$$

The purpose of this note is to propose a family of curves $C_{p,q}$ of degree pq (p, q : coprime integers), enjoying the following properties.

(I) $C_{p,q}$ has pq cusp singularities each of which is locally defined by the equation:

$$x^p + y^q = 0 .$$

(II) The fundamental group $\pi_1(\mathbb{P}^2 - C_{p,q})$ is isomorphic to $\mathbb{Z}_p * \mathbb{Z}_q$.

(III) Therefore the commutator group of $\pi_1(\mathbb{P}^2 - C_{p,q})$ is a free group of rank $(p-1) \cdot (q-1)$.

For the calculation we use the method of so-called pencil section introduced by Zariski [2]. In the remark (8.1), we will give another family of curves D_{2q} of degree $2q$: D_{2q} has q cusps and the fundamental group $\pi_1(\mathbb{P}^2 - D_{2q})$ is isomorphic to $\mathbb{Z}_{2q}$ (therefore abelian).

1. Definition of $C_{p,q}$

Let $C_{p,q}$ be the following projective curve.

(1.1) $$C_{p,q} : (X^p + Y^p)^q + (Y^q + Z^q)^p = 0 .$$

Here X, Y and Z are homogeneous coordinates of $\mathbb{P}^2$ and p and q are coprime integers. Then the possible singularities of $C_{p,q}$ must satisfy these three equations:

(1.2) $$X^{p-1}(X^p + Y^p)^{q-1} = 0$$

(1.3) $$Y^{p-1}(X^p + Y^p)^{q-1} + Y^{q-1}(Y^q + Z^q)^{p-1} = 0$$

(1.4) $$Z^{q-1}(Y^q + Z^q)^{p-1} = 0 .$$

Thus solving (1.2), (1.3) and (1.4), we find pq singularities in $C_{p,q}$ (if $p \geq 2$, $q \geq 2$):

(1.5) $$P_{\alpha,\beta} = [\alpha ; 1; \beta] ; \quad \alpha^p = -1, \quad \beta^q = -1.$$

To study the local behavior in a neighborhood of $P_{\alpha,\beta}$, we consider the affine coordinates $x = X/Y$ and $z = Z/Y$ then we put $\tilde{x} = x - \alpha$ and $\tilde{z} = z - \beta$. Then it turns out that the equation (1.1) is locally equivalent to the following

(1.6) $$\tilde{x}^q + c\cdot\tilde{z}^p = 0 , \quad (c : \text{non-zero constant}).$$

2. Pencil section

Consider the family of lines $L_\eta : X = \eta Y$, $\eta \in \mathbb{C}$. Each line L_η passes through the point $\infty \cong [0; 0; 1]$. We take ∞ as a base point of $\mathbb{P}^2 - C_{p,q}$. Since the intersection of L_η and $C_{p,q}$ is contained in the affine chart $\{Y \neq 0\}$, we consider the affine coordinates $x = X/Y$ and $z = Z/Y$. In these coordinates, the equation of the intersection points of $L_\eta : \{x = \eta\}$ and $C_{p,q}$ is the following

(2.1) $$(1+\eta^p)^q + (1+z^q)^p = 0 .$$

By solving (2.1), we have:

(2.2) $$z^q = -1 + \sqrt[p]{-(1+\eta^p)^q} .$$

These roots have two special cases.

Case (i). Assume that $\eta^p = -1$. Then we have that $z^q = -1$: Namely L_η passes through the singular points $P_{\eta,\beta}$, $\beta^q = -1$ of (1.5). At each $P_{\eta,\beta}$, the intersection multiplicity is exactly p.

Case (ii). Assume that $(1+\eta^p)^q = -1$ i.e. $\eta^p = -1 + \sqrt[q]{-1}$. In this case, one of the roots of (2.2) is zero. This implies that L_η is tangent to $C_{p,q}$ at ~~the~~ non-singular point $(\eta, 0)$ with the intersection multiplicity q.

For the other η, L_η and $C_{p,q}$ meet at exactly pq-points.

Let $\varphi : \mathbb{C}^2 - C_{p,q} \longrightarrow \mathbb{C}$ be the projection map i.e. $\varphi(x, z) = x$. Let Σ be $\{\eta \in \mathbb{C};\ \eta^p = -1 \text{ or } \eta^p = -1 + \sqrt[q]{-1}\}$. Then it is clear that the restriction of φ to $\varphi^{-1}(\mathbb{C} - \Sigma)$ is a locally trivial fibration.

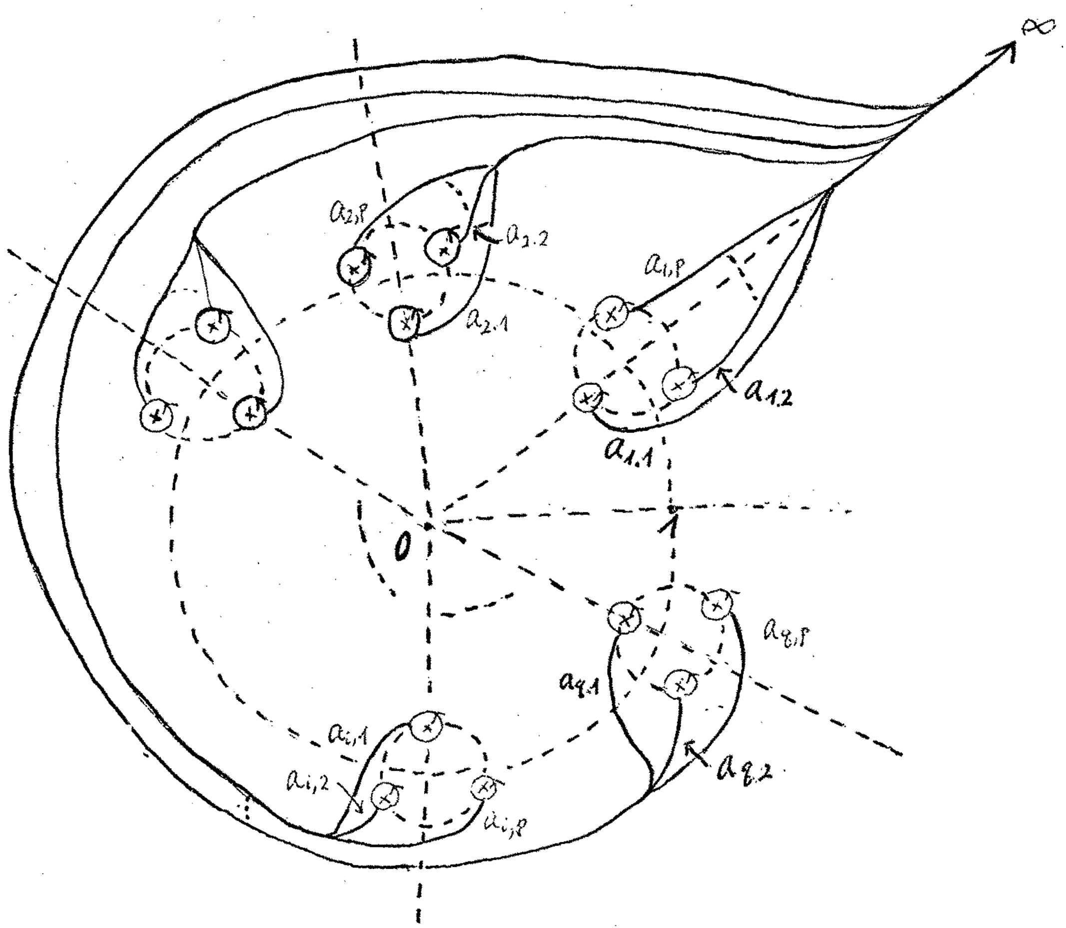
By Van Kampen [1], we have the following properties.

(I) Every loop ℓ in $\mathbb{P}^2 - C_{p,q}$ is deformed into a loop in the compactified fiber $\varphi^{-1}(\eta) \cup \{\infty\} = L_\eta - C_{p,q}$ for any $\eta \notin \Sigma$.

(II) If we fix $\eta_0 \in \mathbb{C} - \Sigma$, and if we choose generators of $\pi_1(\varphi^{-1}(\eta_0) \cup \{\infty\}, \infty)$, the generating relations are obtained by one torsion relation plus monodromy relations i.e. relations derived from the deformations of the generators along the fibers on the small circle $|x-\eta| = \varepsilon$ for every $\eta \in \Sigma$.

It is important to see that these monodromy relations depend upon only the value of η^p by virtue of (2.2) and the fact: $0 \notin \mathbb{C} - \Sigma$.

We take η_0 so that $\eta_0^p = -1 + \varepsilon_0 \exp(\pi i/q)$ where ε_0 is a small positive number. We take generators a_{ij}, $1 \leqq i \leqq q$, $1 \leqq j \leqq p$ in the way sketched in Figure 2.1.



$\varphi^{-1}(\eta_0)$ Figure 2.1

In Figure 2.1, each a_{ij} is oriented in the positive (≡ counter-clockwise) direction and is joined to the base point ∞ along the half line: argument (z) = π/q .

The torsion relation is this:

(2.3) $$\omega_q \omega_{q-1} \cdot \cdots \cdot \omega_1 = e$$

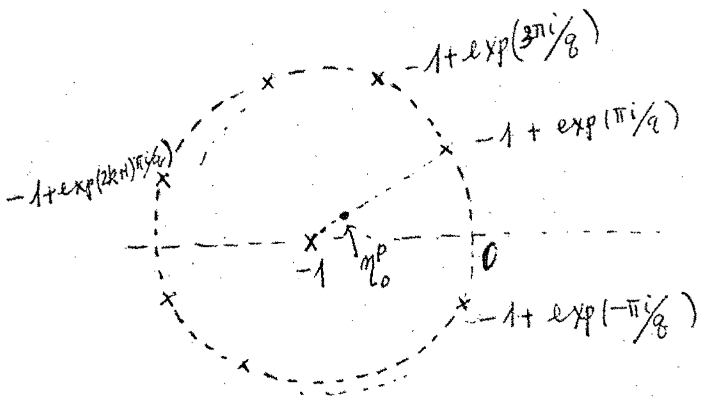
where e is the unit element and ω_i is defined by the following

$(2.4)_i$: $$\omega_i = a_{i,p} \cdot a_{i,p-1} \cdot \cdots \cdot a_{i,1}$$

where $1 \leqq i \leqq q$.

3. Local model I

Figure 3.1 shows the distribution of bad points $\{\eta^p \in \mathbf{C};\ \eta \in \Sigma\}$ in η^p-plane .



η^p-plane

Figure 3.1

First we consider the case (i) i.e. $\eta_1^p = -1$. Then $C_{p,q}$ and L_η are written as follows in a small neighborhood of $P_{\eta_1,\beta}$ ($\beta^q = -1$).

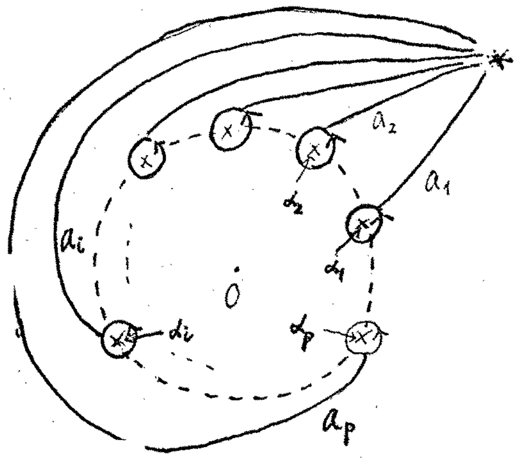
(3.1) $$C_{p,q} : \tilde{x}^q + c\,\tilde{z}^p = 0 \quad (c \neq 0)$$

(3.2) $$L_\eta : \tilde{x} = t \ , \quad t = \eta - \eta_1 .$$

We may assume that:

(3.3) $\quad q = mp + r, \quad 1 \leqq r \leqq p-1 \quad$ and $\quad (p, r) = 1$.

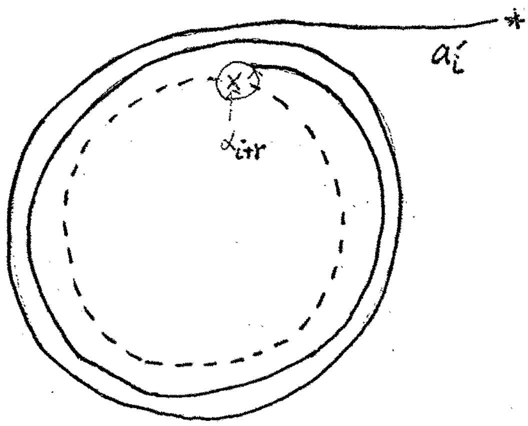
Choosing a small positive number ε , we take generators a_1, a_2, ..., a_p in the plane $\tilde{x} = \varepsilon$. See Figure 3.2.



$\tilde{z}$ - plane , ($\tilde{x} = \varepsilon$)

Figure 3.2

When t moves around the small circle $|t| = \varepsilon$ in the positive direction, a_i is transformed into a_i' in Figure 3.3.



(m = 1 2) $\quad (\alpha_{j+p} \equiv \alpha_j)$

Figure 3.3

Thus we get the following relations.

$$
(3.3)\qquad
\begin{cases}
a_1' = a_1 = \omega^m a_{1+r}\,\omega^{-m} \\
a_2' = a_2 = \omega^m a_{2+r}\,\omega^{-m} \\
\vdots \\
a_{p-r}' = a_{p-r} = \omega^m a_p\,\omega^{-m} \\
a_{p-r+1}' = a_{p-r+1} = \omega^{m+1} a_1\,\omega^{-(m+1)} \\
\vdots \\
a_p' = a_p = \omega^{m+1} a_r\,\omega^{-(m+1)}
\end{cases}
$$

where

$$
(3.4)\qquad \omega = a_p a_{p-1} \cdot \cdots \cdot a_1 .
$$

4. Local model II

Now we consider the case (ii). Fix η_1 such that $\eta_1^p = -1+\sqrt[q]{-1}$. Then in the neighborhood of the tangent point $(\eta_1, 0)$ of L_{η_1} and $C_{p,q}$, we can consider that $C_{p,q}$ and L_η are described by these equations:

$$
(4.1)\qquad C_{p,q} : \quad z^q = cx \qquad (c \neq 0)
$$

$$
(4.2)\qquad L_\eta : \quad x = \eta .
$$

Take generators $b_1, b_2, \ldots, b_q$ as in Figure 4.1.

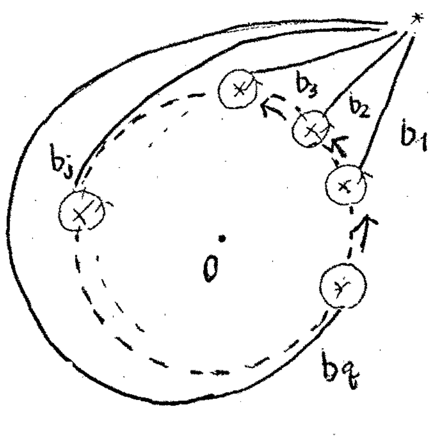


Figure 4.1

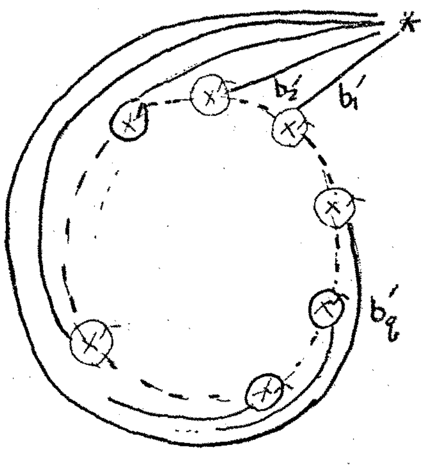


Figure 4.2

Figure 4.2 shows the transformation of $b_1, \ldots, b_q$ along a small circle centered at $\eta = \eta_1$. Namely we get the following monodromy relations.

$$b_1 = b_1' = b_2$$

$$b_2 = b_2' = b_3$$

$$\vdots$$

$$b_{q-1} = b_{q-1}' = b_q$$

$$b_q = b_q' = (b_q b_{q-1} \cdots b_1) b_1 (b_q b_{q-1} \cdots b_1)^{-1} .$$

Thus we obtain the relations:

$$(4.3) \qquad b_1 = b_2 = \cdots = b_q$$

5. Generating relations

Now we are ready to write down the generating relations between a_{ij} $(1 \leqq i \leqq q;\ 1 \leqq j \leqq p)$ of Figure 2.1. Take η_1 such that $\eta_1^p = -1$. By the deformation over the circle $|\eta^p - \eta_1^p| = \varepsilon$ (ε : small enough), each group of the elements $\{a_{i,1}, a_{i,2}, \ldots, a_{i,p}\}$ $(1 \leqq i \leqq q)$ gets the same relations as (3.3) and (3.4). Therefore we get the following relations.

$$(5.1)_i \qquad \begin{cases} a_{i,1} = \omega_i^{m} a_{i,1+r} \omega_i^{-m} \\ a_{i,2} = \omega_i^{m} a_{i,2+r} \omega_i^{-m} \\ \vdots \\ a_{i,p-r} = \omega_i^{m} a_{i,p} \omega_i^{-m} \\ a_{i,p-r+1} = \omega_i^{m+1} a_{i,1} \omega_i^{-(m+1)} \\ \vdots \\ a_{i,p} = \omega_i^{m+1} a_{i,r} \omega_i^{-(m+1)} \end{cases}$$

where $1 \leqq i \leqq q$.

Now we take η_k such that $\eta_k^p = -1 + \exp(-(2k-1)\pi i/q)$ where $0 \leqq k \leqq q-1$. We consider the following path $\ell_k \; S_k$ in η^p-plane for the translation of the monodromy relations at $\eta = \eta_k$ into the words of a_{ij} $(1 \leqq i \leqq q;\; 1 \leqq j \leqq p)$.

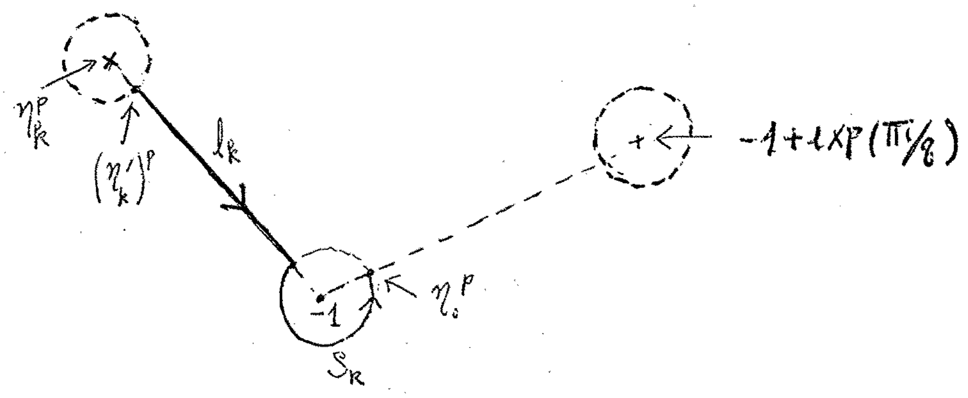


Figure 5.1 (η^p-plane)

Here S_k is an arc of the sphere $|\eta^p + 1| = \varepsilon_0$ and ℓ_k is the following line segment.

(5.2) $$\eta^p = t\eta_k^p + (1-t)\cdot(-1)$$

where $\varepsilon_0 \leqq t \leqq 1 - \varepsilon_1$ (ε_1 is a small positive number). The intersection of L_η and $C_{p,q}$ (η satisfies (5.2)) is the following

(5.3) $$z^q = -1 + \sqrt[p]{t^q} .$$

We take loops $b_1, b_2, \ldots, b_q$ in $\varphi^{-1}(\eta_k') \cup \{\infty\} = L_{\eta_k'} - C_{p,q}$ as in Figure 5.2 where $(\eta_k')^p = -1 + (1-\varepsilon_1)\eta_k^p$.

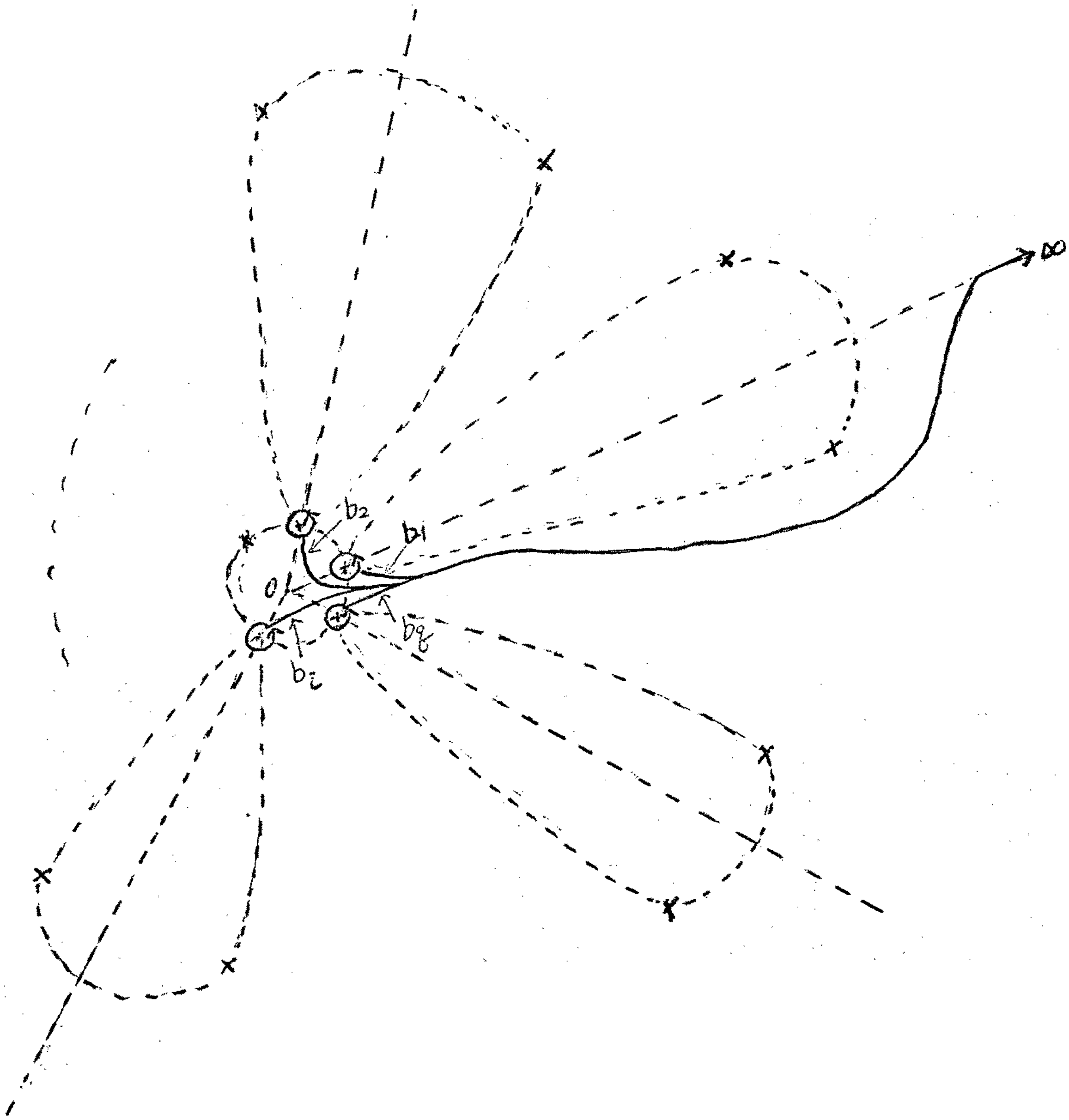


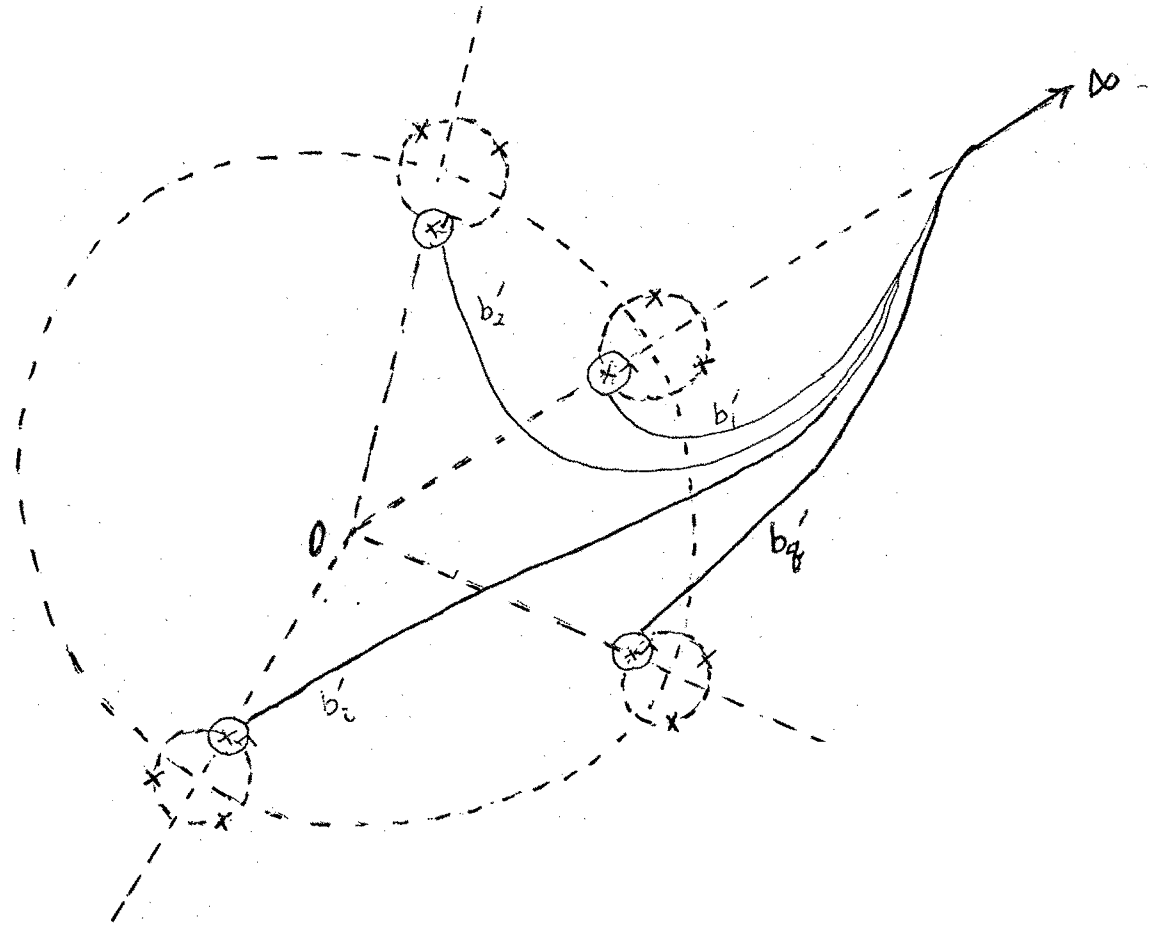
Figure 5.2

Each b_i is chosen so that the other roots of (5.3) do not meet any b_i when t moves for $\varepsilon_0 \leq t \leqq 1 - \varepsilon_1$.

By the consideration in the local model II, we have:

(5.4) $$b_1 = b_2 = \cdots = b_q \ .$$

When t moves from $1 - \varepsilon_1$ to ε_0, b_i, $1 \leqq i \leqq q$, are transformed into b_i' as in Figure 5.3.



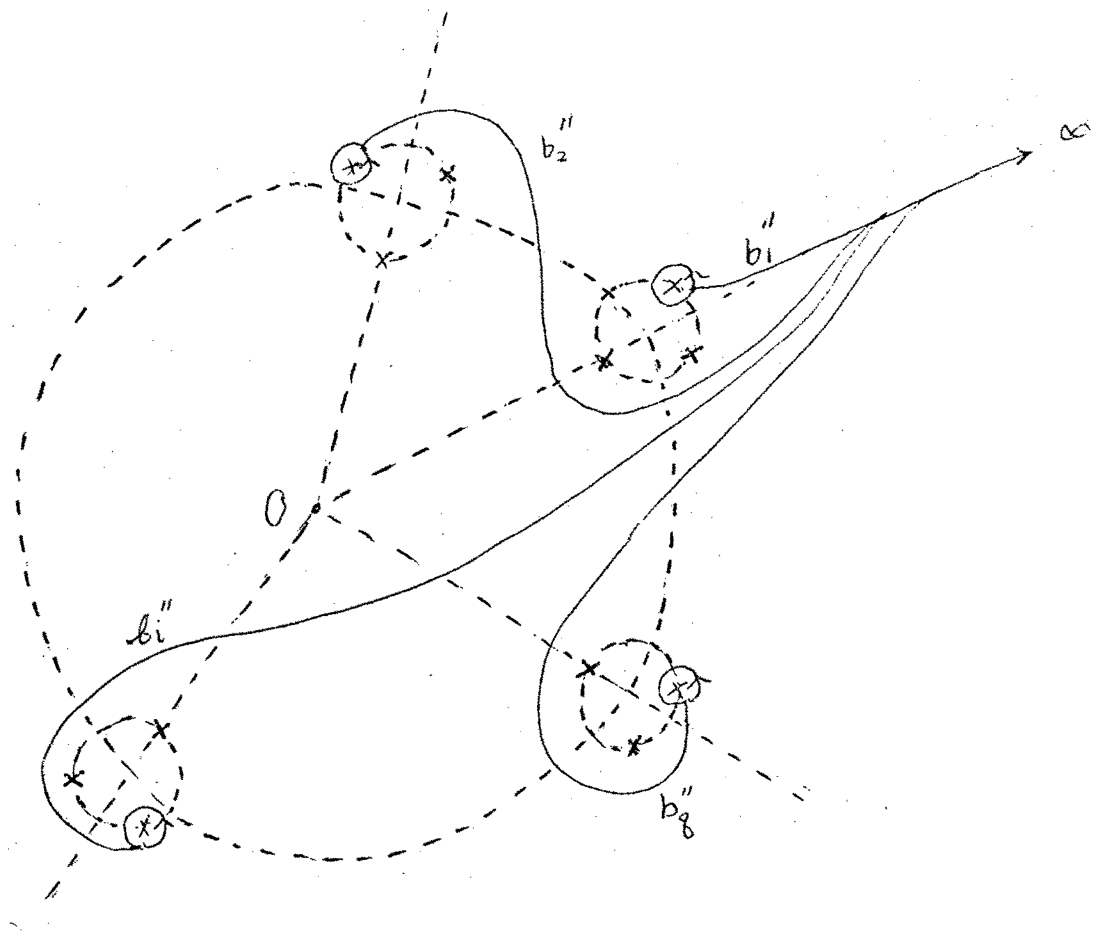
$L_\eta \quad (\eta^p = -1 + \varepsilon_0 \cdot \eta_k^p)$

Figure 5.3

Now we must pull back $b_1', \ldots, b_q'$ along S_k to $\varphi^{-1}(\eta_0)$ $\cup \{\infty\}$. Let $1 + \eta^p = \varepsilon_0 \exp(i\theta)$ where $-(2k-1)\pi/q \leq \theta \leq \pi/q$. By (2.2), we have:

$$z^q = -1 + \sqrt[p]{-\varepsilon_0^q \exp(iq\theta)}\ . \tag{5.5}$$

Thus it is easy to see that each b_i' is rotated along the respective small circle in Figure 5.3. These deformations are sketched in Figure 5.4.



L_{η_0} Figure 5.4

Translating in the words of $\{a_{ij}\}$ and $\{\omega_i\}$ we have:

$$b_1'' = a_{1,1+k}$$

$$b_2'' = \omega_1^{-1} a_{2,1+k}\, \omega_1 \qquad 0 \leq k \leq p-1$$

$$\vdots$$

$$b_q'' = (\omega_{q-1}\omega_{q-2}\cdots\omega_1)^{-1} a_{q,1+k}(\omega_{q-1}\omega_{q-2}\cdots\omega_1).$$

Thus (5.4) implies the following relations

(5.5) $$a_{1,j} = \omega_1^{-1} a_{2,j}\, \omega_j = \cdots = (\omega_{q-1}\cdots\omega_1)^{-1} a_{q,j}(\omega_{q-1}\cdots\omega_1$$

for $1 \leq j \leq p$.

6. Representation of the group

Thus $\pi_1(\mathbb{P}^2 - C_{p,q}, \infty)$ is generated by $pq+q$ elements a_{ij}, ω_i $(1 \leqq i \leqq q;\ 1 \leqq j \leqq p)$ and the generating relations are these:

$$(2.4)_i \qquad \omega_i = a_{i,p}\, a_{i,p-1} \cdots a_{i,1}\ , \quad 1 \leqq i \leqq q.$$

$$(2.3) \qquad \omega_q \cdot \omega_{q-1} \cdot \ \cdots\ \cdot \omega_1 = e$$

$$(5.1)_i \qquad \begin{cases} a_{i,1} = \omega_i^{m}\, a_{i,1+r}\, \omega_i^{-m} \\ a_{i,2} = \omega_i^{m}\, a_{i,2+r}\, \omega_i^{-m} \\ \vdots \\ a_{i,p-r} = \omega_i^{m}\, a_{i,p}\, \omega_i^{-m} \\ a_{i,p-r+1} = \omega_i^{m+1}\, a_{i,1}\, \omega_i^{-(m+1)} \\ \vdots \\ a_{i,p} = \omega_i^{m+1}\, a_{i,r}\, \omega_i^{-(m+1)} \end{cases} ; \quad 1 \leqq i \leqq q$$

and

$$(5.5) \quad a_{1,j} = \omega_1^{-1} a_{2,j}\, \omega_1 = \cdots$$
$$= (\omega_{q-1}\omega_{q-2}\cdots\omega_1)^{-1} a_{q,j} (\omega_{q-1}\omega_{q-2}\cdots\omega_1), \quad 1 \leqq j \leqq p.$$

(5.5) is equivalent to the following

$$(6.1) \qquad \begin{cases} a_{2,j} = \omega_1\, a_{1,j}\, \omega_1^{-1} \\ a_{3,j} = \omega_2\, a_{2,j}\, \omega_2^{-1} \\ \vdots \\ a_{q,j} = \omega_{q-1}\, a_{q-1,j}\, \omega_{q-1}^{-1} \end{cases} , \quad 1 \leqq j \leqq p\ .$$

Assume that $\omega_i = \omega_{i-1} = \cdots = \omega_1$. Then we have

$$\omega_{i+1} \overset{(2.4)_{i+1}}{=\!=\!=} a_{i+1,p} \cdot a_{i+1,p-1} \cdot \cdots \cdot a_{i+1,1}$$

$$\overset{(6.1)}{=\!=\!=} (\omega_i a_{i,p} \omega_i^{-1}) \cdot (\omega_i a_{i,p-1} \omega_i^{-1}) \cdots (\omega_i a_{i,1} \omega_i^{-1})$$

$$=\!=\!= \omega_i a_{i,p} a_{i,p-1} \cdot \cdots \cdot a_{i,1} \cdot \omega_i^{-1}$$

$$\overset{(2.4)_i}{=\!=\!=} \omega_i .$$

Therefore by the induction we get:

(6.2): $\omega_q = \omega_{q-1} = \cdots = \omega_1$

or

$(6.2)_i$: $\omega_i = \omega_{i-1}$; $2 \leqq i \leqq q$.

Conversely we can see that $(6.2)_{i+1} + (6.1) + (2.4)_i$ implies $(2.4)_{i+1}$. Thus an induction argument gives us the following equivalence

(6.3) $(2.4)_i \ (1 \leqq i \leqq q) + (6.1) \Longleftrightarrow (2.4)_1 + (6.1) + (6.2)$.

Now we consider the relations $(5.1)_i$:

For each k, $1 \leqq k \leqq p-r$, we have:

$$\omega_i^m a_{i,k+r} \omega_i^{-m} \overset{(6.1)+(6.2)}{=\!=\!=} \omega_{i-1}^m (\omega_{i-1} a_{i-1,k+r} \omega_{i-1}^{-1}) \omega_i^{-m}$$

$$\overset{(5.1)_{i-1}}{=\!=\!=} \omega_{i-1} a_{i-1,k} \omega_{i-1}^{-1}$$

$$\overset{(6.1)}{=\!=\!=} a_{i,k} .$$

Similarly for each k $(p-r+1 \leqq k \leqq p)$, (6.1), (6.2) and $(5.1)_{i-1}$ implies $(5.1)_i$. Therefore by the induction and (6.3), the generating relations are equivalent to $(2.3) + (2.4)_1 + (5.1)_1 + (6.1) + (6.2)$. Now (6.1) and (6.2) implies that each $a_{i,j}$ $(i \geqq 2)$ and ω_i $(i \geqq 2)$ can be expressed in the words of $a_{1,1}, a_{1,2}, \ldots, a_{1,p}$ and ω_1.

Therefore $\pi_1(\mathbb{P}^2 - C_{p,q}, \infty)$ is generated by $a_{1,1}, \ldots, a_{1,p}$ and ω_1. The generating relations are reduced to $(2.4)_1 + (5.1)_1$ plus

(2.3)': $$\omega_1^q = e .$$

Putting $a_j = a_{1j}$ $(1 \leq j \leq p)$ and $\omega = \omega_1$, we obtain the following.

Lemma 6.1. The fundamental group $\pi_1(\mathbb{P}^2 - C_{p,q}, \infty)$ has the following representation:

$a_1, a_2, \ldots, a_p$ and ω generate $\pi_1(\mathbb{P}^2 - C_{p,q}, \infty)$ and

(6.4) $$\omega = a_p a_{p-1} \cdots a_1$$

(6.5) $$\omega^q = e$$

(6.6) $$\begin{cases} a_1 = \omega^m a_{1+r}\, \omega^{-m} \\ a_2 = \omega^m a_{2+r}\, \omega^{-m} \\ \vdots \\ a_{p-r} = \omega^m a_p\, \omega^{-m} \\ a_{p-r+1} = \omega^{m+1} a_1\, \omega^{-(m+1)} \\ \vdots \\ a_p = \omega^{m+1} a_r\, \omega^{-(m+1)} . \end{cases}$$

7 Group structure

First we introduce elements a_i for any integer $i \in \mathbb{Z}$ by

(7.1) $$a_{j+kp} = \omega^k a_j \omega^{-k} \qquad \text{for } 1 \leq j \leq p, \ k \in \mathbb{Z} .$$

Then one can see that (7.1) implies

(7.2) $$a_{j+p} = \omega\, a_j \omega^{-1} \qquad \text{for } j \in \mathbb{Z} .$$

Using (7.2), we can rewrite (6.6) by this

(7.3) $$a_{j+q} = a_j \qquad \text{for } j \in \mathbb{Z} .$$

Therefore we get the representation :

$$\pi_1(\mathbb{P}^2 - C_{pq}, \infty) = \langle \omega, a_i (i \in \mathbb{Z}) ; (6.4), (6.5), (7.2), (7.3) \rangle .$$

Because p and q are coprime, we can write

(7.4) $$1 = p_1 p + q_1 q \qquad \text{for some } p_1, q_1 \in \mathbb{Z} .$$

Then

$$\begin{aligned} a_{i+1} &= a_{i+p_1p+q_1q} \\ &= \omega^{p_1} a_i \omega^{-p_1} \qquad \text{by (7.2) and (7.3)} . \end{aligned}$$

Thus one gets :

(7.5) $$a_{i+1} = \omega^{ip_1} a_1 \omega^{-ip_1} \qquad \text{for } i \in \mathbb{Z} .$$

By (7.5) and (6.4) ,

$$\omega = \omega^{(p-1)p_1} a_1 \omega^{-(p-1)p_1} \omega^{(p-2)p_1} a_1 \omega \ldots\ldots\ldots a_1 .$$

Namely by (7.4) and (6.5) ,

(7.6) $$(\omega^{-p_1} a_1)^p = e$$

Conversely (6.5), (7.5) and (7.6) implies (6.4), (7.2) and (7.3) :

$$\begin{aligned} a_p a_{p-1} \cdots a_1 &= \omega^{(p-1)p_1} a_1 \omega^{-(p-1)p_1} a_1 \omega^{-(p-2)p_1} a_1 \omega^{-(p-2)p_1} \ldots\ldots a_1 \quad \text{by (7.5)} \\ &= \omega (\omega^{-p_1} a_1)^p \qquad \text{by (6.5)} \\ &= \omega \qquad \text{by (7.6)} . \end{aligned}$$

$$a_{i+q} = \omega^{(i+q-1)p_1} a_1 \omega^{-(i+q-1)p_1} \qquad \text{by (7.5)}$$

$$= a_i \qquad \text{by (6.5) and (7.5)}.$$

$$a_{i+p} = \omega^{(i+p-1)p_1} a_1 \omega^{-(i+p-1)p_1} \qquad \text{by (7.5)}$$

$$= \omega^{pp_1} a_i \omega^{-pp_1} \qquad \text{by (7.5)}$$

$$= \omega\, a_i \omega^{-1} \qquad \text{by (7.9) and (6.5)}$$

Therefore one gets

$$\pi_1(\mathbb{P}^2 C_{pq}, \infty) \cong \langle \omega, a_i (i \in \mathbb{Z}) ;\ (6.5)\ ,\ (7.5),\ (7.6) \rangle$$

$$\cong \langle \omega, a_1 ;\ (6.5)\ ,\ (7.6) \rangle$$

by eliminating generators $a_i (i \neq 1)$.

Taking ω and $b = \omega^{-p_1} a_1$ as generators, we obtain

$$\pi_1(\mathbb{P}^2 C_{pq}, \infty) \cong \langle \omega, b ;\ \omega^q = e\ ,\ b^p = e \rangle$$

$$\cong \mathbb{Z}_p * \mathbb{Z}_q$$

8. Conclusion

Let us restate the result.

Let $C_{p,q}$: $(X^p + Y^p)^q + (Y^q + Z^q)^p = 0$ where p and q are coprime, $p \geq 2$, $q \geq 2$.

Theorem. The fundamental group $\pi_1(\mathbb{P}^2 - C_{p,q})$ is isomorphic to $\mathbb{Z}_p * \mathbb{Z}_q$.

Corollary. The commutator group D of $\pi_1(\mathbb{P}^2 - C_{p,q})$ is a free group of rank $(p-1)(q-1)$.

Proof. This is a well-known fact. A geometric sketch of the proof is the following: Let X be $\{$a 2-disk minus two small open 2-disks$\}$.

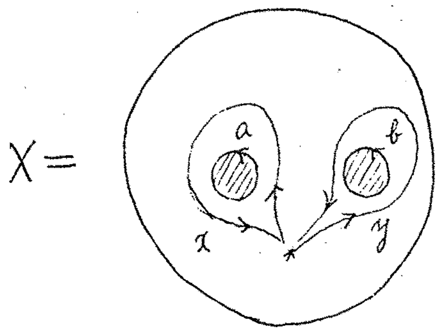


Figure 8.1

Let Y be the space obtained by attaching two 2-disks along a^p and b^q. Then the fundamental group of X is a free group generated by x and y in Figure 8.1 and the fundamental group of Y is isomorphic to $\mathbb{Z}_p * \mathbb{Z}_q$. Consider a surjective homomorphism $\varphi : \pi_1(X) \longrightarrow \mathbb{Z}_p \oplus \mathbb{Z}_q$ such that $\varphi(x)$ and $\varphi(y)$ are respective generators of $\mathbb{Z}_p$ and $\mathbb{Z}_q$. We can construct a finite covering space $\pi : \tilde{X} \longrightarrow X$ corresponding to the kernel of φ. Then the lift of a^p (b^q respectively) is q-copies (p-copies respectively) of embedded circles. Attaching $(p+q)$ 2-disks along these circles we obtain a Riemann surface $\tilde{Y}$ with boundary. (We may assume that the attaching maps are compatible with the action of $\pi_1(X)$.) By the construction, we can extend $\{\pi : \tilde{X} \longrightarrow X\}$ to $\{\pi' : \tilde{Y} \longrightarrow Y\}$ so that $\{\pi' : \tilde{Y} \longrightarrow Y\}$ is a covering space corresponding to the commutator group of $\pi_1(Y)$. Therefore one can see that the commutator group of $\pi_1(Y)$, which is isomorphic to $\pi_1(\tilde{Y})$, is a free group. The rank of $\pi_1(\tilde{Y})$ is easily calculated by the Hurewicz formula. (One can also prove the corollary purely group theoretically: If a and b are generators of $\mathbb{Z}_p$ and $\mathbb{Z}_q$ respectively, then $x_{ij} \equiv a^i b^j a^{p-i} b^{q-j}$, $1 \leq i \leq p-1$ and $1 \leq j \leq q-1$, are free basis of D.)

Remark (8.1). Consider the following curve:

$$D_{2q} : \ X^{2q-1}Y + (Y^q + Z^q)^2 = 0$$

where $q \geq 2$. This curve D_{2q} has q cusps at $P_\beta = [0;1;\beta]$, $\beta^q = -1$. Using the same pencil $L_\eta : X = \eta Y$ ($\eta \in \mathbb{C}$), one can see easily that $\pi_1(\mathbb{P}^2 - D_{2q})$ is isomorphic to $\mathbb{Z}_{2q}$. The calculation is done in the similar way. What is important is the technique to minimize the generating relations and generators.

Question 1. Take any irreducible curve C in $\mathbb{P}^2$. Is there a normal subgroup of the fundamental group $\pi_1(\mathbb{P}^2 - C)$ with a finite index which is isomorphic to a finitely generated free group?

Question 1'. If $\pi_1(\mathbb{P}^2 - C)$ is infinite, is the commutator group of $\pi_1(\mathbb{P}^2 - C)$ a free group? (cf. [2])

References

[1] Kampen, E. R. Van: On the fundamental group of an algebraic curve, Amer. J. Math. 55 (1933), 255-260 .

[2] Zariski, O.: On the problem of existence of algebraic function of two variables possessing a given branch curve, Amer. J. Math. 51 (1929), 305-328 .